Explicit Forwarding Controllers—Beyond Linearizable Class

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Abstract— **Following an explicit design for linearizable feedforward systems in [Krstic, "Integrator forwarding... ," ACC'04], we find closed form control laws for several classes of systems that are not linearizable, including some in the feedforward form and 'block-feedforward' form.**

I. INTRODUCTION

Their "true nonlinear" nature, their intolerance to aggressive controls, and the pioneering work done in the early–mid 1990's by Teel [19], [20], Mazenc and Praly [12], and Sepulchre, Jankovic, and Kokotovic [15], [16], have made feedforward systems one of the most intriguing, challenging, and active research topics in nonlinear control. A stream of contributions over the past ten years [1], [3], [8], [9], [10], [11], [13], [14], [17], [18], [21] has addressed a wide range of problems.

While it was believed that feedforward systems are "generically not feedback linearizable," in [5] we revealed a subclass linearizable by coordinate transformation. The control methods in [12], [16] require analytical solution of a series of nonlinear ODEs and integrals w.r.t. time. For two subclasses of feedforward systems, we generated explicit linearizing coordinate transformations, control laws, and even closed-loop solutions [6].

In this paper we develop control formulas for two classes of feedforward systems that are *not* linearizable, followed, in Section III, by an example similar to (but more challenging than) the Kokotovic-Teel third-order 'benchmark' example. The forwarding design procedure is extended to a class of 'block-feedforward' systems in Section IV, where closed-form feedback laws are also developed for two subclasses. Following an idea in [15], interlacing of forwarding and backstepping is pursued for two classes of systems for which feedback linearization formulas are given in Section V. Block-forwarding and interlacing are then all illustrated on an example (which is not linearizable) in Section VI.

II. ALGORITHMS FOR FEEDFORWARD SYSTEMS THAT ARE NOT LINEARIZABLE

In this section we expand upon the Type I and II feedforward systems [6], to develop algorithms for feedforward systems that are not linearizable.

Consider the following extension of the Type I strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u \tag{1}$$

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u \tag{2}$$

$$\dot{x}_i = x_{i+1}\,, \qquad i = 2,\ldots,n-1 \tag{3}$$

$$\dot{x}_n = u\,, \tag{4}$$

where $x = [x_1,\ldots,x_n]^T$ (i.e., x_0 is not included in x),

$$\psi_0(0) = \phi_0(0) = \pi_j(0) = 0\,, \quad j = 2,\ldots,n \tag{5}$$

and $\frac{\partial\psi_0(0)}{\partial x_i} = 0, i = 1,\ldots,n$. The subsystem (2)–(4) is linearizable. This makes it possible to develop a closed-form control law.

Design algorithm. Start by computing the expressions (110)–(112) in the Appendix. Then, calculate

$$\begin{aligned}\beta_1(x) &= -\int_0^\infty [\xi_1(\tau,x) + \psi_0(\xi(\tau,x)) \\ &\quad +\phi_0(\xi(\tau,x))\tilde{\alpha}_1(\tau,x)]\,d\tau \end{aligned}\tag{6}$$

$$w_0(x) = \phi_0(x) - \frac{\partial\beta_1(x)}{\partial x_1}\pi_n(x_n) - \frac{\partial\beta_1(x)}{\partial x_n} \tag{7}$$

$$\begin{aligned} u &= \alpha_0(x_0,x) \\ &= -w_0(x)(x_0-\beta_1(x)) \\ &\quad -\sum_{i=1}^{n}\begin{pmatrix} n \\ i-1\end{pmatrix} x_i + \sum_{i=2}^{n}\int_0^{x_i}\pi_i(s)ds\,. \end{aligned}\tag{8}$$

Theorem 1: The feedback system (1)–(4), (8) is globally asymptotically stable at the origin.

Proof: Lengthy calculations verify that $\frac{d}{dt}\sum_{i=0}^n z_i^2 = -w_0^2 z_0^2 - \sum_{i=1}^n z_i^2 - (w_0 z_0 + \sum_{i=1}^n z_i)^2$, where $w_0(0) = 1$, $z_0 = x_0 - \beta_1$, and

$$z_i = \sum_{j=i}^{n}\begin{pmatrix} n-i \\ j-i\end{pmatrix} x_j - \delta_{i,1}\sum_{j=2}^{n}\int_0^{x_j}\pi_j(s)ds \tag{9}$$

for $i = 1,\ldots,n$. ■

Next, consider the following extension of the Type II strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u \tag{10}$$

$$\dot{x}_i = x_{i+1} + \phi_i(\underline{x}_{i+1})u\,, \qquad i = 1,\ldots,n-1 \tag{11}$$

$$\dot{x}_n = u\,, \tag{12}$$

where the ϕ_i's satisfy the condition

$$\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1}\gamma_{j-i}(x_n)x_j + \phi_i(0,\ldots,0,x_n) \tag{13}$$

$\forall x, i = 1, \ldots, n-2$, where

$$\mu_n(x_n) = \frac{\int_0^{x_n} \phi_{n-1}(s)ds}{x_n} \quad (14)$$

$$\mu_i(x_n) = \frac{1}{x_n}\int_0^{x_n} \Big[\phi_{i-1}(0,\ldots,0,s) - \sum_{j=i+1}^{n} \mu_j(s)\phi_{i+n-j}(0,\ldots,0,s)\Big] ds \quad (15)$$

for $i = n-1, n-2, \ldots, 2$, and

$$\gamma_1(x_n) = \mu_n'(x_n) \quad (16)$$

$$\gamma_k(x_n) = \sum_{l=1}^{k-1} \gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \frac{d\mu_{n+1-k}(x_n)}{dx_n} \quad (17)$$

for $k = 2, \ldots, n-2$.

Design algorithm. Start by computing the expressions in (113)–(114) in the Appendix. Then, calculate

$$\beta_1(x) = -\int_0^{\infty} [\xi_1(\tau,x) + \psi_0(\xi(\tau,x)) + \phi_0(\xi(\tau,x))\tilde{\alpha}_1(\tau,x)]\, d\tau \quad (18)$$

$$w_0(x) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial \beta_1(x)}{\partial x_i}\phi_i(\underline{x}_{i+1}) - \frac{\partial \beta_1(x)}{\partial x_n} \quad (19)$$

$$u = \alpha_0(x_0, x) = -w_0(x)(x_0 - \beta_1(x)) - x_1 - \sum_{m=2}^{n} x_m \times \left[\binom{n}{m-1} - \sum_{j=1}^{m}\binom{n}{j-1}\mu_{j+1+n-m}(x_n)\right] \quad (20)$$

Theorem 2: The feedback system (10)–(12), (20) is globally asymptotically stable at the origin.

Proof: Same as proof of Theorem 1, except that

$$z_i = x_i + \sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m}\binom{n-i}{j-i}\mu_{j+1+n-m}(x_n)\right] \quad (21)$$

for $i = 1, \ldots, n-1$. ■

III. 3rd Order Example (Not Feedback Linearizable)

To illustrate the construction in Section II, consider the following example

$$\dot{x}_1 = x_2 + x_3^2 \quad (22)$$

$$\dot{x}_2 = x_3 + x_3 u \quad (23)$$

$$\dot{x}_3 = u. \quad (24)$$

The second order (x_2, x_3) subsystem is linearizable and is of both Type I and Type II. Like the "benchmark problem" $\dot{x}_1 = x_2 + x_3^2, \dot{x}_2 = x_3, \dot{x}_3 = u$, the overall system (22)–(24) is not feedback linearizable.

While the "benchmark" requires only two steps of forwarding because the (x_2, x_3) subsystem is linear, the system (22)–(24) requires three steps. The first two steps are already precomputed in Theorem 2 in [6],

$$\xi_3 = \left(x_3 - \tau\left(x_2 + x_3 - \frac{x_3^2}{2}\right)\right)e^{-\tau} \quad (25)$$

$$\xi_2 = \left((1+\tau)\left(x_2 + x_3 - \frac{x_3^2}{2}\right) - x_3\right)e^{-\tau} + \frac{1}{2}\left(x_3 - \tau\left(x_2 + x_3 - \frac{x_3^2}{2}\right)\right)^2 e^{-2\tau} \quad (26)$$

and $\tilde{\alpha}_2 = -\xi_2 - \xi_3 + \frac{\xi_3^2}{2}$.

The third step of forwarding is to calculate (6), (7),

$$\beta_2 = -2x_2 - x_3 + \frac{5}{8}x_3^2 - \frac{3}{8}\left(x_2 - \frac{x_3^2}{2}\right)^2, \quad (27)$$

$$w_1 = 1 + \frac{3}{4}x_3, \quad (28)$$

and the final control law

$$u = -w_1(x_1 - \beta_2) - \left(x_2 + x_3 - \frac{x_3^2}{2}\right) - x_3 = -x_1 - 3x_2 - 3x_3 - \frac{3}{8}x_2^2 + \frac{3}{4}x_3\Big(-x_1 - 2x_2 + \frac{1}{2}x_3 + \frac{x_2 x_3}{2} + \frac{5}{8}x_3^2 - \frac{1}{4}x_3^3 - \frac{3}{8}\left(x_2 - \frac{x_3^2}{2}\right)^2\Big). \quad (29)$$

In the remainder of this section we indicate one way to make the designs from the previous section applicable to a broader class of systems. Consider the example

$$\dot{x}_1 = x_2 + x_3^2 \quad (30)$$

$$\dot{x}_2 = \sinh x_3 + x_3 u \quad (31)$$

$$\dot{x}_3 = u. \quad (32)$$

which, although only a slight variation from (22)–(24), is not represented in the class (1)–(4). The difference in (31) is easily accommodated by the coordinate/pre-feedback change $X_3 = \sinh x_3, v = \sqrt{1 + (\sinh x_3)^2}u$, which converts (30)–(32) into

$$\dot{x}_1 = x_2 + \left(\sinh^{-1}(X_3)\right)^2 \quad (33)$$

$$\dot{x}_2 = X_3 + \frac{\sinh^{-1}(X_3)}{\sqrt{1 + X_3^2}}v \quad (34)$$

$$\dot{X}_3 = v. \quad (35)$$

This system fits the forms in Section II.

However, the system

$$\dot{x}_i = \sin(x_{i+1}), \qquad i = 1, \ldots, n-1 \quad (36)$$

$$\dot{x}_n = u, \quad (37)$$

suggested to us by Teel, motivated by the ball-and-beam problem [19], cannot be brought into those forms, except in the case $n = 2$ where the resulting control is

$$u = -x_2 - \frac{\sin x_2}{x_2}\left(x_1 - \int_0^{x_2} \frac{\sin \xi}{\xi} d\xi\right). \quad (38)$$

IV. Block-Forwarding

In this section we extend the class of systems to which the SJK forwarding procedure is applicable. Then we present our explicit controllers for this class of systems.

Consider the *block-strict-feedforward* systems[1]

$$\dot{x}_i = x_{i+1} + \psi_i\left(\underline{x}_{i+1}, \underline{q}_{i+1}\right) + \phi_i\left(\underline{x}_{i+1}, \underline{q}_{i+1}\right) u \quad (42)$$

$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right) \quad (43)$$

where $i = 1, 2, \ldots, n$, each x_i is scalar valued, each q_i is r_i-vector valued,

$$\underline{x}_i = [x_i,\ x_{i+1}, \ldots, x_n]^T \quad (44)$$

$$\underline{q}_i = [q_i^T,\ q_{i+1}^T, \ldots, q_n^T]^T\,, \quad (45)$$

A_i is a Hurwitz matrix for all $i = 1, 2, \ldots, n$, $x_{n+1} = u, q_{n+1} = 0, \phi_n = 0$, and $\frac{\partial \psi_i(0)}{\partial x_j} = \phi_i(0) = \omega_i(0) = 0$ for $i = 1, 2, \ldots, n-1, j = i+1, \ldots, n$. This class of systems should be understood as a dual of the *block-strict-feedback* systems in Section 4.5.2 of [7].

The control law is designed as follows. Let

$$\beta_{n+1} = \alpha_{n+1} = 0\,. \quad (46)$$

For $i = n, n-1, \ldots, 2, 1$

$$z_i = x_i - \beta_{i+1} \quad (47)$$

$$w_i(\underline{x}_{i+1}, \underline{q}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j}\phi_j - \frac{\partial \beta_{i+1}}{\partial x_n} \quad (48)$$

$$\alpha_i(\underline{x}_i, \underline{q}_{i+1}) = \alpha_{i+1} - w_i z_i \quad (49)$$

$$\begin{aligned} \beta_i(\underline{x}_i, \underline{q}_i) = -\int_0^\infty \Big[& \xi_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \\ & + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right) \\ & + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right) \\ & \times \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \underline{\eta}_{i+1}^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right)\Big] d\tau, \end{aligned} \quad (50)$$

where the notation in the integrand of (50) refers to the solutions of the (sub)system(s)

$$\begin{aligned} \frac{d}{d\tau}\xi_j^{[i]} = \; & \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right) \\ & + \phi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}, \underline{\eta}_{i+1}^{[i]}\right) \end{aligned} \quad (51)$$

$$\frac{d}{d\tau}\eta_j^{[i]} = A_j \eta_j^{[i]} + \omega_j\left(\underline{\xi}_j^{[i]}, \underline{\eta}_{j+1}^{[i]}\right) \quad (52)$$

for $j = i-1, i, \ldots, n$, at time τ, starting from the initial condition $(\underline{x}_i, \underline{q}_i)$. The control law is

$$u = \alpha_1\,. \quad (53)$$

Theorem 3: The feedback system (42), (43), (53) is globally asymptotically stable at the origin.

Proof: The Lyapunov function $V = \frac{1}{2}\sum_{i=1}^n z_i^2$, has a negative definite derivative,

$$\dot{V} = -\frac{1}{2}\sum_{i=1}^n w_i^2 z_i^2 - \frac{1}{2}\left(\sum_{i=2}^n z_i w_i\right)^2. \quad (54)$$

This implies that $x_n(t)$ converges to zero. Since $\omega_n(0) = 0$, we have that $\omega_n(x_n(t))$ converges to zero. Because A_n is Hurwitz, $q_n(t)$ converges to zero. One can show recursively that $w_i(0) = 1$ and $\beta_i(0) = 0$. It then follows that $w_{n-1}(x_n(t), q_n(t))$ converges to one. Since (54) guarantees that $w_{n-1}z_{n-1}$ goes to zero, $z_{n-1}(t)$ also goes to zero. Hence, $x_{n-1}(t) = z_{n-1}(t) + \beta_n(x_n(t), q_n(t))$ converges to zero. Continuing in the same fashion, one shows that $x(t), q(t) \to 0$ as $t \to \infty$. This establishes that the equilibrium $x = 0, q = 0$ is (uniformly) attractive. Global stability is argued in a similar, recursive fashion, using (54) and the fact that the subsystems (43) are input-to-state stable. In conclusion, the origin is globally asymptotically stable. ■

The solution $\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right)$, needed in the integral (50), is impossible to obtain analytically. Hence, we consider two classes of block-feedforward systems, inspired by systems of Types I and II, for which a closed-form controller can be obtained.

Consider the class of systems we refer to as *Type I block-feedforward* systems:

$$\dot{x}_0 = x_1 + \psi_0(x, q) + \phi_0(x, q)u \quad (55)$$

$$\dot{q}_0 = A_0 q_0 + \omega_0(x_0, x, q) \quad (56)$$

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u \quad (57)$$

$$\dot{q}_1 = A_1 q_1 + \omega_1\left(x, \underline{q}_2\right) \quad (58)$$

$$\dot{x}_i = x_{i+1}\,, \qquad i = 2, \ldots, n-1 \quad (59)$$

$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right) \quad (60)$$

$$\dot{x}_n = u \quad (61)$$

$$\dot{q}_n = A_n q_n + \omega_n(x_n)\,, \quad (62)$$

where x denotes $[x_1, \ldots, x_n]^T$, q denotes $[q_1^T, \ldots, q_n^T]^T$ (i.e., it does not include q_0), $\psi_0(0) = \phi_0(0) = \omega_0(0) = \omega_1(0) = \omega_j(0) = \pi_j(0) = 0, j = 2, \ldots, n$ and $\frac{\partial \psi_0(0)}{\partial x_i} = 0, i = 1, \ldots, n$. The subsystem $(x_1, \ldots, x_n)$ is linearizable, which makes it possible to develop a closed-form formula. The first step in the design algorithm is to compute the expressions in (110)–(111). It is worth noting that $\xi(\tau, x)$ and $\tilde{\alpha}_1(\tau, x)$ are both

[1] The blocks considered here are less general than those in [20], [12], [4]. We can generalize the idea we are presenting (even somewhat beyond the classes considered [20], [12], [4]), to include blocks q_i that are merely input-to-state stable with respect to $\left(\underline{x}_i, \underline{q}_{i+1}\right)$, rather than being linear in q_i. A simple example is the system

$$\dot{q} = -q^3 + x_2 \quad (39)$$

$$\dot{x}_1 = x_2 + qu \quad (40)$$

$$\dot{x}_2 = u\,. \quad (41)$$

This generalization would, however, preclude closed-form solvability of the problem; the result would be only an extension of [16].

independent of q. Then, for $i = n, n-1, \ldots, 2$ we calculate

$$\begin{aligned}\eta_i\left(\tau, \underline{q}_i, x\right) &= e^{A_i\tau} q_i + \int_0^\tau e^{A_i(\tau-\sigma)} \\ &\quad \times \omega_i\left(\underline{\xi}_i(\sigma, x), \underline{\eta}_{i+1}\left(\sigma, \underline{q}_{i+1}, x\right)\right) d\sigma,\end{aligned} \tag{63}$$

$$\begin{aligned}\beta_1(x,q) &= -\int_0^\infty [\xi_1(\tau,x) + \psi_0(\xi(\tau,x), \eta(\tau,q,x)) \\ &\quad + \phi_0(\xi(\tau,x), \eta(\tau,q,x))\,\tilde{\alpha}_1(\tau,x)]\,d\tau \quad (64)\\ w_0(x,q) &= \phi_0(x) - \frac{\partial\beta_1(x,q)}{\partial x_1}\pi_n(x_n) - \frac{\partial\beta_1(x,q)}{\partial x_n} \quad (65)\\ u &= \alpha_0(x_0,x,q) \\ &= -w_0(x,q)(x_0 - \beta_1(x,q)) \\ &\quad - \sum_{i=1}^n \begin{pmatrix} n \\ i-1 \end{pmatrix} x_i + \sum_{i=2}^n \int_0^{x_i} \pi_i(s)ds. \quad (66)\end{aligned}$$

Theorem 4: The feedback system (55)–(62), (66) is globally asymptotically stable at the origin.

Proof: The same expressions hold as in the proof of Theorem 1. Convergence to the origin is proved in the following order: $x_n, x_{n-1}, \ldots, x_1, q_n, q_{n-1}, \ldots, q_1, x_0, q_0$. Global stability is argued similarly. Hence, $x_0 = q_0 = 0, x = 0, q = 0$ is globally asymptotically stable. ■

Finally, consider the class of systems we refer to as *Type II block-feedforward* systems:

$$\begin{aligned}\dot{x}_0 &= x_1 + \psi_0(x,q) + \phi_0(x,q)u && (67)\\ \dot{q}_0 &= A_0 q_0 + \omega_0(x_0,x,q) && (68)\\ \dot{x}_i &= x_{i+1} + \phi_i(x_{i+1})\,u, \qquad i = 1,\ldots,n-1 && (69)\\ \dot{q}_i &= A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right) && (70)\\ \dot{x}_n &= u && (71)\\ \dot{q}_n &= A_n q_n + \omega_n(x_n), && (72)\end{aligned}$$

where the ϕ_i's satisfy the conditions (13)–(17) With $\xi(\tau,x)$ and $\tilde{\alpha}_1(\tau,x)$ calculated as in (112)–(114), η_i's and β_1 calculated as in (63), (64), respectively, the algorithm's final step is to calculate

$$w_0(x,q) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial\beta_1(x,q)}{\partial x_i}\phi_i(\underline{x}_{i+1}) - \frac{\partial\beta_1(x,q)}{\partial x_n} \tag{73}$$

$$\begin{aligned}u &= \alpha_0(x_0,x,q) \\ &= -w_0(x,q)(x_0 - \beta_1(x,q)) - x_1 - \sum_{m=2}^n x_m \\ &\quad \times \left[\begin{pmatrix} n \\ m-1 \end{pmatrix} - \sum_{j=1}^m \begin{pmatrix} n \\ j-1 \end{pmatrix} \mu_{j+1+n-m}(x_n)\right]\end{aligned} \tag{74}$$

Theorem 5: The feedback system (67)–(72), (74) is globally asymptotically stable at the origin.

V. Interlaced Feedforward-Feedback Systems

The ability to stabilize systems that are neither in the strict-feedback nor in the strict-feedforward form was illustrated in [15]. We present designs for two classes of systems obtained by interlacing strict-feedback systems [7] with feedforward systems of Type I and II.

First, consider the *interlaced systems of Type I*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u && (75)\\ \dot{x}_i &= x_{i+1}, \qquad i = 2,\ldots,n && (76)\\ \dot{x}_{n+1} &= x_{n+2} + f_1(\underline{x}_1, x_{n+1}) && (77)\\ \dot{x}_{n+j} &= x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j}), \qquad j = 2,\ldots,N && (78)\end{aligned}$$

where $x_{n+N+1} = u$. In this system, $\bar{x}_{n+j}$ denotes $[x_{n+1},\ldots,x_{n+j}]^T$, and, as before, $\underline{x}_j$ denotes $[x_j,\ x_{j+1},\ldots,x_n]^T$ (which means, in particular, that $\underline{x}_1 = [x_1,\ldots,x_n]^T$). It is clear from the above notation that the overall system order is $n+N$, where the feedforward part (top) is of order n and the feedback part (bottom) is of order N. We assume that $\pi_i(0) = 0, i = 2,\ldots,n, f_i(0) = 0, i = 1,\ldots,N$. The control synthesis is given in the following theorem.

Theorem 6: The control law given by

$$z_i = x_i + \sum_{j=i+1}^n \begin{pmatrix} n-i \\ j-i \end{pmatrix} x_j - \delta_{i,1}\sum_{j=2}^n \int_0^{x_j} \pi_j(s)ds \tag{79}$$

$$\alpha_1(\underline{z}_1) = -\sum_{i=1}^n z_i \tag{80}$$

for $i = 1,\ldots,n$,

$$\begin{aligned}z_{n+1} &= x_{n+1} - \alpha_1 && (81)\\ \alpha_{n+1}(\underline{z}_1, z_{n+1}) &= -(n+1)z_{n+1} + \sum_{l=1}^n (n-l)z_l \\ &\quad - f_1(\underline{x}_1, x_{n+1}) && (82)\\ z_{n+j} &= x_{n+j} - \alpha_{n+j-1}(\underline{z}_1, z_{n+j-1}) && (83)\\ \alpha_{n+j} &= -z_{n+j-1} - z_{n+j} - f_j(\underline{x}_1, \bar{x}_{n+j}) \\ &\quad + \sum_{l=1}^n \frac{\partial\alpha_{n+j-1}}{\partial z_l}\left(-\sum_{k=1}^i z_k + z_{n+1}\right) \\ &\quad + \frac{\partial\alpha_{n+j-1}}{\partial z_{n+1}}\left(-\sum_{k=1}^{n+1} z_k + z_{n+2}\right) \\ &\quad + \sum_{l=2}^{j-1} \frac{\partial\alpha_{n+j-1}}{\partial z_{n+l}}(-z_{n+l} + z_{n+l+1}) && (84)\end{aligned}$$

for $j = 2,\ldots,n$, and

$$u = \alpha_{n+N} \tag{85}$$

globally asymp. stabilizes (75)–(78) at the origin.

Proof: It can be verified that the closed loop system in the z coordinates is

$$\dot{z}_i = -\sum_{k=1}^{i} z_k + z_{n+1}, \qquad i = 1, \dots, n \tag{86}$$

$$\dot{z}_{n+1} = -\sum_{k=1}^{n} z_k - z_{n+1} + z_{n+2} \tag{87}$$

$$\dot{z}_{n+j} = -z_{n+j} + z_{n+j+1}, \qquad j = 2, \dots, N \tag{88}$$

where $z_{n+N+1} = 0$. The Lyapunov function $V = \sum_{i=1}^{n+N} z_i^2$ satisfies $\dot{V} = -\sum_{i=1}^{n+N} z_i^2 - \sum_{i=n+2}^{n+N} (z_i - z_{i-1})^2 - \left(\sum_{i=1}^{n} z_i\right)^2$, which proves the result. ■

Next, consider *interlaced systems of Type II*:

$$\dot{x}_1 = x_2 + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, \dots, n-2 \tag{89}$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u \tag{90}$$

$$\dot{x}_n = x_{n+1} \tag{91}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\underline{x}_1, x_{n+1}) \tag{92}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j}), \qquad j = 2, \dots, N \tag{93}$$

where $x_{n+N+1} = u$. We assume that $\phi_i(0) = f_j(0) =$ and the ϕ_i's satisfy the conditions (13)–(17).

Theorem 7: The control law given by $z_n = x_n$,

$$z_i = x_i + \sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right] \tag{94}$$

for $i = 1, \dots, n-1$, and (80)–(85) globally asymptotically stabilizes the system (89)–(93) at the origin.

With the following result one can pursue a full-state feedback linearization design by conversion to the Brunovsky form.

Theorem 8: The systems (75)–(78) and (89)–(93) are of relative degree $n+N$ from u to the respective outputs

$$y_1 = x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)\,ds, \tag{95}$$

$$y_1 = x_1 - \sum_{j=2}^{n} \mu_{2+n-j}(x_n) x_j. \tag{96}$$

VI. Example: Combining Block-Backstepping and Block-Forwarding

In this section we show that *block*-backstepping and *block*-forwarding can be combined in a similar manner on an example that is outside of the forms considered in Section V (and also outside of those in [15]):

$$\dot{q} = -2q + x_2^2 \tag{97}$$

$$\dot{x}_1 = x_2 + qx_3 \tag{98}$$

$$\dot{x}_2 = x_3 + q \tag{99}$$

$$\dot{x}_3 = u + qx_1. \tag{100}$$

This system is neither block-strict-ffwd (because of qx_1 in the x_3-equation) nor block-strict-fbk (because of qx_3 in the x_1-equation). However, the x_1, x_2, q-subsystem is block-strict-ffwd if one views x_3 as control, and the x_2, x_3, q-subsystem is block-strict-fbk with u as control. Hence, we will derive a controller using one step of forwarding, followed by one step of backstepping.

Following the design from Section IV, $\xi_2^{[2]}(\tau, x_2) = x_2 e^{-\tau}$ and $\eta^{[2]}(\tau, x_2, q) = (q + \tau x_2^2) e^{-\tau}$. Then we derive

$$\beta_2(x_2, q) = -x_2 + \frac{qx_2}{3} + \frac{qx_2^2}{8} + \frac{q^2}{4} + \frac{x_2^3}{9} + \frac{x_2^4}{32} \tag{101}$$

$$w_1(x_2, q) = 1 + \frac{2}{3}q - \frac{qx_2}{4} - \frac{x_2^2}{3} - \frac{x_2^3}{8}. \tag{102}$$

The system is converted from the x_1, x_2, x_3 coordinates into z_1, x_2, z_3 (note that x_2 is unaltered), where

$$z_1 = x_1 - \beta_2 \tag{103}$$

$$z_3 = x_3 + q + w_1 z_1 + x_2. \tag{104}$$

Note that (103) corresponds to one step of forwarding, resulting in a 'virtual control' $-q - w_1 z_1 - x_2$ for x_3 as a control input, whereas (104) corresponds to one step of backstepping. The control law

$$\begin{aligned} u = {} & -z_3 - x_2 - w_1 z_1 - x_1 q + 2q - x_2^2 - w_1^2 (x_3 + q + x_2) \\ & -(x_3 + q) + z_1 \left[\left(\frac{x_2}{4} - \frac{2}{3} \right) (-2q + x_2^2) \right. \\ & \left. + \left(\frac{q}{4}{}_{\frac{2}{3}} x_2 + \frac{3}{8} x_2^2 \right) (x_3 + q) \right] \end{aligned} \tag{105}$$

results in the system being transformed into

$$\dot{z}_1 = -w_1^2 z_1 + w_1 z_3 \tag{106}$$

$$\dot{x}_2 = -w_1 z_1 - x_2 + z_3 \tag{107}$$

$$\dot{z}_3 = -w_1 z_1 - x_2 - z_3. \tag{108}$$

The stability of this system follows from the Lyapunov function $V(x, q) = z_1(x_1, x_2, q)^2 + x_2^2 + z_3(x_1, x_2, x_3, q)^2$ because

$$\dot{V} = -w_1^2 z_1^2 - x_2^2 - (w_1 z_1 + x_2)^2 - 2z_3^2. \tag{109}$$

The convergence to zero can be seen in the following order: x_2 [from (109)], q [from (97)], x_1 [from (103) and (101)], x_3 [from (104)].

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APPENDIX

The following expressions, derived in [6, Thm. 2], are used in the first control algorithm in Section II:

$$\xi_i(\tau,x) = \mathrm{e}^{-\tau}\left[\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l + (-1)^i\sum_{j=i}^{n}\binom{n-i}{j-i}\frac{\tau^{j-1}}{(j-1)!}\times\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right] \tag{110}$$

for $i = 2,\ldots,n$,

$$\xi_1(\tau,x) = \mathrm{e}^{-\tau}\left[\sum_{j=1}^{n}\binom{n-1}{j-1}(-1)^{j-1}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l - \sum_{j=1}^{n}\binom{n-1}{j-1}\frac{\tau^{j-1}}{(j-1)!}\times\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right] + \sum_{j=2}^{n}\int_0^{\xi_j(\tau,x)}\pi_j(s)ds, \tag{111}$$

and

$$\begin{aligned}\tilde{\alpha}_1(\tau,x) \quad = \quad & -\mathrm{e}^{-\tau}\sum_{i=1}^{n}\binom{n}{i-1}\left[\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\right.\\ & \times\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l\\ & +(-1)^i\sum_{j=i}^{n}\binom{n-i}{j-i}\frac{\tau^{j-1}}{(j-1)!}\\ & \left.\times\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right].\end{aligned} \tag{112}$$

The following expressions, derived in [6, Thm. 4], are used in the second control algorithm in Section II:

$$\begin{aligned}\xi_i(\tau,x) = \mathrm{e}^{-\tau}&\left[\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\right.\\ &\times\binom{n-j+k}{l-j+k}\left(x_l-\sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\\ &+\sum_{p=i+1}^{n}\lambda_{i+1+n-p}\left(\mathrm{e}^{-\tau}\sum_{k=0}^{n-1}\frac{(-\tau)^k}{k!}\sum_{l=n-k}^{n}\binom{k}{l-n+k}\right.\\ &\left.\times\left(x_l-\sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\right)\sum_{j=p}^{n}\binom{n-p}{j-p}(-1)^{j-p}\\ &\times\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}\\ &\left.\times\left(x_l-\sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\right],\end{aligned} \tag{113}$$

where $i = 1,\ldots,n$, and

$$\begin{aligned}\tilde{\alpha}_1(\tau,x) \quad = \quad & -\mathrm{e}^{-\tau}\sum_{i=1}^{n}\binom{n}{i-1}\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\\ & \times\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}\\ & \times\left(x_l-\sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right).\end{aligned} \tag{114}$$