Inverse Optimality and Performance in Forwarding

MIROSLAV KRSTIC
University of California, San Diego

***Abstract*— In a pair of companion ACC'04 papers [1], [2] we revealed that the family of feedforward systems contains a substantial class that is linearizable by a coordinate change and presented explicit control formulae for two subclasses. For those controllers, we derive inverse optimal cost functionals and give quantitative performance bounds.**

I. CONTROL DESIGNS FOR LINEARIZABLE FEEDFORWARD SYSTEMS OF TYPE I AND II

Consider the Type I class of strict-feedforward systems,

$$\begin{aligned} \dot{x}_1 &= x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u & (1)\\ \dot{x}_i &= x_{i+1}, \qquad i = 2,\ldots,n-1 & (2)\\ \dot{x}_n &= u, & (3) \end{aligned}$$

where $\pi_j(0) = 0$.

Theorem 1 ([2]): The diffeomorphic transformation

$$\begin{aligned} y_1 &= x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds & (4)\\ y_i &= x_i, \qquad i = 2,\ldots,n & (5) \end{aligned}$$

converts the strict-feedforward system (1)–(3) into

$$\begin{aligned} \dot{y}_i &= y_{i+1}, \qquad i = 1,2,\ldots,n-1 & (6)\\ \dot{y}_n &= u. & (7) \end{aligned}$$

The feedback law[1]

$$u = \alpha_1(x) = -\sum_{i=1}^{n} \begin{pmatrix} n \\ i-1 \end{pmatrix} y_i \qquad (8)$$

globally asymptotically stabilizes the origin of (1)–(3).

Consider the Type II strict-feedforward systems,

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \phi_i(\underline{x}_{i+1})u, \qquad i = 1,\ldots,n-1 & (9)\\ \dot{x}_n &= u, & (10) \end{aligned}$$

where $\phi_i(0) = 0$ and $\underline{x}_{i+1} = [x_{i+1} \cdots x_n]$. To characterize the linearizable subclass, consider the following sequence of functions for $i = n-1, n-2, \ldots, 2$ and $k = 2,\ldots,n-2$:

$$\begin{aligned} \mu_n(x_n) &= \frac{\int_0^{x_n} \phi_{n-1}(s)ds}{x_n} & (11)\\ \mu_i(x_n) &= \frac{1}{x_n}\int_0^{x_n} \left[\phi_{i-1}(0,\ldots,0,s) - \sum_{j=i+1}^{n} \mu_j(s)\phi_{i+n-j}(0,\ldots,0,s)\right] ds & (12) \end{aligned}$$

$$\begin{aligned} \gamma_1(x_n) &= \mu_n'(x_n) & (13)\\ \gamma_k(x_n) &= \sum_{l=1}^{k-1} \gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \frac{d\mu_{n+1-k}(x_n)}{dx_n} & (14) \end{aligned}$$

Theorem 2 ([2]): If

$$\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \gamma_{j-i}(x_n)x_j + \phi_i(0,\ldots,0,x_n) \qquad (15)$$

$\forall x, i = 1,\ldots,n-2$, then the diffeomorphism $y_n = x_n$,

$$y_i = x_i - \sum_{j=i+1}^{n} \mu_{i+1+n-j}(x_n)x_j, \quad i = 1,\ldots,n-1 \qquad (16)$$

converts the strict-feedforward system (9)–(10) into the chain of integrators (6)–(7). The feedback law (8) globally asymptotically stabilizes the origin of (9)–(10).

II. INVERSE OPTIMALITY

Definition 1: The control law

$$u^* = 2\alpha_1(x) = -2\sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right) \qquad (17)$$

is said to be *inverse optimal* if it minimizes the cost functional

$$J = \int_0^{\infty} \left(l(x(t)) + u(t)^2\right) dt \qquad (18)$$

along the solutions of the plant, where

$$l(x) = \sum_{j=1}^{n} (x_j - \beta_{j+1})^2 + \left(\sum_{j=1}^{n} (x_j - \beta_{j+1})\right)^2 \qquad (19)$$

and $\beta_j(\underline{x}_j)$ are C^0 and zero at zero.

One of the most important consequences of inverse optimality is that the control law (17) remains globally asymptotically stabilizing at the origin in the presence of input unmodeled dynamics of the form

$$a(I + \mathcal{P}), \qquad (20)$$

where $a \geq \frac{1}{2}$ is a constant, $\mathcal{P}u$ is the output of any strictly passive nonlinear system[2] with u as its input, and I denotes the identity operator [3].

Theorem 3: The control law (17), with $\alpha_1(x)$ defined in (8), applied to the Type I plant (1)–(3) is inverse optimal with $\beta_{n+1} = 0$ and

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{j=i+1}^{n} \begin{pmatrix} n-i \\ j-i \end{pmatrix} x_j + \delta_{i,1}\sum_{j=2}^{n}\int_0^{x_j} \pi_j(s)ds. \qquad (21)$$

[1] Many other choices for stabilizing (6)–(7) are of course possible. This one allows to also achieve inverse optimality with the least amount of notation.

[2] with possibly non-zero initial conditions

Theorem 4: The control law (17), with $\alpha_1(x)$ defined in (8), applied to the Type II plant (9)–(10), (11), (12), (13), (14), (15) is inverse optimal with $\beta_{n+1} = 0$ and

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right] \quad (22)$$

It is easy to see that the form of the β's in the above theorems guarantees that $l(x)$ is positive definite and radially unbounded.

III. Performance

In the next two theorems, which are proved using

$$\int_0^\infty \tau^p e^{-q\tau} d\tau = \frac{p!}{q^{p+1}} \quad (23)$$

$$\tau^p e^{-q\tau} \leq \left(\frac{p}{qe}\right)^p, \quad (24)$$

and, respectively, (29) and (30), we calculate explicit $\mathcal{L}^1$ and $\mathcal{L}^\infty$ bounds on the control effort in stabilizing feedforward systems of Types I and II.

Theorem 5: The control law (8) applied to the plant (1)–(3), (4), (5) expends the control effort in the amount bounded by

$$\|u\|_{\mathcal{L}^1} \leq \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \times \left[\sum_{k=0}^{j-1} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} |x_l(0)| + \left| \sum_{m=2}^{n} \int_0^{x_m(0)} \pi_m(s) ds \right| \right] \quad (25)$$

and

$$\|u\|_{\mathcal{L}^\infty} \leq \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \times \left[\sum_{k=0}^{j-1} \frac{k^k}{e^k k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} |x_l(0)| + \frac{(j-1)^{j-1}}{e^{j-1}(j-1)!} \left| \sum_{m=2}^{n} \int_0^{x_m(0)} \pi_m(s) ds \right| \right], \quad (26)$$

where $x_i(0)$ are the initial conditions of the state.

Theorem 6: The control law (8) applied to the plant (9)–(15) expends the control effort in the amount bounded by

$$\|u\|_{\mathcal{L}^1} \leq \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \times \sum_{k=0}^{j-1} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} \left| x_l(0) - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n(0)) x_m(0) \right| \quad (27)$$

and

$$\|u\|_{\mathcal{L}^\infty} \leq \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \times \sum_{k=0}^{j-1} \frac{k^k}{e^k k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} \left| x_l(0) - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n(0)) x_m(0) \right|, \quad (28)$$

where $x_i(0)$ are the initial conditions of the state.

References

[1] M. Krstic, "Feedforward systems linearizable by coordinate change," *2004 American Control Conference*.

[2] M. Krstic, "Integrator forwarding control laws for some classes of linearizable feedforward systems," *ACC'04*.

[3] M. Krstic and H. Deng, *Stabilization of Nonlinear Uncertain Systems*, Springer, 1998.

Appendix

The closed-loop control signal resulting from the control algorithm in Theorem 1 has been derived in [2, Thm. 2] and is used in proving Theorem 5:

$$u = -e^{-t} \sum_{i=1}^{n} \binom{n}{i-1} \left[\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \sum_{k=0}^{j-1} \frac{(-t)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l(0) + (-1)^i \sum_{j=i}^{n} \binom{n-i}{j-i} \frac{t^{j-1}}{(j-1)!} \times \left(\sum_{m=2}^{n} \int_0^{x_m(0)} \pi_m(s) ds \right) \right]. \quad (29)$$

The closed-loop control signal resulting from the control algorithm in Theorem 1 has been derived in [2, Thm. 4] and is used in proving Theorem 6:

$$u = -e^{-t} \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \sum_{k=0}^{j-1} \frac{(-t)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} \left(x_l(0) - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n(0)) x_m(0) \right) \quad (30)$$