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Kalman Filter for Continuous State-Space System with Continuous, Multirate, Randomly Sampled and Delayed Measurements

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***Abstract*— This paper presents an optimal, in the Kalman sense, filter for linear, continuous, stochastic state-space system with continuous, multirate, randomly sampled and delayed measurements. A general theorem on optimal filter of Itô-Volterra system with discontinuous measure is presented and then specialized to standard state-space model with both continuous and discrete measurements. The discontinuity of the measurement vector leads to the optimal filter with continuous and impulsive inputs, causing the discontinuity of the filter equations. The size of the jumps in state and covariance matrix can be explicitly calculated using the theory of vibrosolutions. A previously unknown optimal filter for continuous state space systems with continuous and sampled measurements, including multirate, randomly sampled and delayed measurements, is obtained. Under additional assumption, it is shown that the derived optimal filter recovers several known results, including the Kalman-Bucy and Jazwinski filters (continuous process with discrete measurements). The developed and the previously reported filters are compared using Monte Carlo simulations, which show that the optimal result gives the least-mean-square-error estimates of the states, and correctly predicts the goodness of the obtained estimates; the alternative filters tend to be overly optimistic in calculating the quality of the generated state estimates. Numerical simulations demonstrate that the proposed approach is convenient in practice as it neither requires implementation of multirate filters, nor any approximations to handle measurements arriving with different and, possibly, random sampling rates, as often is the case with human-in-the-loop and networked measurements.**

I. INTRODUCTION

Most processes of practical interest are continuous in nature, while the available measurements used to probe the current state of continuous processes are either sampled (discrete), or the combination of sampled and continuous measurements. In approaching the problem of state estimation of a continuous process with the combination of continuous and discrete measurements, we have three fundamental options, summarized in Figure 1:

(1) Discrete state estimator approach requires approximation of the continuous model of the process and sampling of the available continuous measurements. Subsequently, one of the known state estimators for discrete systems (i.e. discrete Kalman filter, KF) is applied to obtain an estimate of the states of the continuous process at certain discrete moments.

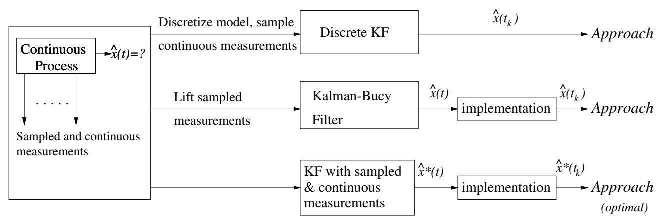

Fig. 1. State estimation problem for continuous process with continuous and sampled measurements.

(2) The second alternative is to lift the discrete measurements into the space of continuous functions (e.g. by using the polynomial fit of several discrete measurements in a sliding window) and then apply one of the known results for state estimation in the continuous system (i.e., Kalman-Bucy filter).

(3) The final option is to directly consider the state estimation problem with a continuous model and the combination of the discrete and continuous measurements.

The applicability of classical state estimation methods after simple approximations led to a wide acceptance of the first two approaches, both of which give suboptimal solutions. The optimal state estimation is obtained if we follow a more theoretically challenging Approach 3, which results in the optimal filter in the form of a continuous system with impulsive inputs.

There are many examples of multirate state estimation algorithms based on the application of discrete KF (Approach 1): Ellis et al. [1] used a multirate extended KF (EKF) to estimate the unmeasurable process states using frequently available measurements of temperature and density and the infrequent and delayed measurements of average molecular weights. Shah et al. [2] implemented a multirate formulation of the iterated EKF on a bioreactor. Mutha et al. [3] proposed fixed-lag smoothing-based EKF algorithm, which was applied to the case of emulsion copolymerization batch reactor. The Kalman filter has also been the basis of multirate digital filters (decimators and interpolators) and filter banks (analysis/synthesis filter banks) [4]. Shah [5] used a lifting technique to transform a multirate single-input single-output system to a single-rate MIMO system, which allowed them to use slow-rate measurements to generate

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high rate control inputs.

The purpose of this paper is to extend the previous result [6] to optimal filter of state-space system with continuous, multirate, randomly sampled and delayed measurements. In the next section, We first show that the Kalman filter [7] for dynamic systems in the integral Itô-Volterra (IV) form can be adapted for the case of the linear stochastic systems with discontinuous measurements. We then show that the optimal filter for the IV systems with discontinuities in measurements can be specialized for the case of state space systems. Based on theory of vibrosolutions, which is used to explicitly computed the size of jumps, an optimal filter for continuous state space systems with an arbitrary combination of continuous and sampled measurements is obtained, including multirate, randomly sampled and delayed cases. The paper is concluded with numerical examples illustrating the application of the optimal filter, and its comparison with traditional alternatives.

II. Itô-Volterra Description of Dynamic Systems

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes. Here Ω is the sample space, F is a set of subsets on which the probability measure (or, simply, probability) is defined, and P is the probability defined on F. All subsets of F form a σ-algebra, and F_t denotes a family of subsets (σ-algebra) for each t such that for $t_1 < t_2$, $F_{t_1} \subset F_{t_2}$. The partly observed F_t-measurable random process $(x(t), z(t))$ can be described using the Itô-Volterra equations:

$$x(t)=\int_0^t (A(t,s)x(s)+B(t,s)u(s))ds+\int_0^t G(t,s)dW_1(s) \quad (1)$$

$$z(t) = \int_0^t C(t,s)x(s)ds + \int_0^t H(t,s)dW_2(s) \quad (2)$$

where $x(t)\in R^n$ is the state vector and $z(t)\in R^m$ is a vector of measurements *integrated* over the interval $[0, t]$. Function $B(t,s)u(s)$ describes known system inputs. $B(t,s)$ is a smooth function of t uniformly in s. Functions $C(t,s)$ and $H(t,s)$ of appropriate dimensions are continuous in t and s with $H(t,s)H^T(t,s)>0$. To simplify notation, $\Upsilon(t,s)=(H(t,s)H^T(t,s))^{-1}$ and $G^2(t,s) = G(t,s)G^T(t,s)$ throughout the paper. Both t and s are independent (time) variables with $t\geq s\geq 0$. All coefficients in the equations (1) and (2) are deterministic functions. Without loss of generality, we assume zero initial conditions.

The estimation problem is to find the estimate of the system state $x(t)$ described by the Itô–Volterra model (1) based on the observation process $Z(t) = \{z(s), 0 \leq s \leq t\}$, which minimizes the 2-norm $E[(x(t)-\hat{x}(t))^T(x(t)-\hat{x}(t))]$ at each time moment t. Alternatively, the objective is to find the conditional expectation $m(t) = \hat{x}(t) = E(x(t) \mid F_t^Z)$. As usual, the matrix function $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Z]$ is the estimation error covariance matrix. The state space model is recovered when all functional parameters in (1)–(2) depend only on s.

In the context of the current paper, the main advantage of the integral formulation is the ability to introduce discontinuities into the vector of measurement by using discontinuous measures, which allows us to consistently model an arbitrary combination of sampled and continuous measurement available for the continuous process (1).

The solution of the optimal filtering problem for the system (1)–(2) was reported in [7], which generalized results of [8], [9]. The explicit solution of the optimal state estimation problem is obtained in terms of the integral cross-correlation function $f(t,s)$, which characterizes the deviation of the optimal estimate $m(t)$ from unknown true state $x(t)$, and is defined as:

$$f(t,s) = E[(x_s^t - m_s^t)(x(s) - m(s))^T \mid F_{t,s}^Z] \quad (3)$$

where x_s^t can be viewed as a state with independent (time) variable s and parameter t:

$$x_s^t=\int_0^s [A(t,r)x(r)+B(t,r)u(r)]dr+\int_0^s G(t,r)dW_1(r) \quad (4)$$

The governing equation for x_s^t can be differentiated with respect to s to yield the state space form of equation (4). $F_{t,s}^Z$ is the σ-algebra generated by the stochastic process z_s^t:

$$z_s^t = \int_0^s C(t,r)x(r)dr + \int_0^s H(t,r)dW_2(r) \quad (5)$$

and $m_s^t = E[x_s^t \mid F_{t,s}^Z]$. Function f is a generalization of the variance P since $f(t,t) = P(t)$. Furthermore for $s = t$, $x_s^t = x(t)$ and $z_s^t = z(t)$.

Next, we introduce IV measurement model with discontinuous measure and obtain the optimal filter for this case.

III. Optimal Filtering for Itô-Volterra Systems with Discontinuous Measure

Consider a nondecreasing vector-valued function of bounded variation $\mu(t) \in R^m$:

$$\mu(t) \quad = \quad \{\mu_k^c(t) + \mu_k^d(t)\}, \quad k = \overline{1,m} \quad (6)$$

where $\mu_k^c(t)$ is a continuous nondecreasing function (such as $\mu_k^c(t) = t$), $\mu_k^d(t)$ is a sum of bounded jumps $\Delta\mu_i$ occurring at $t_i - h$: $\mu_k^d(t) = \sum_{i=1}^N \Delta\mu_i \chi(t-(t_i-h))$, where h is the delay and χ is the Heaviside step function. In this paper, the sampled measurements are modeled assuming $\Delta\mu_i = \mu((t_i-h)^+) - \mu((t_i-h)^-) = 1$. In practical applications, each channel k will typically be either continuous (e.g. $\mu_k(t) = t$) or sampled (e.g., $\mu_k(t) = \mu_k^d(t)$), but not both. The set of discontinuity points of $\mu(t)$ is considered to be a countable set of isolated points.

Using $\mu(t)$ as a measure in equation (2), both continuous and sampled measurements can be modeled as follows:

$$z(t) = \int_0^t C(t,s)x(s)d\mu(s) + \int_0^t H(t,s)dW_2(\mu(s)) \quad (7)$$

where we use componentwise multiplication by $d\mu(s)$. If $\mu(s)$ includes continuous and discontinues components, then measurements $z(t)$ also has continuous, z^c, and discontinuous, z^d components:

$$z(t) = \{z_k^c(t) + z_k^d(t)\} \quad (8)$$

where z^c is given by equation (2). For purely continuous measurements in k-th channel $z_k^d \equiv 0$; if only discrete measurements are available, then $z_k^c \equiv 0$. In general, equation (7) describes an arbitrary combination of discrete and continuous measurements. In the standard differential notation, the i-th continuous measurement $y_i(t) = \frac{dz_i(t)}{dt}$. The relationship between j-th discrete measurement in differential and integral forms is given by the summation $z_j(t_k) = \sum_{l=0}^{k} y_j(t_l)$, and $y_j(t_k) = \delta z_j(t_k)$.

Theorem 1: [7], [10] The optimal in the Kalman sense estimate $m(t)$ of the states of system (1) with discontinuous measurements (7) satisfies the following filter equation

$$\begin{aligned} m(t) = \int_0^t \big[A(t,s)m(s) + B(t,s)u(s)\big]ds \\ + \int_0^t K_{tttt}(s)\big[dz(s) - C(t,s)m(s)d\mu(s)\big] \end{aligned} \quad (9)$$

where the filter gain is equal

$$K_{abcd}(e) = f(a,e)C^T(b,e)\big(H(c,e)H^T(d,e)\big)^{-1} \quad (10)$$

and $f(\cdot,\cdot)$ satisfies the following Riccati-like equation:

$$\begin{aligned} f(t,s) = &\int_0^s \big[A(t,r)f^T(s,r) + f(t,r)A^T(s,r) \\ &+\tfrac{1}{2}(G(t,r)G^T(s,r) + G(s,r)G^T(t,r))\big]dr \\ &- \int_0^s [K_{tsss}(r)C(s,r)f^T(s,r) + K_{tttt}(r)C(t,r)f^T(s,r) \\ &-\tfrac{1}{2}K_{ttts}(r)C(s,r)f^T(s,r) \\ &-\frac{1}{2}K_{ssst}(r)C(t,r)f^T(t,r)]d\mu(r) \end{aligned} \quad (11)$$

where r and s are independent variables. The error covariance $P(t) = f(t,t)$, and satisfies the following equation:

$$\begin{aligned} P = &\int_0^t [f(t,s)A^T(t,s) + A(t,s)f^T(t,s) + G^2(t,s)]dr \\ &-\int_0^t f(t,s)C^T(t,s)\Upsilon(t,s)C(t,s)f^T(t,s)d\mu(s) \end{aligned} \quad (12)$$

IV. Optimal Filter for State-space Systems with Continuous and Discrete Measurements

Consider the case of A, B, and G independent of t. Differentiation of (1) yields the state-space system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w_1(t) \quad (13)$$

where $\omega_1 dt = dW_1(t)$, $\omega_1(t) \sim N(0, \tilde{Q}(t))$ is $l\times 1$ white Gaussian process. Without loss of generality, set $\tilde{Q}(t) = I$. Further assume that measurements are memoryless (C and H are independent of t). Then (7) can be written as:

$$z(t) = \int_0^t C(s)x(s)d\mu(s) + \int_0^t H(s)dW_2(\mu(s)) \quad (14)$$

For the state-space system with continuous and discrete measurements, described by (13)–(14), the filter equations (9) and (12) takes the following simplified form:

$$\begin{aligned} m(t) = &\int_0^t (A(s)m(s) + B(s)u(s))ds \\ &+ \int_0^t P(s^-)C^T(s)\Upsilon(s)[dz(s) - C(s)m(s^-)d\mu(s)] \end{aligned} \quad (15)$$

$$\begin{aligned} P(t) = &\int_0^t \big[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)\big]ds \\ &- \int_0^t P(s^-)C^T(s)\Upsilon(s)C(s)P(s^-)d\mu(s) \end{aligned} \quad (16)$$

where we accounted that for system (13)–(14), $x_s^t = x(s)$, $m_s^t = \mathrm{E}[x(s)|F_s^Y] = m(s)$, $z_s^t = z(s)$, $f(t,s) = P(s)$.

We will use unit-step jumps in $\mu(s)$ to model sampled measurements. Impulsive discontinuity in $d\mu(s)$ is a cause of jumps in $m(t)$, $f(t,s)$ and $P(t)$ when sampled measurements arrive. The following theorem gives the vibrosolution [11], [12] the optimal filter equations (15)–(16), from which the explicit expressions for these jumps can be found.

Theorem 2: The vibrosolution of the optimal filter (15)–(16) for the state-space system with continuous and discrete measurements is given by the following equations:

$$\begin{aligned} m(t) = &\int_0^t (A(s)m(s) + B(s)u(s))ds \\ &+ \int_0^t P(s^-)C^T(s)\Upsilon(s)[dz^c(s) - C(s)m(s^-)d\mu^c(s)] \\ &+ \sum_{i=1}^N \kappa(t_i-h)\,[y(t_i) - C(t_i-h)m(t_i-h)]\,, \end{aligned} \quad (17)$$

$$\begin{aligned} P(t) = &\int_0^t \big[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)\big]ds \\ &- \int_0^t P(s^-)C^T(s)\Upsilon(s)C(s)P(s^-)d\mu^c(s) \\ &- \sum_{i=1}^N \kappa(t_i-h)C(t_i-h)P(t_i-h), \end{aligned} \quad (18)$$

$$\begin{aligned} \kappa(t_i-h) = &P(t_i-h)[I + C(t_i-h)\Upsilon(t_i-h) \\ &\times C(t_i-h)P(t_i-h)]^{-1}C^T(t_i-h)\Upsilon(t_i-h) \end{aligned} \quad (19)$$

Proof: The proof for the general case of IV systems is given in [10]. For state space systems, the proof is presented in Appendix for scalar $\mu^d(s)$ only, which simplifies and clarifies the development. The vector case is proved using a componentwise multiplication by vector $d\mu^d(s)$.

The optimal filter (17)–(18) is a continuous system with discontinuities at the time of arrival of discrete measurements. The fusion of discrete and continuous measurements to calculate optimal state estimates is direct and explicit. Each sampled measurement is processed as soon as it becomes available, so that the case of multirate and randomly sampled measurements is allowed without additional complications. Delays can be time varying, random and *a priori* unknown as long as measurements are time-labelled.

The following examples illustrate the implementation of the optimal filter given by Theorem 2.

a) Continuous system with continuous measurements: This is the case when $\mu(t)=\{\mu_k^c(t)\}=t$, yielding for the state-space system the following optimal filter equations:

$$\begin{aligned} m(t) = &\int_0^t (A(s)m(s) + B(s)u(s))ds \\ &+ \int_0^t P(s)C^T(s)\Upsilon(s)[dz(s) - C(s)m(s)ds] \end{aligned} \quad (20)$$

$$\begin{aligned} P(t) = &\int_0^t \big[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)\big]ds \\ &- \int_0^t P(s)C^T(s)\Upsilon(s)C(s)P(s)ds \end{aligned} \quad (21)$$

which are identical to the Kalman-Bucy filter.

b) Continuous systems with delayed sampled measurements: For purely discrete measurements, $\mu(t) = \{\mu_k^d(t)\}$. The optimal filter is obtained by setting $\mu^c(s) = 0$ in equations (17)-(18), and its implementation requires iterative execution of the following steps:

S1. Initialize $m(t)$ and $P(t)$ as $m(t_{i-1}^+)$ and $P(t_{i-1}^+)$. At $i = 0$, initialize with $m(t_0^+) = E[x(t_0) \mid F_{t_0}^Y]$ and $P(t_0^+) = E[(x(t_0) - m(t_0)(x(t_0) - m(t_0)^T \mid F_{t_0}^Y]$.

S2. Time update for $t \in [t_{i-1}, t_i - h)$:

$$m(t)= m(t_{i-1}^{+})+ \int_{t_{i-1}}^{t} (A(s)m(s) +B(s)u(s))ds \quad (22)$$

$$P= P(t_{i-1}^{+})+\int_{t_{i-1}}^{t}[A(s)P(s)+P(s)A^T(s)+G(s)G^T]ds \quad (23)$$

S3. Measurement update for new delayed measurements, which become available at t_i:

$$m((t_i - h)^+) = m((t_i - h)^-) + \delta m((t_i - h)^+) \quad (24)$$

$$P((t_i - h)^+) = P((t_i - h)^-) + \delta P((t_i - h)^+) \quad (25)$$

where the jumps are calculated explicitly as

$$\delta m((t_i-h)^+) = \kappa(t_i-h)\,[y(t_i)-C(t_i-h)m(t_i-h)] \quad (26)$$

$$\delta P((t_i - h)^+) = -\kappa(t_i - h)C(t_i - h)P(t_i - h) \quad (27)$$

S4. Time update for $t \in (t_i - h, t_i]$:

$$m(t)=m((t_i - h)^+)+\int_{t_i-h}^{t}(A(s)m(s) + B(s)u(s))ds \quad (28)$$

$$P(t) = P((t_i - h)^+) + \int_{t_i-h}^{t}[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)]ds \quad (29)$$

If A, B, C, D, G, H and u are time-invariant in $[t_i - h,\ t_i]$, then (28) and (29) have analytical solution; e.g., instead of using equation (28), the state estimate can be calculated as

$$m(t_i) = e^{Ah}m((t_i - h)^+) - A^{-1}[I - e^{Ah}]Bu \quad (30)$$

The optimal filter (17)–(18) is applicable to all practically important cases of continuous processes with an arbitrary combination of continuous and sampled measurements, including multirate, randomly sampled, and delayed measurements. The developed filter can be used without modification in the case of time varying delays. Without delays ($h = 0$), the optimal filter for continuous systems with sampled measurements is recovered. If we further assume that C is square and invertible, then the obtained result is identical to the Jazwinski's filter (Theorem 7.1[13]).

The implementation of the developed optimal filters is transparent: The continuous filter (e.g., equations (20)–(21) in the case of continuous measurements without delay) is used with continuous measurements until sampled measurement becomes available in one of the channels. At that time, the jumps in m and P are computed explicitly using (26)–(27), which give the optimal state estimate at the time when the delayed measurement was taken. The estimate up to the current time are then calculated using equations (20)–(21) and stored continuous measurements.

V. Examples

The following stable, continuous LTI system with continuous and discrete measurements is selected to compare the optimal approach with the alternatives:

$$A_t=\begin{bmatrix}-1 & -.02 & -.03 & 0\\ -.03 & -2 & .05 & 1\\ -.05 & -6 & -3 & 1\\ -1 & .5 & .8 & -.9\end{bmatrix},\ B_t=\begin{bmatrix}1\\2.5\\1\\1\end{bmatrix},\ C_t=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

The model used in all filters is given by: $A = 0.9A_t,\ B = B_t,\ C = C_t$. The plant-model mismatch is introduced to amplify the effect of approximations on the performance of state estimators derived following Approaches 1 and 2. We assume the continuous process disturbance $\omega(t) \sim N(0, \tilde{Q})$, where $\tilde{Q}$ is known. In all examples, y_1 is available continuously while y_2 and y_3 are sampled, and $\tilde{R}(1,1),\ R(j,j),\ j = 2,3$ are known. The values of $\tilde{Q}$, $\tilde{R}$ and R characterize either the actual Gaussian disturbances and measurement noises, or represent model and measurement uncertainties in L_2 norm.

Each realization of the state estimation is a stochastic process driven by random measurements z. Consequently, characterization of the filter performance based on a single realization may be misleading. In this paper, we use Monte Carlo approach to assess the performance of different filters. Multiple realizations of state trajectories (realizations of true values of states) are calculated using the model stochastic differential equation $dx(t) = (A_t x + B_t u)dt + dW(t)$, where $\omega = dW/dt$ and the Brownian process $W(t)$ is approximated as a random walk [14]. The corresponding realization of stochastic measurements are by adding zero-mean white measurement noises, generated to match covariances of continuous and sampled measurements. The obtained measurements are used as inputs to all filters.

A total of $N = 1000$ realizations of the state trajectories and the corresponding state estimates were obtained with the filters described in the paper. A sample mean of the estimate for each filter is calculated as an ensemble average $\hat{x}_s(t) = \frac{1}{N}\sum_i \hat{x}^i(t)$ summed over all realizations. Assuming that $\hat{x}^i(t)$ are independent and identically distributed, $\hat{x}_s(t) = x_m(t)$ as $N \to \infty$, where the true mean of the states is calculated from $\dot{x}_m(t) = A_t x_m(t) + B_t u$, with $x_m(t_0) = \mathrm{E}[x(0)]$; $\hat{x}_s(t)$ for different filters is then compared with the theoretical mean $x_m(t)$.

Each filter produces estimation error covariance matrix, which self-characterizes the quality of generated state estimates. Accuracy of filter-generated P is assessed by calculating root mean square errors of filter estimated x_j using ensemble averaging : $RMSE(x_j)$= $\sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_j^i(t)\text{-}\hat{x}_j^i(t))^2}$, and $P(t)$= $\mathrm{E}[(x(t)\text{-}\hat{x}(t))(x(t)\text{-}\hat{x}(t))^T]\approx \frac{1}{N-1}\sum_{i=1}^{N}[(x(t)\text{-}\hat{x}(t))(x(t)\text{-}\hat{x}(t))^T]$.

Case 1: Continuous process with continuous and single-rate sampled measurements: Both y_2 and y_3 are sampled at the same uniform rate of $20{\times}\Delta t$. The discrete KF is applied after approximating the continuous model and measurement y_1 using discretization step Δt. The continuous KF (Approach 2) is implemented after fitting discrete measurements to a polynomial.

The first row of Fig. 2 compares the theoretical mean of x_2, x_3 and x_4 with the sample mean obtained for the three filter and shows that the optimal filter results are most accurate. The second row gives the RMSE for all filters, which shows that the optimal filter has the smallest RMSE, as expected. The third row gives the filter-generated values of RMSE. In the case of the optimal filter, the filter-generated values closely agree with the values obtained using ensemble averaging. The two alternative filters over-

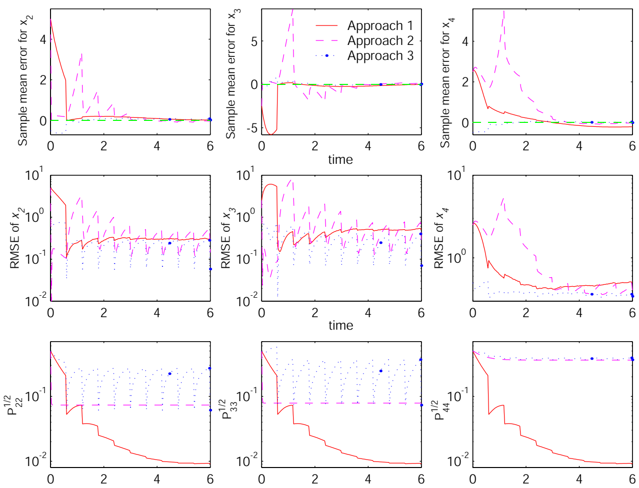

Fig. 2. Case 1.

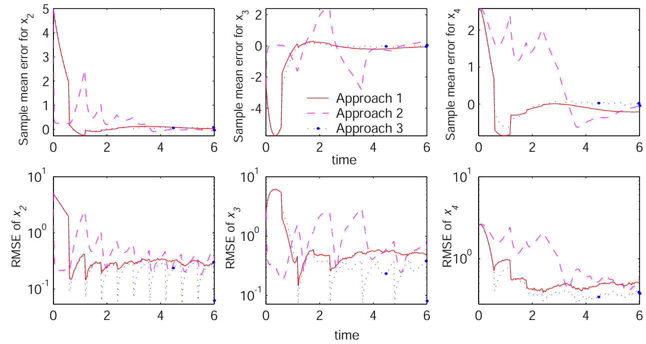

Fig. 3. Case 2.

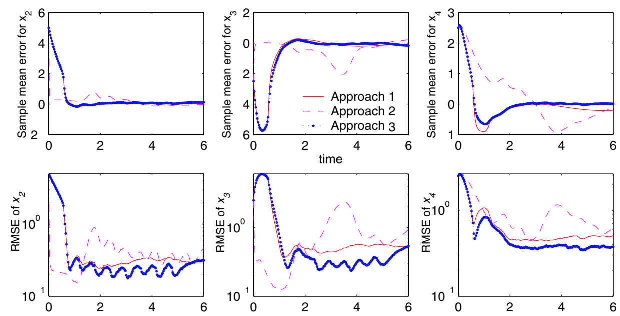

Fig. 4. Case 3.

estimate the goodness of generated state estimates, with Approach 1 giving the largest overestimation. Such behavior of the discrete KF is due to the approximation [6], which makes $Q(t_i)$ too small, leading to unjustifiably low Kalman gain and excessive reliance of the mismatched model in generating the state estimates. Using filter "tuning," which would require a substantial increase in Q, the discrete KF could be adjusted to yield satisfactory results. Note that to correctly tune discrete and Kalman-Bucy filters, the optimal results must be known.

Case 2: Continuous process with continuous and multirate measurements: In this case, measurement y_2 is sampled every $20\times\Delta t$, while y_3 is available every $40\times\Delta t$ units. A multirate discrete filter is implemented following [2]. To apply the continuous Kalman-Bucy filter, the linear extrapolation was used to approximate y_2 and y_3 as piecewise linear functions, which may introduces significant errors. Fig. 3 shows the results for all filters, which indicate that:

1. The gain of the discrete KF is too small (consequence of approximations), which leads to a biased estimation for x_4. At the same time, the estimation error covariance generated by the discrete KF (not shown) continues to have unjustifiably low values.

2. The optimal filter has a superior performance in terms of RMSE.

3. In cases 2 and 3, close to the initial time, the Kalman-Bucy filter estimates have the smallest RMSE, which is the consequence of using true values of x_2 and x_3 before the first measurements y_2 and y_3 become available.

Case 3: Continuous process with continuous and randomly sampled measurements: In this case, y_2 is sampled every $N_1 dt$, where the integer N_1 is obtained by rounding N_1=(20+4ω); y_3 is sampled at $N_2 dt$, where N_2 is obtained by rounding (40+8ω), where $\omega \in N(0,1)$ and $dt = 0.05$. the implementation of the discrete and continuous KF was similar to Case 2. Figure 4 shows that the optimal filter continues to have the best performance in terms of RMSE.

Case 4: Continuous process with continuous and multirate measurements subject to delays: This case is similar to case 2, except that y_3 is delayed by $10\times\Delta t$. With time delays, the extrapolation of sampled measurements needed to implement the Kalman-Bucy filter leads to significant estimation errors. For this reason, only the results of the optimal and discrete KF are compared in this case. The discrete KF with fixed-lag smoothing is implemented by augmenting the state vector with delayed states [3]. Smoothing process utilize the same measurements multiple times, which improves the performance of the discrete KF, Figure 5. Note that smoothing becomes computationally expensive with long delays, as they require the introduction of a large augmented state vector. The discrete KF is not applicable in the general case of time-varying delays.

VI. Conclusions

The state estimation problem for continuous processes with continuous and sampled measurements, including multirate and randomly sampled cases, attracted considerable attention because of its practical importance. Most of previously proposed methods can be classified as belonging to either Approach 1 or 2, and require the approximation of the problem at hand as the state estimation for either discrete or continuous systems. In this paper, we develop

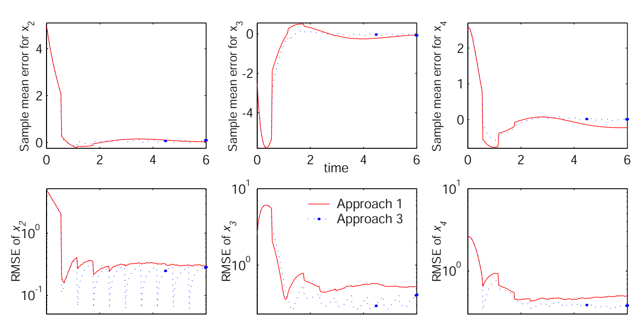

Fig. 5. Case 4.

an optimal (in the Kalman sense) state estimator without reverting to an approximation as a first step of the state estimation procedure. The derived optimal filter is the continuous system with discontinuous inputs appearing each time a new sampled measurement becomes available. It can be used with an arbitrary combination of continuous, multirate, randomly sampled or delayed measurements, and is convenient in applications since each sampled measurement is processed immediately and explicitly whene it becomes available, thus eliminating the need for multirate filters. The numerical Monte-Carlo experiments show that the optimal filter produces the smallest state estimation error (as expected), and provides accurate information about the goodness of obtained state estimates by generating correct estimation error covariances. The alternative methods tend to mislead the user by suggesting higher-quality estimates than actually achieved.

APPENDIX

Proof of Theorem 2: A vibrosolution [12], [10] is defined as a limiting result of a series of conventional solutions obtained with the sequence of absolutely continuous functions that, in the limit, approximate discontinuous bounded variation function $\mu(t)$ at the point of its discontinuity. The vibrosolutions are composed of two parts describing the effect of continuous $\mu^c(t)$ and discontinuous function $\mu^d(t)$. The continuous part of the vibrosolution for the state estimate $m(t)$ and covariance $P(t)$ (equations (17) and (17)) are easy to recover. The jumps in $m(t)$ and $P(t)$, caused by the arrival of discrete measurements, are computed by solving the following equations in differential "in the point of discontinuity" $\mu \in [\mu(t_i-h)^-, \mu(t_i-h)^+]$:

$$\frac{dm(\mu)}{d\mu} = P(\mu)C^T\Upsilon[\frac{dz(\mu)}{d\mu} - Cm(\mu)] \quad (31)$$

$$\frac{dP(\mu)}{d\mu} = -P(\mu)C^T\Upsilon CP(\mu) \quad (32)$$

where $C = C(t_i - h)$ and $\Upsilon = [H((t_i-h)H^T((t_i-h)]^{-1}$. The initial condition for the equation (31) is the value of the state estimate just prior to the arrival of the sampled measurement: $m(\mu((t_i-h)^-)) = m((t_i-h)^-) = m^-$. Similarly, $P(\mu((t_i-h)^-)) = P((t_i-h)^-) = P^-$.

The equation for $P(\mu)$ is independent of $m(\mu)$ and is solved first. It is easy to verify that the analytical solution of (32) is given by the following equation

$$P(\mu) = P((t_i-h)^-) \times[I + C^T\Upsilon CP(P((t_i-h)^-)(\mu - \mu((t_i-h)^-))]^{-1} \quad (33)$$

To verify it in the matrix case, note that for any square A with an appropriately defined matrix inverse $\frac{dA^{-1}}{dt}$=$-A^{-1}\frac{dA}{dt}A^{-1}$, which is proved by the following sequence of equalities: $\frac{dA(t)A^{-1}(t)}{dt} = \frac{dA}{dt}A^{-1} + A\frac{dA^{-1}}{dt} = \frac{dI}{dt} = 0$.

The jumps in P are computed by evaluating (33) before and after the arrival of a sampled measurement at $t = t_i - h$:

$$\begin{aligned}\delta P((t_i-h)^+) &= P\big(\mu((t_i-h)^+)\big) - P\big(\mu((t_i-h)^-)\big)\\ &= P\big((t_i-h)^+\big) - P\big((t_i-h)^-\big)\\ &= P^-[I + C^T\Upsilon CP^-]^{-1} - P^-\\ &= -P^-[I + C^T\Upsilon CP^-]^{-1}C^T\Upsilon CP^-\end{aligned}$$

where we took into account that $[I{+}A]^{-1}$-I=-$A[I{+}A]^{-1}$, and that sampled measurements are modeled using $\Delta\mu_i$=1.

With found P, the directly verification shows that the analytical solution of (31) is given by

$$m(\mu) = m(\mu((t_i-h)^-)) + P(\mu)C^T\Upsilon[y(\mu) - Cm(\mu((t_i-h)^-))(\mu - \mu((t_i-h)^-))] \quad (34)$$

The jumps in the optimal state estimates caused by the arrival of discrete measurements can now be computed as by evaluating (34) at $m\big(\mu((t_i-h)^+)\big) = m\big((t_i-h)^+\big)$:

$$\begin{aligned}\delta m((t_i-h)^+) &= m\big(\mu((t_i-h)^+)\big) - m\big(\mu((t_i-h)^-)\big)\\ &= m\big((t_i-h)^+\big) - m\big((t_i-h)^-\big)\\ &= P((t_i-h)^+)C^T\Upsilon[y((t_i-h)^+) - Cm^-)\Delta\mu_i]\\ &= P^-[I + C^T\Upsilon CP^-]^{-1}C^T\Upsilon[y((t_i-h)^+) - Cm^-]\end{aligned} \quad (35)$$

Following Proposition 2 in [11], the result of Theorem 2, equations (17)–(18), are immediately recovered, thus completing the proof.

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