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Optimal State Estimation of the General Linear ODE with Multiplicative and Additive Wiener Noises

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Abstract— **The problem of the optimal state estimation for systems described by the continuous, linear, n-dimensional ordinary differential equation with multiplicative and additive Wiener noises is solved. The solution essentially relies on the recently developed optimal filtering theory for Itô-Volterra systems.**

I. Motivation

We are interested in the state estimation of the continuous stochastic systems subject to additive and multiplicative Wiener processes. The classical example of a system with additive Wiener noises is the case of Brownian motion [1]. The example when a Wiener disturbance is multiplicative is given by the model of re-entry vehicle when the variance of the drag coefficient increases linearly with time as vehicle descends into a denser atmosphere.

To our knowledge, the state estimation problem for the systems described by a high-dimensional ordinary differential model with additive and multiplicative Wiener noises has not been previously studied. The case of additive and multiplicative white noises is studied in the companion paper [2]. The case considered in [2] allows the reformulation of the problem in the state space form with state dependent noise intensities. For the case of multiplicative and additive Wiener noises, the state space re-formulation is not possible, and the result of the paper substantially depends on a recently developed optimal filtering theory for the Itô-Volterra systems.

II. Optimal Filter for Itô-Volterra Systems

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes with the unit variance intensities. Here Ω is the sample space, F is a set of subsets on which the probability measure (or, simply, probability) is defined, and P is the probability defined on F. All subsets of F form a σ-algebra, and F_t denotes a family of subsets (σ-algebra) for each t such that for $t_1 < t_2$, $F_{t_1} \subset F_{t_2}$. The partially observed F_t-measurable random process $(x(t), z(t))$ is described by the following Itô-Volterra model:

$$x(t) = \int_0^t (A(t,s)x(s) + B(t,s)u(s))ds + \int_0^t G(x,t,s)dW_1(s) \quad (1)$$

$$z(t) = \int_0^t C(t,s)x(s)ds + \int_0^t H(t,s)dW_2(s) \quad (2)$$

where $x(t) \in R^n$ is the state vector, and $z(t) \in R^m$ is a vector of measurements *integrated* over the time interval $[0,t]$. The vector-valued function $B(t,s)u(s)$ describes the effect of known system inputs. Functions $A(t,s)$, $B(t,s)$ are smooth in t uniformly in s. Functions $C(t,s)$, $H(t,s)$ are of appropriate dimensions and continuous in t and s. $C(t,s)$ is a nonzero matrix and $G(x,t,s)G^T(x(s),t,s) \geq 0$, $H(t,s)H^T(t,s) > 0$. To simplify notation,

$$(H(t,s)H^T(t,s))^{-1} = \Upsilon(t,s) \quad (3)$$

throughout the paper. Except for state-dependent function G, all coefficients in the equations (1) and (2) are deterministic functions of t and s, both of which are independent (time) variables with $t \geq s \geq 0$. Without loss of generality, zero initial conditions are assumed.

The estimation problem is to find the estimate $\hat{x}(t)$ of the system state $x(t)$ described by the Itô–Volterra model (1) based on the observation process $Z(t) = \{z(s), 0 \leq s \leq t\}$, such that the Euclidean 2-norm

$$J = \mathrm{E}[(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))] \quad (4)$$

is minimized at each time moment t. In alternative formulation, the objective is to find the conditional expectation $m(t) = \hat{x}(t) = E(x(t) \mid F_t^Z)$. As usual, $P(t) = \mathrm{E}[(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Z]$ is the estimation error covariance matrix.

Our formulation is, in fact, the Kalman filtering problem for the Itô-Volterra systems. This formulation is more general than the problem considered in [3], [4], since the Itô-Volterra measurements model, equation (2), cannot be reduced to the standard differential form. The standard state space formulation can be recovered by making all functional parameters in (1) and (2) dependent on s only.

The solution of the optimal filtering problem for the system (1)–(2) was first reported in [5], which generalized

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results [3], [4] obtained for systems with Itô-Volterra dynamics and standard differential measurements. It is shown in [3], [4] that the variance matrix $P(t)$ alone is not sufficient to completely characterize the state estimation process and to obtain a closed form of filtering equations for dynamic systems in the integral form. Instead, for the systems with the Itô-Volterra dynamic model, equation (1), the explicit solution can be obtained in terms of the integral cross-correlation function $f(t,s)$, which characterizes the deviation of the optimal estimate $m(t)$ from unknown true state $x(t)$, and defined as:

$$f(t,s) \quad = \quad \mathrm{E}[(x_s^t - m_s^t)(x(s)-m(s))^T \mid F_{t,s}^Z] \qquad (5)$$

where x_s^t can be viewed as a state with independent (time) variable s and parameter t:

$$\begin{aligned} x_s^t \quad &= \quad \int_0^s [A(t,r)x(r) + B(t,r)u(r)]dr \\ &+ \quad \int_0^s G(x(r),t,r)dW_1(r) \end{aligned} \qquad (6)$$

The governing equation for x_s^t can be differentiated with respect to s to yield the state space form of equation (6). The conditional mean $m_s^t = E[x_s^t \mid F_{t,s}^Z]$ is the estimate of x_s^t, where $F_{t,s}^Z$ is the σ-algebra generated by the stochastic "measurement" process z_s^t:

$$z_s^t = \int_0^s C(t,s)x(s)ds + \int_0^s H(t,s)dW_2(s) \qquad (7)$$

Note that function f is the generalization of the variance P since $f(t,t) = P(t)$. Furthermore for $s = t$, $x_s^t = x(t)$ and $z_s^t = z(t)$.

Theorem 1: [5], [6] The optimal in the Kalman sense estimate $m(t)$ of the states of system (1) with measurements (2) satisfies the following optimal filter equation

$$\begin{aligned} m(t) \quad &= \quad \int_0^t \big[A(t,s)m(s) + B(t,s)u(s)\big]ds \\ &+ \quad \int_0^t K_{tttt}(s)\big[dz(s) - C(t,s)m(s)ds\big] \end{aligned} \qquad (8)$$

where the componentwise multiplication by the m-dimensional measure $\mu(t)$ is used, and the gain

$$K_{abcd}(e) = f(a,e)C^T(b,e)(H(c,e)H^T(d,e))^{-1} \qquad (9)$$

The function $f(t,s)$ is found from the following Riccati-like equation

$$\begin{aligned} f(t,s) = &\int_0^s \big[A(t,r)f^T(s,r) + f(t,r)A^T(s,r) + \Psi\big]dr \\ -&\int_0^s \Big[K_{tsss}(r)C(s,r)f^T(s,r) + K_{tttt}(r)C(t,r)f^T(s,r) \\ -&\frac{K_{tttt}(r)H(t,r)H^T(s,r)\Upsilon(s,r)C(s,r)f^T(s,r)}{2} \\ -&\frac{K_{ssss}(r)H(s,r)H^T(t,r)\Upsilon(t,r)C(t,r)f^T(t,r)}{2}\Big]dr \end{aligned} \qquad (10)$$

where $\Psi = \mathrm{E}[G(m,t,s)G^T(m,t,s)|F_t^Z]$.

III. Main Results

The problem is to find optimal (in the Kalman sense) estimates of the state $\xi(t)$ and its derivatives up to the order $(n-1)$ for the linear stochastic system described by n-th order ordinary differential equation with additive and multiplicative Wiener noises. It is assumed that the measurements of the linear combinations of $\xi(t)$ and its derivatives $\xi^{(1)}(t), \cdots, \xi^{(n-1)}(t)$ are available, but affected by additive white measurement noises. We begin by considering the case of the model with time-invariant deterministic coefficients.

A. Case 1: Time-Invariant Coefficients

Consider a general linear ODE with time-invariant deterministic coefficients, and multiplicative and additive Wiener noises:

$$\begin{aligned} &\xi^{(n)}(t) + (a_1 + W_1^1(t))\xi^{(n-1)}(t) + \cdots + (a_i + W_1^i(t))\xi^{(i)}(t) \\ &\quad + \cdots + (a_n + W_1^n(t))\xi(t) = \lambda(t) + W_1^0(t) \end{aligned} \qquad (11)$$

where $W_1^i(t)$, $i = 0, 1, \ldots, n$ are the independent Wiener processes. Assuming zero initial conditions $\xi(0) = \xi'(0) = \ldots = \xi^{(n-1)}(0) = 0$ and zero forcing before the initial time ($\lambda(t) = 0$ for $t < 0$), the integration of the ODE model yields

$$\begin{aligned} &\xi^{(n-1)}(t) = \\ &\quad -\int_0^t a_1\xi^{(n-1)}(\nu)d\nu - \int_0^t W_1^1(\nu)\xi^{(n-1)}(\nu)d\nu - \cdots \\ &\quad -\int_0^t a_i\xi^{(n-i)}(\nu)d\nu - \int_0^t W_1^i(\nu)\xi^{(n-i)}(\nu)d\nu - \cdots \\ &\quad -\int_0^t a_n\xi(\nu)d\nu - \int_0^t W_1^n(\nu)\xi(\nu)d\nu \\ &\quad +\int_0^t \lambda(\nu)d\nu + \int_0^t W_1^0(\nu)d\nu \end{aligned} \qquad (12)$$

Lemma 1: For $i = 2, \cdots, n$

$$\xi^{(n-i)}(\nu) = \int_0^\nu \frac{(\nu - s)^{i-2}}{(i-2)!}\xi^{(n-1)}(s)ds \qquad (13)$$

Proof: For $i = 2$,

$$\xi^{(n-i)}(\nu) = \xi^{(n-2)}(\nu) = \int_0^\nu \xi^{(n-1)}(s)ds \qquad (14)$$

For $i = 3$,

$$\begin{aligned} \xi^{(n-i)}(\nu) \quad &= \quad \int_0^\nu \int_0^\mu \xi^{(n-1)}(s)dsd\mu \\ &= \quad \int_0^\nu \int_s^\nu \xi^{(n-1)}(s)d\mu ds \end{aligned} \qquad (15)$$

obtained by changing the order of integration. The integration with respect to μ gives that

$$\xi^{(n-i)}(\nu) = \int_0^\nu (\nu - s)\xi^{(n-1)}(s)ds \qquad (16)$$

By induction, assume that

$$\xi^{(n-i-1)}(\nu) = \int_0^\nu \frac{(\nu - s)^{i-3}}{(i-3)!}\xi^{(n-1)}(s)ds$$

is true. Then for an arbitrary $i \leq n$

$$\begin{aligned}
\xi^{(n-i)}(\nu) &= \int_0^\nu \xi^{(n-i-1)}(\mu)d\mu \\
&= \int_0^\nu \int_0^\mu \frac{(\mu - s)^{i-3}}{(i-3)!}\xi^{(n-1)}(s)dsd\mu \\
&= \int_0^\nu \xi^{(n-1)}(s) \int_s^\nu \frac{(\mu - s)^{i-3}}{(i-3)!}d\mu ds \\
&= \int_0^\nu \frac{(\nu - s)^{i-2}}{(i-2)!}\xi^{(n-1)}(s)ds
\end{aligned}$$

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The integration of equation (12) on $\nu \;\epsilon\; [0, t]$ gives

$$\begin{aligned}
\int_0^t \xi^{(n-i)}(\nu)d\nu &= \int_0^t \Big[\int_0^\nu \frac{(\nu - s)^{(i-2)}}{(i-2)!}\xi^{(n-1)}(s)ds\Big]d\nu \\
&= \int_0^t \frac{(t-s)^{i-1}}{(i-1)!}\xi^{(n-1)}(s)ds \qquad (17)
\end{aligned}$$

where the result of Lemma 1 was used.

In equation (12), consider terms of the form $W_1^i(\nu)\xi^{(n-i)}(\nu)$, for $i = 1, \ldots, n$. The following chain of equalities is obtained using the obvious change of the integration variable followed by integration by parts:

$$\begin{aligned}
\int_0^t W_1^i(\nu)\xi^{(n-i)}(\nu)d\nu &= \int_0^t W_1^i(\nu)d\xi^{(n-i-1)}(\nu) \\
&= W_1^i(t)\xi^{(n-i-1)}(t) - \int_0^t \xi^{(n-i-1)}(\nu)dW_1^i(\nu) \\
&= \int_0^t \xi^{(n-i-1)}(t)dW_1^i(\nu) - \int_0^t \xi^{(n-i-1)}(\nu)dW_1^i(\nu) \\
&= \int_0^t \big(\xi^{(n-i-1)}(t) - \xi^{(n-i-1)}(\nu)\big)dW_1^i(\nu) \\
&= \int_0^t [\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\
&\quad - \int_0^\nu \frac{(\nu - r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\big]dW_1^i(\nu) \qquad (18)
\end{aligned}$$

Therefore, each term in (12), which includes the Wiener process as a part of the integrand, can be transformed into the integral with respect to the differential of the Wiener process $W_1^i(\nu)$.

Using (17) and (18) in equation (12) obtain:

$$\begin{aligned}
\xi^{(n-1)}(t) &= -a_1 \int_0^t \xi^{(n-1)}(\nu)d\nu \\
&- \int_0^t \Big[\int_0^t \xi^{(n-1)}(r)dr - \int_0^\nu \xi^{(n-1)}(r)dr\Big]dW_1^1(\nu) \\
&- \cdots - a_i \int_0^t \frac{(t-s)^{i-1}}{(i-1)!}\xi^{(n-1)}(s)ds \\
&-\int_0^t \Big[\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\
&- \int_0^\nu \frac{(\nu-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr\Big]dW_1^i(\nu) - \cdots \\
&-a_n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\xi^{(n-1)}(s)ds \\
&- \int_0^t \Big[\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr \\
&- \int_0^\nu \frac{(\nu-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr\Big]dW_1^n(\nu) \\
&+ \int_0^t \lambda(\nu)d\nu + \int_0^t (t-\nu)dW_1^0(\nu) \qquad (19)
\end{aligned}$$

where the last term of (12) is transformed into the last term of the above equation following the same procedure as used to obtain equation (18). Setting $\nu = s$, obtain the description of the system with additive and multiplicative Wiener noises in the standard Itô-Volterra form of equation (1) with $\xi^{(n-1)}$ as a state:

$$\begin{aligned}
\xi^{(n-1)}(t) &= \int_0^t \Big[\big(-\sum_{i=1}^n a_i \frac{(t-s)^{i-1}}{(i-1)!}\big)\xi^{(n-1)}(s) + \lambda(s)\Big]ds \\
&+ \int_0^t G(\xi^{(n-1)}, t, s)dW_1(s) \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
&G(\xi^{(n-1)}, t, s) = [G_0 \;\; G_1 \cdots G_i \cdots G_n] \qquad (21) \\
&G_0(t, s) = (t - s) \\
&G_1(\xi^{(n-1)}, t, s) = -\int_0^t \xi^{(n-1)}(r)dr + \int_0^s \xi^{(n-1)}(r)dr \\
&\quad \vdots \\
&G_i(\xi^{(n-1)}, t, s) = -\int_0^t \frac{(t-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\
&\qquad + \int_0^s \frac{(s-r)^{i-1}}{(i-1)!}\xi^{(n-1)}(r)dr \\
&\quad \vdots \\
&G_n(\xi^{(n-1)}, t, s) = -\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr \\
&\qquad + \int_0^s \frac{(s-r)^{n-1}}{(n-1)!}\xi^{(n-1)}(r)dr \\
&W_1^T(s) = [W_1^0(s) \;\; W_1^1(s) \;\; W_1^2(s) \cdots W_1^{n-1}(s) \;\; W_1^n(s)] \qquad (22)
\end{aligned}$$

and $W_1(t)$ is the $(n + 1)$-dimension column vector of independent Wiener noises with zero mean ($\mathrm{E}[W_1(t)] = 0$) and covariance $cov[W_1(t)] = tI$.

We now turn our attention to casting the measurement model in the Itô-Volterra form. Suppose $\xi^{(n-j)}(t)$, $(j = 1, 2, \cdots, n)$ is measured, i.e.

$$y(t) = \xi^{(n-j)}(t) + H(t)\omega_2(t) \tag{23}$$

Then the *integral* measurement

$$z(t) = \int_0^t y(s)ds$$

can be written in the form of equation (2) with $\xi^{(n-1)}(t)$ as the state variable:

$$z(t) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!}\xi^{(n-1)}(s)ds + \int_0^t H(s)dW_2(s) \tag{24}$$

The optimal filter for the system (11) with measurements (23) is obtained by applying Theorem 1 to the system in the Itô-Volterra form of equations (20), (24). The optimal estimate of $\hat{\xi}^{(n-1)}(t) = m(t)$ is given by the following filter equations:

$$m(t) = \int_0^t \Big[\big(-\sum_{i=1}^n a_i \frac{(t-s)^{i-1}}{(i-1)!}\big)m(s) + \lambda(s)\Big]ds$$
$$+\int_0^t f(t,s)\frac{(t-s)^{j-1}}{(j-1)!}\Upsilon(s)\big[dz(s) - \frac{(t-s)^{j-1}}{(j-1)!}m(s)ds\big] \tag{25}$$

where

$$f(t,s) = \int_0^s \Big[\big(-\sum_{i=1}^n a_i \frac{(t-r)^{i-1}}{(i-1)!}\big)f(s,r)$$
$$+f(t,r)\big(-\sum_{i=1}^n a_i \frac{(s-r)^{i-1}}{(i-1)!}\big) + \Psi\Big]dr$$
$$-\int_0^s f(t,r)\Upsilon(r)\Big[\big(\frac{(s-r)^{j-1}}{(j-1)!}\big)^2$$
$$+\big(\frac{(t-r)^{j-1}}{(j-1)!}\big)^2 - \frac{(s-r)^{j-1}(t-r)^{j-1}}{((j-1)!)^2}\Big]f(s,r)dr \tag{26}$$

and $\Psi = \mathrm{E}[G^T(m,t,r)G(m,s,r) \mid F_t^Z]$. Each component of $G(m,t,s)$ can be written as a sum of two terms: The first one a function of t and $m(t)$ only, while the second one depends only on s and $m(s)$. Therefore, if we express G as

$$G(m,t,r) = f_1(t) + f_2(r) \tag{27}$$

then

$$\begin{aligned}\Psi &= \mathrm{E}[(f_1(t)+f_2(r))(f_1(s)+f_2(r)) \mid F_t^Z] \\ &= \mathrm{E}[f_1(t)f_1(s) + f_1(t)f_2(r) + f_1(s)f_2(r) \\ &+ f_2(r)f_2(r) \mid F_t^Z]\end{aligned} \tag{28}$$

The constructive expression for calculating Ψ is obtained by taking the expectation operator of the four additive terms in the last equation. Consider $\mathrm{E}[f_1(t)f_1(s) \mid F_t^Z]$ as an example. Since for any random $x_1(t)$ and $x_2(t)$

$$\begin{aligned}&\mathrm{E}[(x_1(t)x_2(t) \mid F_t^Z] \\ &\quad = \mathrm{E}[(x_1(t)-\hat{x}_1(t))(x_2(t)-\hat{x}_2(t)) \mid F_t^Z] \\ &\quad +\hat{x}_1(t)\hat{x}_2(t)\end{aligned} \tag{29}$$

obtain that

$$\begin{aligned}&\mathrm{E}[f_1(t)f_1(s) \mid F_t^Z] \\ &= \sum_{i=1}^n \mathrm{E}\Big[\int_0^r \frac{(t-r_1)^{i-2}}{(i-2)!}\xi^{(n-1)}(r_1)dr_1 \\ &\times \int_0^r \frac{(s-r_2)^{i-2}}{(i-2)!}\xi^{(n-1)}(r_2)dr_2 \mid F_t^Z\Big] \\ &= \sum_{i=1}^n \int_0^t\int_0^s \frac{(t-r_1)^{i-2}}{(i-2)!}\mathrm{E}\Big[\xi^{(n-1)}(r_1)\xi^{(n-1)}(r_2) \mid F_t^Z\Big] \\ &\times\frac{(s-r_2)^{i-2}}{(i-2)!}dr_1dr_2 \\ &= \sum_{i=1}^n \int_0^t\int_0^s \frac{(t-r_1)^{i-2}(s-r_2)^{i-2}}{((i-2)!)^2} \\ &\times[K(r_1,r_2)+m(r_1)m(r_2)]dr_2dr_2\end{aligned} \tag{30}$$

The expressions for the other three expectation operators can be obtained similarly to yield the following expression for Ψ:

$$\begin{aligned}\Psi &= (t-r)(s-r) \\ &+ \sum_{i=1}^n \Big\{\int_0^t\int_0^s \frac{(t-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(s-r_2)^{i-1}}{(i-1)!}dr_2dr_1 \\ &+ \int_0^t\int_0^r \frac{(t-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2 \\ &+ \int_0^s\int_0^r \frac{(s-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2 \\ &+ \int_0^r\int_0^r \frac{(r-r_1)^{i-1}}{(i-1)!}[K(r_1,r_2) \\ &+ m(r_1)m(r_2)]\frac{(r-r_2)^{i-1}}{(i-1)!}dr_1dr_2\Big\}\end{aligned} \tag{31}$$

Assuming $r_1 \geq r_2$, the cross-covariance matrix

$$\begin{aligned}&K(r_1,r_2) \\ &\quad = \mathrm{E}[(\xi^{(n-1)}(r_1)-m(r_1))(\xi^{(n-1)}(r_2)-m(r_2))|F_r^t]\end{aligned} \tag{32}$$

can be found using Theorem 2 in [3]:

$$K(r_1,r_2) = f(r_1,r_2) + \int_{r_2}^{r_1}\Big[-\sum_{i=1}^n a_i\frac{(r_1-r_3)^{i-1}}{(i-1)!}$$
$$-\frac{(r_1-r_3)^{2(j-1)}}{((j-1)!)^2}\Upsilon(r_3)f(r_1,r_3)\Big]K(r_3,r_2)dr_3 \tag{33}$$

We have shown that the optimal estimate $m(t) = \hat{\xi}^{(n-1)}(t)$ must satisfy the closed system of filter equations (25), (26), (31) and (33). If we are interested in estimating derivatives $\xi^{(i)}(t)$ any order from 0 to $(n-2)$, the optimal estimate is found using equation (13) without the need for multiple integrations of $m(t)$.

B. Case 2: Time-Varying Coefficients

The system with multiplicative and additive Wiener noises now depends on known, time-varying deterministic coefficients $a_i(t)$:

$$\begin{aligned} &\xi^{(n)}(t) + (a_1(t) + W_1^1(t))\xi^{(n-1)}(t) + \cdots \\ &\quad + (a_i(t) + W_1^i(t))\xi^{(i)}(t) + \cdots + (a_n(t) + W_1^n(t))\xi'(t) \\ &\quad = \lambda(t) + W_1^0(t) \end{aligned} \tag{34}$$

where $W_1^i(t)$ are independent Wiener processes. As in the case of time invariant coefficient, assuming $\xi(0) = \xi'(0) = \cdots = \xi^{(n-1)}(0) = \lambda(0) = 0$, the integration of the model gives

$$\begin{aligned} &\xi^{(n-1)}(t) = \\ &-\int_0^t a_1(\nu)\xi^{(n-1)}(\nu)d\nu - \int_0^t W_1^1(\nu)\xi^{(n-1)}(\nu)d(\nu) - \cdots \\ &-\int_0^t a_i(\nu)\xi^{(n-i)}(\nu)d\nu - \int_0^t W_1^i(\nu)\xi^{(n-i)}(\nu)d(\nu) - \cdots \\ &-\int_0^t a_n(\nu)\xi(\nu)d\nu - \int_0^t W_1^n(\nu)\xi(\nu)d\nu \\ &+\int_0^t \lambda(\nu)d\nu + \int_0^t W_1^0(\nu)d\nu \end{aligned} \tag{35}$$

Using Lemma 1 obtain that

$$\begin{aligned} I_a &= \int_0^t a_i(\nu)\xi^{(n-i)}(\nu)d\nu \\ &= \int_0^t a_i(\nu)\Big[\int_0^\nu \frac{(\nu-s)^{i-2}}{(i-2)!}\xi^{(n-1)}(s)ds\Big]d\nu \\ &= \int_0^t \Big[\int_s^t a_i(\nu)\frac{(\nu-s)^{i-2}}{(i-2)!}d\nu\Big]\xi^{(n-1)}(s)ds \end{aligned} \tag{36}$$

Introducing new coefficients $\tilde{a}_i(t,s)$ defined by the following equation

$$\tilde{a}_i(t,s) = \int_s^t a_i(\nu)\frac{(\nu-s)^{i-2}}{(i-2)!}d\nu, \quad i = 2, 3, \cdots, n \tag{37}$$

obtain that

$$I_a = \int_0^t \tilde{a}_i(t,s)\xi^{(n-1)}(s)ds \tag{38}$$

The standard Itô-Volterra form of equation (35) can now be written as

$$\begin{aligned} \xi^{(n-1)}(t) &= \int_0^t \Big[\big(-\sum_{i=1}^n \tilde{a}_i(t,s)\xi^{(n-1)}(s) + \lambda(s)\Big]ds \\ &+ \int_0^t G(\xi^{(n-1)}, t, s)dW_1(s) \end{aligned} \tag{39}$$

where $G(\xi^{(n-1)}, t, s)$ and $W_1(s)$ are defined by equations (21) and (22).

Limiting the consideration to the simplest case of the measurement model (24), the optimal filter for (39) is obtained by applying the general optimal state estimation result of Theorem 1:

$$\begin{aligned} m(t) = &\int_0^t \Big[\big(-\sum_{i=1}^n \tilde{a}_i(t,s)m(s) + \lambda(s)\Big]ds \\ &+ \int_0^t f(t,s)\frac{(t-s)^{j-1}}{(j-1)!}\Upsilon(s) \\ &\times\big[dz(s) - \frac{(t-s)^{j-1}}{(j-1)!}m(s)ds\big] \end{aligned} \tag{40}$$

$$\begin{aligned} f(t,s) = &\int_0^s \Big[\big(-\sum_{i=1}^n \tilde{a}_i(t,s)f(s,r) \\ &+ f(t,r)\big(-\sum_{i=1}^n \tilde{a}_i(t,s) + \Psi\Big]dr \\ &-\int_0^s f(t,r)\Upsilon(r)\Big[(\frac{(s-r)^{j-1}}{(j-1)!})^2 + (\frac{(t-r)^{j-1}}{(j-1)!})^2 \\ &-\frac{(s-r)^{j-1}(t-r)^{j-1}}{((j-1)!)^2}\Big]f(s,r)dr \end{aligned} \tag{41}$$

where Ψ is given by (31).

IV. Conclusions

In this paper, an optimal, in the Kalman sense, filter for linear n-dimensional ODE system with multiplicative and additive Wiener disturbances is developed. Though only a simple measurement model is considered in the paper, the extension for the case of vector measurements in the form of arbitrary linear combinations of derivatives of the state $\xi(t)$ of any order between 0 and $n-1$ can be easily obtained. The developed filter provides the optimal estimate of the $(n-1)$-th order derivative of the state of the ODE model. It is shown that the estimation of a state derivative of any order between 0 to $(n-2)$ can be obtained with the single integration of $\hat{\xi}^{(n-1)}(t)$.

Unlike the optimal state estimation for the general linear ODE with multiplicative and additive white Gaussian noises considered in [6], the case of multiplicative and additive Wiener noises cannot be reduced to the state space estimation problem with state-dependent noise intensities. Therefore, the obtained result substantially relies on the recently developed theory on the optimal filtering for the Itô-Volterra systems.

References

[1] Karl J. Åström. *Introduction to Stochastic Control Theory*, volume 70 of *Mathematics in Science and Engineering*. Academic Press, New York, 1970.

[2] Huichai Zhang, Michael V. Basin, and Mikhail Skliar. Optimal filtering for the linear systems with multiplicative and additive white noises. *Journal of Computational and Applied Mathematics*, submitted.

[3] M. L. Kleptsina and A. Yu. Veretennikov. On filtering and properties of conditional laws of Itô-Volterra processes. In *Statistics and Control of Stochastic Processes*, pages 179–196. New York, 1985.

[4] L. E. Shaikhet. On an optimal control problem of partly observable stochastic Volterra processes. *Problems of Control and Inform. Theory*, 16:439–448, 1987.

[5] M. V. Basin. On filtering over Itô-Volterra observations. *Journal of Applied Mathematics and Stochastic Analysis*, 13:347–365, 2000.

[6] H. Zhang, M. V. Basin, and M. Skliar. An Itô-Volterra optimal state estimation with continuous, multirate, randomly sampled and delayed measurements. *IEEE Trans. Aut. Control*, submited, manuscript at http://implicit.che.utah.edu/∼skliar/TAC2004.pdf.