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Robust stabilization of discrete-time systems with time-varying delays

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Abstract— **This paper introduces a new approach to robust stabilization of discrete-time delay systems under non-linear perturbations. For the discrete-time systems with state/input delays, the robust stabilization problem is transformed to a constrained convex optimization one. Sufficient conditions on the existence of state feedback controllers are established in terms of linear matrix inequality (LMI), which guarantee stability of the closed-loop system and at the same time maximize the non-linearity bound. Finally, numerical examples are presented to illustrate the efficiency and feasibility of our proposed approach.**

Keywords: LMIs, discrete-time systems, state delays, input delays, robust stabilization.

I. INTRODUCTION

Time delays ubiquitously occur in many dynamical systems such as biological systems, chemical systems and electrical networks. Since delays are frequently the main sources of deterioration of system performance and stability, there has been increasing interest in the field of analysis and synthesis of time-delay systems in recent years, see, e.g. [2]–[8]. Two types of stability conditions have been reported in the literature: the so-called delay-dependent conditions (the condition containing delay information) and delay-independent conditions (the condition without containing delay information).

During the last decade, continuous-time linear systems with delays have been extensively studied. Stability results have been obtained for both of delay-dependent and delay-independent cases in terms of Ricatti equation [7] or linear matrix inequalities (LMIs) [3], [12]. More recently, a descriptor approach has been introduced to study continuous-time systems with delays [4], [6], [5], where the conservatism that comes from the transformation of the system and bounding for certain terms has been reduced. Meanwhile, less attention has been paid to the study of discrete-time systems with delays. For the constant time delay case, a discrete-time system can be transformed into a system with no time delay via augmentation approach [8], and hence can be dealt with by using the control theory of discrete-time linear systems. However, this theory cannot be directly applied to the case of time-varying delays. Ref. [13] has considered control problems for discrete-time systems with time-varying delays in state, where output feedback controllers have been designed in terms of LMI. In fact, this result is not LMI in the strict sense because it has supposed that the matrix in the Lyapunov equation should be predetermined. Meanwhile, these work has not considered the case of non-linear perturbation in the delayed system.

Recently, a new approach to dealing with stability and stabilization for linear continuous-time and discrete-time systems under non-linear perturbations have been obtained in terms of LMI in [9], [10], which provides a possibility to reduce the conservativeness in computations of maximal bounds on non-linear terms. But when this method is applied to the discrete-time systems, it can only be used for single-input delay and there are structured restrictions on the Lyapunov matrix. Ref. [15] has presented a less conservative result with the above drawbacks removed and has extended the result to discrete-time systems with constant delays. However, these approaches do not work in the case with time-varying delays.

Based on the research mentioned above, inspired by Fridman's recent works (see, e.g., [4], [5]), we investigate robust stabilization of discrete-time systems with time-varying delays and non-linear perturbations. Our new approach is based on an equivalent descriptor form of the discrete-time system. A new type of Lyapunov function is introduced for the development of delay-dependent methods on designing linear memoryless state feedback controllers, where there are fewer bounds applied. New delay-dependent sufficient conditions on the existence of robust state feedback controllers which guarantee the stability of the closed-loop system and maximize the non-linearity bound are obtained in terms of LMI.

Notation: In this paper, $\mathbf{R}$ is the set of all real numbers, $\mathbf{R}^n$ is the set of all n-tuples of real numbers. A^T and A^{-1} Denote the transpose and the inverse of a matrix A, respectively. I_m denotes the unit matrix with m rows and m columns, $\mathrm{diag}\{\cdots\}$ is a block-diagonal matrix. $A>0$ $(A<0)$ means that A is symmetric positive definite (negative definite). Z^+ denotes the set of non-negative integer and * represents blocks that are readily inferred by symmetry.

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II. Robust stabilization of discrete-time systems with state delays

We first consider the following discrete-time system with time-varying delays in state

$$x(k+1) = Ax(k) + A_d x(k-\tau(k)) + Bu(k) + h(k,x(k)), \quad (1)$$

where $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^m$ are the plant state and the plant input, respectively, A, B, A_d are known real constant matrices with appropriate dimensions. $h:\ Z^+ \times \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear perturbation function that satisfies the quadratic constraint condition

$$h^T(k,x(k))h(k,x(k)) \le \alpha^2 x^T(k)H^T Hx(k), \quad (2)$$

where $\alpha > 0$ is the bounding parameter on the uncertain perturbation function h and H is a constant matrix. Note that constraint (2) is equivalent to

$$\begin{bmatrix} x \\ h \end{bmatrix}^T \begin{bmatrix} -\alpha^2 H^T H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \le 0. \quad (3)$$

For any given H, we define the set

$$H_\alpha = \{h:\ \mathbf{R}^{n+1} \to \mathbf{R}^n | h^T(k,x(k))h(k,x(k)) \le \alpha^2 x^T(k)H^T \times Hx(k) \text{ for all } (k,x) \in Z^+ \times \mathbf{R}^n\}. \quad (4)$$

It is assumed that the pair (A,B) is stabilizable and the time-varying delay $\tau(k)$ satisfies the following assumption:

$$0 < \tau_1 \le \tau(k) \le \tau_2 < \infty, \quad (5)$$

where τ_1 and τ_2 are known positive integers.

In the sequel of this paper, we will use the following concept of robust stabilization for the system (1).

Definition 1: Given positive integers τ_1, τ_2 such that $0 < \tau_1 < \tau_2$, the system (1) is robustly stabilizable with some degree α if there exists a control law $u(k) = Fx(k)$ such that the resulting closed-loop system is robustly stable for any time-varying delay satisfying (5) and for all $h(k,x(k)) \in H_\alpha$.

In this section, we shall develop method for designing a linear memoryless state feedback controller for the system (1) which both ensures robust stability and maximizes the non-linearity bound for any time-varying delay satisfying condition (5).

Since the robust stability of the system (1) is crucial to this solution, we shall address it first.

Lemma 1: Consider unforced system (1) with $u(k) \equiv 0$. Given positive integers τ_1, τ_2 such that $0 < \tau_1 < \tau_2$, the system is stable with degree α if for any time-varying delay $\tau(k)$ satisfying (5) there exist matrices $\bar{P}_1 > 0$, $\bar{P}_2$, $\bar{P}_3$, $\bar{Q} > 0$ and a scalar γ satisfying the following LMI:

$$\begin{bmatrix} (\tau_2-\tau_1+1)\bar{Q} - \bar{P}_1 - A^T\bar{P}_2 - \bar{P}_2^T A & -A^T\bar{P}_3 + \bar{P}_2^T & -\bar{P}_2^T A_d & -\bar{P}_2 & H^T \\ * & \bar{P}_1 + \bar{P}_3 + \bar{P}_3^T & -\bar{P}_3^T A_d & -\bar{P}_3^T & 0 \\ * & * & -\bar{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma I \end{bmatrix} < 0, \quad (6)$$

where $\gamma = \alpha^{-2}$.

Proof: We represent (1) with $u(k) \equiv 0$ in an equivalent descriptor form [5]

$$\begin{cases} x(k+1) = y(k), \\ 0 = -y(k) + Ax(k) + A_d x(k-\tau(k)) + h(k,x(k)), \end{cases} \quad (7)$$

which is valid in the case of $0 < \tau_1 \le \tau(k) \le \tau_2 < \infty$ for $k \ge 0$. Given initial condition $x(k) = \phi(k)$ $(k \in [-\tau_2, 0])$, where ϕ is a discrete-time function, $x(k)$ satisfies (1) for $k \ge 0$ iff it satisfies (7).

We use the following Lyapunov-Krasovskii functional

$$V(k) = V_1(k) + V_2(k) + V_3(k), \quad (8)$$

where

$$V_1(k) = \bar{x}(k)^T EP\bar{x}(k),\ V_2(k) = \sum_{i=k-\tau(k)}^{k-1} x^T(i)Qx(i),$$

$$V_3(k) = \sum_{j=-\tau_2+2,}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l),$$

with

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ y(k) \\ x(k-\tau(k)) \\ h(k,x(k)) \end{bmatrix},\ P = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ P_2 & P_3 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

$$E = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $P_1 = P_1^T > 0$, P_2, P_3 are matrices with appropriate dimensions.

It can be easily verified that (8) satisfies the following inequalities:

$$a|x(k)|^2 \le V(k) \le b \sup_{s\in[-\tau_2,0]} |\bar{x}(k+s)|^2,\ a > 0,\ b > 0.$$

Note that

$$\bar{x}^T(k)EP\bar{x}(k) = x(k)^T P_1 x(k).$$

Then the difference of the first term of (8) with respect to k is given by

$$\begin{aligned}\Delta V_1(k) &= x(k+1)^T P_1 x(k+1) - x(k)^T P_1 x(k) \\ &= y^T(k) P_1 y(k) - 2\bar{x}(k)^T P^T \begin{bmatrix} \frac{1}{2}x(k) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{9}$$

Substituting the first 0 in the right-hand side of (9) by the expressions in (7), we obtain

$$\Delta V_1(k) = \bar{x}^T(k) W \bar{x}(k), \tag{10}$$

where

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - P^T \begin{bmatrix} \frac{1}{2}I & 0 & 0 & 0 \\ A & -I & A_d & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}I & A^T & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & A_d^T & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} P.$$

The difference of the second term of (8) is given by

$$\begin{aligned}\Delta V_2(k) &= \sum_{i=k+1-\tau(k+1)}^{k} x^T(i) Q x(i) - \sum_{i=k-\tau(k)}^{k-1} x^T(i) Q x(i) \\ &= \sum_{i=k+1-\tau(k+1)}^{k-\tau_1} x^T(i) Q x(i) + x^T(k) Q x(k) \\ &\quad - x^T(k-\tau(k)) Q x(k-\tau(k)) \\ &\quad + \sum_{i=k+1-\tau_1}^{k-1} x^T(i) Q x(i) - \sum_{i=k+1-\tau(k)}^{k-1} x^T(i) Q x(i). \end{aligned} \tag{11}$$

Since $\tau(k) \geq \tau_1$, we have

$$\sum_{i=k+1-\tau_1}^{k-1} x^T(i) Q x(i) - \sum_{i=k+1-\tau(k)}^{k-1} x^T(i) Q x(i) \leq 0.$$

This together with (11) implies

$$\begin{aligned}\Delta V_2(k) \leq \sum_{i=k+1-\tau(k+1)}^{k-\tau_1} x^T(i) Q x(i) + x^T(k) Q x(k) \\ - x^T(k-\tau(k)) Q x(k-\tau(k)). \end{aligned}$$

For the third term of (8), we derive

$$\begin{aligned}\Delta V_3(k) &= \sum_{j=-\tau_2+2}^{-\tau_1+1} \left(x^T(k) Q x(k) - x^T(k+j-1) Q x(k+j-1) \right) \\ &= (\tau_2-\tau_1) x^T(k) Q x(k) - \sum_{i=k+1-\tau_2}^{k-\tau_1} x^T(i) Q x(i). \end{aligned} \tag{12}$$

Noting that $\tau(k) \leq \tau_2$, we have

$$\sum_{i=k+1-\tau(k+1)}^{k-\tau_1} x^T(i) Q x(i) - \sum_{i=k+1-\tau_2}^{k-\tau_1} x^T(i) Q x(i) \leq 0.$$

Then it follows that

$$\begin{aligned}\Delta V_2(k) + \Delta V_3(k) \leq (\tau_2-\tau_1+1) x^T(k) Q x(k) \\ - x^T(k-\tau(k)) Q x(k-\tau(k)). \end{aligned}$$

So, we obtain

$$\Delta V(k) \leq \bar{x}^T(k) \Psi \bar{x}(k),$$

where

$$\Psi = \begin{bmatrix} (\tau_2-\tau_1+1)Q & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - P^T \begin{bmatrix} \frac{1}{2}I & 0 & 0 & \\ A & -I & A_d & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}I & A^T & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & A_d^T & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} P.$$

A sufficient condition on the stability of the system (1) is given by

$$\Psi < 0. \tag{13}$$

By the well-known S-procedure [14], inequality (13) with constraint (3) is equivalent to the existence of matrices $P > 0$, $Q > 0$ and a scalar $\varepsilon \geq 0$ such that

$$\begin{bmatrix} (\tau_2-\tau_1+1)Q + \varepsilon\alpha^2 H^T H & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & -\varepsilon I \end{bmatrix} - P^T \begin{bmatrix} \frac{1}{2}I & 0 & 0 & \\ A & -I & A_d & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}I & A^T & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & A_d^T & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} P < 0.$$

This can be written as

$$\begin{bmatrix} \Lambda & -A^T P_3 + P_2^T & -P_2^T A_d & -P_2 \\ * & P_1 + P_3 + P_3^T & -P_3^T A_d & -P_3^T \\ * & * & -Q & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0, \quad P > 0,\ Q > 0,\ \varepsilon \geq 0, \tag{14}$$

where $\Lambda = (\tau_2-\tau_1+1)Q + \varepsilon\alpha^2 H^T H - P_1 - A^T P_2 - P_2^T A$.

It should be noted that inequalities (14) represent non-strict LMIs since $\varepsilon \geq 0$. It is well known that minimization under non-strict LMI constraints gives the same result as minimization under strict LMI constraints when both strict and non-strict LMI constraints are feasible [1]. This is true for (14) because if there is a solution for $\varepsilon = 0$ there is a solution for some $\varepsilon > 0$ and sufficiently small α. Thus we can substitute $\varepsilon > 0$ for $\varepsilon \geq 0$ in (14). Therefore (14) is equivalent to the existence of matrices $\bar{P} := P/\varepsilon > 0$, $\bar{Q} := Q/\varepsilon > 0$ such that

$$\begin{bmatrix} \bar{\Lambda} & -A^T \bar{P}_3 + \bar{P}_2^T & -\bar{P}_2^T A_d & -\bar{P}_2 \\ * & \bar{P}_1 + \bar{P}_3 + \bar{P}_3^T & -\bar{P}_3^T A_d & -\bar{P}_3^T \\ * & * & -\bar{Q} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{15}$$

where $\bar{\Lambda} = (\tau_2-\tau_1+1)\bar{Q} + \alpha^2 H^T H - \bar{P}_1 - A^T \bar{P}_2 - \bar{P}_2^T A$. Using Schur complement formula, (15) is equivalent to (6). ■

To obtain the maximal parameter α for the bound of non-linear perturbations on the system, we propose the optimization problem

$$\left.\begin{array}{ll}\text{minimize} & \gamma \\ \text{subject to} & \bar{P}_1 > 0,\ \bar{P}_2,\ \bar{P}_3,\ \bar{Q} > 0 \text{ and (6)}\ .\end{array}\right\} \quad (16)$$

We arrive at the following theorem.

Theorem 1: Consider unforced system (1) with $u(k) \equiv 0$. Given integers τ_1, τ_2 such that $0 < \tau_1 < \tau_2$, this system is robustly stable with maximal non-linear bound $\alpha = 1/\sqrt{\gamma}$ if there exist matrices $\bar{P}_1 > 0$, $\bar{P}_2$, $\bar{P}_3$, $\bar{Q} > 0$ such that problem (16) is feasible for any time-varying delay $\tau(k)$ satisfying condition (5).

Then, we focus our attention on designing linear constant feedback control laws to robustly stabilize plant (1).

Theorem 2: Given integers τ_1, τ_2 such that $0 < \tau_1 < \tau_2$, the system (1) is robustly stabilizable with degree α if for any time-varying delay $\tau(k)$ satisfying condition (5) there exist matrices $W_1 > 0$, $Z > 0$, W_2, W_3, Y and a scalar $\gamma > 0$ satisfying the following LMI:

$$\left[\begin{array}{cccccccccc} -W_1 & W_2^T - W_1 A^T - Y^T B^T & 0 & 0 & W_1 H^T & W_2^T & W_1 \\ * & W_3^T + W_3 & -A_d Z & -I & 0 & W_3^T & 0 \\ * & * & -Z & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & -W_1 & 0 \\ * & * & * & * & * & * & -\eta Z \end{array}\right] < 0, \quad (17)$$

where $\eta = 1/(\tau_2 - \tau_1 + 1)$, $\gamma = \alpha^{-2}$. Furthermore, the state feedback control law is given by

$$u(k) = Y W_1^{-1} x(k).$$

Proof: Following the proof of Lemma 1, the robust stability of the system (1) with $u(k) = Fx(k)$ can be guaranteed by the existence of matrices $\bar{P}_1 > 0$, $\bar{P}_2$, $\bar{P}_3$, $\bar{Q} > 0$ and a scalar $\gamma > 0$ satisfying the following LMI:

$$\left[\begin{array}{ccccc} (\tau_2 - \tau_1 + 1)\bar{Q} - \bar{P}_1 - \bar{A}^T \bar{P}_2 - \bar{P}_2^T \bar{A} & -\bar{A}^T \bar{P}_3 + \bar{P}_2^T & -\bar{P}_2^T A_d & -\bar{P}_2 & H^T \\ * & \bar{P}_1 + \bar{P}_3 + \bar{P}_3^T & -\bar{P}_3^T A_d & -\bar{P}_3^T & 0 \\ * & * & -\bar{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma I \end{array}\right] < 0, \quad (18)$$

where $\bar{A} = A + BF$. In order to design the controller and obtain LMI we use P^{-1}. It can be easily verified in (18) that $\bar{P}_3 + \bar{P}_3^T < 0$. Because $\bar{P}_1 > 0$, we know

$$\bar{P} = \left[\begin{array}{cccc} \bar{P}_1 & 0 & 0 & 0 \\ \bar{P}_2 & \bar{P}_3 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right]$$

is nonsingular. Define

$$\bar{P}^{-1} := W = \left[\begin{array}{cccc} W_1 & 0 & 0 & 0 \\ W_2 & W_3 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right].$$

Multiplying $\mathrm{diag}\{W^T, I\}$ and its transpose on the left and on the right sides of (18), respectively, using Schur complement, we know that (18) is equivalent to

$$\left[\begin{array}{ccccc} \Xi & W_2^T - W_1 \bar{A}^T & 0 & 0 & W_1 H^T \\ * & W_3^T - W_3 & -A_d & -I & 0 \\ * & * & -\bar{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma I \end{array}\right] + \left[\begin{array}{c} W_2^T \\ W_3^T \\ 0 \\ 0 \end{array}\right] \bar{P}_1 [W_2 \ \ W_3 \ \ 0 \ \ 0] < 0, \quad (19)$$

where $\Xi = (\tau_2 - \tau_1 + 1) W_1 \bar{Q} W_1 - W_1$.

Denoting $Y := F W_1$, $Z = \bar{Q}^{-1}$, Pre- and post-multiplying $\mathrm{diag}\{I,\ I,\ \bar{Q},\ I,\ I\}$ on both sides of (19), using Schur complements, we obtain that (19) is equivalent to (17). Furthermore, the feedback gain is given by $F = Y W_1^{-1}$. ■

Theorem 2 provides a delay dependent condition on robust stabilization of discrete-time linear state delayed systems in terms of the solvability of LMI. One important advantage of Theorem 2 is that the largest uncertain perturbation parameter α which preserves the robust stability of the closed-loop system can be computed by solving the following optimization problem in W_1, Z, W_2, W_3, Y for given positive integers τ_1, τ_2:

$$\left.\begin{array}{ll}\text{minimize} & \gamma \\ \text{subject to} & W_1 > 0,\ W_2,\ W_3,\ Z > 0,\ Y \\ & \text{and (17).}\end{array}\right\} \quad (20)$$

Theorem 3: Given integers τ_1, τ_2 such that $0 < \tau_1 < \tau_2$, the system (1) is robustly stabilizable with maximal non-linear bound $\alpha = 1/\sqrt{\gamma}$ if there exist matrices $W_1 > 0$, W_2, W_3, $Z > 0$, Y such that problem (20) is feasible for any time-varying delay $\tau(k)$ satisfying condition (5). Furthermore, the state feedback control law is given by

$$u(k) = Y W_1^{-1} x(k).$$

III. ROBUST STABILIZATION OF DISCRETE-TIME SYSTEMS WITH INPUT DELAYS

Now, we pay our attention to the following discrete-time system with time-varying input delays

$$x(k+1) = Ax(k) + Bu(k - \tau(k)) + h(k, x(k)), \quad (21)$$

where $\tau(k)$ and $h(k,x(k))$ are defined as that of Section II, and satisfy conditions (5) and (4), respectively.

In this section, we shall develop delay-dependent conditions on the robust stabilization of the system (21).

Theorem 4: Given integers τ_1, τ_2 such that $0<\tau_1<\tau_2$, the system (21) is robustly stabilizable with degree α if for any time-varying delay $\tau(k)$ satisfying condition (5) there exist matrices $W_1>0$, $Z>0$, W_2, W_3, Y and a positive scalar γ satisfying the following LMI:

$$\begin{bmatrix} -W_1 & W_2^T-W_1A^T & 0 & 0 & W_1H^T & W_2^T & W_1 \\ * & W_3^T+W_3 & -BY & -I & 0 & W_3^T & 0 \\ * & * & -Z & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & -W_1 & 0 \\ * & * & * & * & * & * & -\eta Z \end{bmatrix}<0, \tag{22}$$

where $\eta=1/(\tau_2-\tau_1+1)$. Furthermore, the state feedback control law is given by

$$u(k)=YZ^{-1}x(k).$$

Proof: Using quadratic function $V(x(k))$ defined by (8), following the proof of Lemma 1, we obtain that the robust stability of the system (21) with $u(k)=Fx(k)$ can be guaranteed by the existence of matrices $\bar{P}_1>0$, $\bar{P}_2$, $\bar{P}_3$ and $\bar{Q}>0$ satisfying

$$\begin{bmatrix} (\tau_2-\tau_1+1)\bar{Q}-\bar{P}_1-A^T\bar{P}_2-\bar{P}_2^TA & -A^T\bar{P}_3+\bar{P}_2^T & -\bar{P}_2^TBF & -\bar{P}_2 & H^T \\ * & \bar{P}_1+\bar{P}_3+\bar{P}_3^T & -\bar{P}_3^TBF & -\bar{P}_3^T & 0 \\ * & * & -\bar{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma I \end{bmatrix}<0. \tag{23}$$

Analogous to the proof of Theorem 2, we can easily obtain the conclusion. ■

Theorem 4 provides a delay-dependent condition on robust stabilization of a class of discrete-time systems with input delays. The smallest γ for a given η in Theorem 4 can be computed by solving the following convex optimization problem:

$$\left.\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & W_1>0,\ W_2,\ W_3,\ Z>0,\ Y \\ & \text{and (22).} \end{array}\right\} \tag{24}$$

Theorem 5: Given integers τ_1, τ_2 such that $0<\tau_1<\tau_2$, the system (21) is robustly stabilizable with maximal non-linear bound $\alpha=1/\sqrt{\gamma}$ if there exist matrices $W_1>0$, W_2, W_3, $Z>0$, Y such that problem (24) is feasible for any time-varying delay $\tau(k)$ satisfying condition (5). Furthermore, the feedback gain is given by

$$F=YZ^{-1}.$$

Remark 1: For a given γ, similar procedure can be applied to the computation of the largest delay-related parameter $\tau_2-\tau_1+1$ by solving a quasi-convex optimization problem using the LMI toolbox [1].

Remark 2: For discrete-time systems with both state and input delays, the robust control problems can be solved by using the approach proposed above.

IV. Numerical Examples

Example 1: Let us consider the discrete-time system with state delays

$$x(k+1)=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}x(k)+\begin{bmatrix} 0.1 & 1 \\ 0 & 1 \end{bmatrix}x(k-\tau(k)) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)+h(k,x(k)) \tag{25}$$

with $H=[2\ 0]$ and the state feedback gain F to be designed.

It is assumed that the time-varying delay $\tau(k)$ satisfies (5) with

$$\tau_1=1,\ \ \tau_2=4.$$

By solving the convex optimization problem (20) using LMI toolbox, we obtain

$$W1=\begin{bmatrix} 14.8818 & -0.3482 \\ -0.3482 & 1.4814 \end{bmatrix},$$

$$W2=\begin{bmatrix} -0.0697 & 0.2098 \\ 0.0035 & -0.0080 \end{bmatrix},$$

$$W3=\begin{bmatrix} -12.5221 & -0.0503 \\ 0.2353 & -1.5296 \end{bmatrix},$$

$$Z=\begin{bmatrix} 125.7780 & -6.5237 \\ -6.5237 & 8.7844 \end{bmatrix},$$

$$Y=[28.7967\ \ 3.4557],\ \gamma_{min}=53.2980.$$

Then it can be easily calculated from Theorem 3 that

$$\alpha_{max}=0.1370,\quad F=[2.0006\ \ 2.8031]$$

with the eigenvalues of matrix $A+BF$ located at 0.0031 and -0.2000.

This example shows that our method can be efficiently applied to the discrete-time system with time-varying state delays and non-linear perturbations.

To further illustrate our approach, we consider a discrete-time system with input delays.

Example 2:

$$\begin{aligned} x(k+1)&=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}x(k)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k-\tau(k))+h(k,x(k)) \\ u(k)&=Fx(k) \end{aligned} \tag{26}$$

with $H = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and state feedback gain F to be designed.

We assumed that the time-varying delay $\tau(k)$ satisfies (5) with

$$\tau_1 = 1, \ \ \tau_2 = 5.$$

By solving the convex optimization problem (24) using LMI toolbox, we obtain the following feasible solution

$$W1 = \begin{bmatrix} 17.4313 & 0.0484 \\ 0.0484 & 6.9765 \end{bmatrix},$$

$$W2 = \begin{bmatrix} 0.0027 & 1.5790 \\ -0.0001 & 0.0027 \end{bmatrix},$$

$$W3 = \begin{bmatrix} -13.0822 & 0.0029 \\ -0.0441 & -5.9243 \end{bmatrix},$$

$$Z = \begin{bmatrix} 235.8203 & 0.6192 \\ 0.6192 & 172.8935 \end{bmatrix},$$

$$Y = [-0.0483 \ \ -6.8897], \ \gamma_{min} = 8.5312.$$

By Theorem 5, we obtain

$$\alpha_{max} = 0.3424, \quad F = [-0.0001 \ \ -0.0398].$$

with eigenvalues of $A+BF$ located at 0.0001 and 0.9600.

Therefore, the unstable input delayed non-linear discrete-time system (26) can be effectively stabilized via the state feedback controller designed in this paper.

V. CONCLUSION

We have investigated the robust stabilization problem for discrete-time delay systems with state/input delays subject to non-linear perturbations. We first deal with the robust stabilization of discrete-time systems with state delays. Then the robust stabilization for discrete-time systems with input delays is studied. Delay-dependent conditions on the stabilization of such discrete-time systems are given in terms of LMI. The designed stabilizing feedback controllers produce a closed-loop system which is maximally tolerant to the uncertain non-linear terms. Numerical examples have been worked out to show how an unstable system can be effectively stabilized via the controller designed in this paper.

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