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l_∞ Performance Analysis of Feedback Systems with Saturation Nonlinearities: An Approach Based on Polytopic Representation[§]

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***Abstract*— In this paper, we show a method of estimating l_∞-norm of an output signal of discrete-time feedback systems with saturation nonlinearities. The analysis condition for estimating l_∞-norm is derived based on the parameter dependent Lyapunov function (PDLF) and is reduced to a condition based on the linear matrix inequality (LMI) representation. Further, we show that the condition is guaranteed to be less conservative than the condition based on the standard sector condition. A numerical example is provided to illustrate effectiveness of the proposed method.**

I. INTRODUCTION

In most of physical control systems, there exist saturation limitations on controller outputs. Estimating control performance that can be achieved under such a situation accurately is an important issue, since it provides a basis for constructing design methods of a high performance controller in the presence of input saturation. Before now, various types of methods have been developed for estimating control performance of such systems (see e.g.,[2], [22]–[26]). In this paper, we address the analysis problem of l_∞-norm of an output signal of feedback systems with saturation nonlinearities. This class of problem typically appears when control performance of the anti-windup control system is studied [30].

Recently, a novel polytopic model of a saturation function was proposed in [12]. Based on the model, several interesting results were reported in [5], [7], [12]–[14], [27]. In [7], [27], it is shown that the analysis condition of $\mathcal{L}_2$ performance of systems with input saturation derived by using the polytopic model is less conservative than that based on the circle criterion. Further, it is shown that, although the analysis condition based on the circle criterion becomes the bilinear matrix inequality (BMI) in general [11], [18], the analysis condition derived by using the polytopic model can be reduced to the complete LMI.

In this paper, we show a method of estimating l_∞-norm of an output signal of discrete-time feedback systems with saturation nonlinearities. Firstly, we derive a simple analysis condition based on the Lyapunov approach and $\mathcal{S}$-procedure. To derive the condition, we utilize the polytopic model of a saturation function of [12] and the PDLF [6], [19], [28]. We show that the analysis condition is reduced to scalar dependent LMIs. Then we show that the proposed condition is guaranteed to be less conservative than the condition based on the standard sector condition. A numerical example is provided to illustrate this fact.

Notations: For a vector $v \in \mathcal{R}^n$, we define the standard multivariable saturation function as $\Phi(v) := (\phi(v_1), \cdots, \phi(v_n))^T$, where

$$\phi(v_i) := \begin{cases} \mathrm{sgn}(v_i), & |v_i| > 1 \\ v_i, & |v_i| \le 1 \end{cases}$$

For a vector $v \in \mathcal{R}^n$, we denote its Euclidean norm as $\|v\|_2 := (v^T v)^{1/2}$. For a signal $v(t)$ defined on $[0, \infty)$, we define its l_∞ norm as $\|v\|_{l_\infty} := \sup_{t \ge 0} \|v(t)\|$. Let $P \in \mathcal{R}^{n \times n}$ be a positive definite matrix. Denote

$$\mathcal{E}(P) := \left\{ x \in \mathcal{R}^n : x^T P x \le 1 \right\} \tag{1}$$

For a matrix $F \in \mathcal{R}^{m \times n}$, denote the ith row of F as $f^{(i)}$ and define

$$\mathcal{L}(F) := \left\{ x \in \mathcal{R}^n : |f^{(i)} x| \le 1, i = 1, \cdots, m \right\} \tag{2}$$

II. PRELIMINARIES

In this section, we introduce a polytopic model of a saturation function of [12]. Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose diagonal element are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as $\mathbf{E}_i, i = 1, 2, \cdots, 2^m$, and denote $\mathbf{E}_i^- := I - \mathbf{E}_i$. Clearly, $\mathbf{E}_i^-$ is also an element of $\mathcal{V}$.

Lemma 1: [12] Let $u, v \in \mathcal{R}^m$. Suppose that $|v_i| \le 1$ for all $i \in [1, m]$, then

$$\Phi(u) \in \mathrm{co}\{\mathbf{E}_i u + \mathbf{E}_i^- v : i \in [1, 2^m]\} \tag{3}$$

where co denotes the convex hull.

This means that, for $|v_i| \le 1$, we can rewrite $\Phi(u)$ as

$$\Phi(u) = \sum_{i=1}^{2^m} \lambda_i (\mathbf{E}_i u + \mathbf{E}_i^- v) \tag{4}$$

where $0 \le \lambda_i \le 1$, $\sum_{i=1}^{2^m} \lambda_i = 1$.

III. l_∞ PERFORMANCE ANALYSIS USING PDLF

Consider the following discrete-time system with the saturation nonlinearity.

$$\begin{aligned} x(t+1) &= Ax(t) + B_1 q(t) + B_2 w(t) && (5) \\ p(t) &= C_1 x(t) \qquad\qquad + D_{12} w(t) && (6) \\ z(t) &= C_2 x(t) + D_{21} q(t) + D_{22} w(t) && (7) \\ q(t) &= \Phi(p(t)) && (8) \end{aligned}$$

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where $x(t) \in \mathcal{R}^n, q(t) \in \mathcal{R}^m, w(t) \in \mathcal{R}^{n_w}, z(t) \in \mathcal{R}^{n_z}$. We assume that $w(t) \in \mathcal{W} := \{w(t) \in \mathcal{R}^{n_w} : \|w\|_{l_\infty} \leq \xi\}$. In this section, we show a method of estimating an upper bound of the l_∞ norm of the signal z of the system (5)–(8).

From Lemma 1, while $[x^T, w^T]^T \in \mathcal{L}([E, F])$, the saturation nonlinearity $\Phi(p)$ can be represented as

$$\Phi(p) = \sum_{i=1}^{2^m} \lambda_i \{\mathbf{E}_i[C_1, D_{12}] + \mathbf{E}_i^-[E, F]\} \begin{bmatrix} x \\ w \end{bmatrix} \tag{9}$$

where $E \in \mathcal{R}^{m\times n}$ and $F \in \mathcal{R}^{m\times n_w}$ are constant matrices. From Eq.(9), while $[x^T, w^T]^T \in \mathcal{L}([E, F])$, the system (5)–(8) can be rewritten as

$$x(t+1) = \mathcal{A}(\lambda(t))x(t) + \mathcal{B}(\lambda(t))w(t) \tag{10}$$

$$z(t) = \mathcal{C}(\lambda(t))x(t) + \mathcal{D}(\lambda(t))w(t) \tag{11}$$

where $\mathcal{A}(\lambda(t)) := \sum_{i=1}^{2^m} \lambda_i(t)\mathcal{A}_i,\ \mathcal{A}_i := A + B_1(\mathbf{E}_iC_1 + \mathbf{E}_i^- E),\ \mathcal{B}(\lambda(t)) := \sum_{i=1}^{2^m} \lambda_i(t)\mathcal{B}_i,\ \mathcal{B}_i := B_2 + B_1(\mathbf{E}_iD_{12} + \mathbf{E}_i^- F),\ \mathcal{C}(\lambda(t)) := \sum_{i=1}^{2^m} \lambda_i(t)\mathcal{C}_i,\ \mathcal{C}_i := C_2 + D_{21}(\mathbf{E}_iC_1 + \mathbf{E}_i^- E),\ \mathcal{D}(\lambda(t)) := \sum_{i=1}^{2^m} \lambda_i(t)\mathcal{D}_i$ and $\mathcal{D}_i := D_{22} + D_{21}(\mathbf{E}_iD_{12} + \mathbf{E}_i^- F)$. Then, the following theorem can be stated.

Theorem 1: Consider the system (5)–(8). For given positive definite matrices $Q_i, (i = 1, \cdots, 2^m)$, assume that there exist matrices G, Z, F and positive scalars γ and $\alpha \in [0, 1)$ satisfying the following matrix inequalities.

$$\begin{bmatrix} (1-\alpha)(G + G^T - Q_i) & * & * \\ 0 & \frac{\alpha}{\xi^2} I & * \\ AG + B_1(\mathbf{E}_iC_1G + \mathbf{E}_i^- Z) & B_2 + B_1(\mathbf{E}_iD_{12} + \mathbf{E}_i^- F) & Q_j \end{bmatrix} \geq 0 \quad \forall i, j \in [1, 2^m] \tag{12}$$

$$\begin{bmatrix} G + G^T - Q_i & 0 & * \\ 0 & \frac{1}{\xi^2} I & * \\ z^{(l)} & f^{(l)} & \frac{1}{2} \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m],\ \forall l \in [1, m] \tag{13}$$

$$\begin{bmatrix} G + G^T - Q_i & * & * \\ 0 & \frac{1}{\xi^2} I & * \\ C_2G + D_{21}(\mathbf{E}_iC_1G + \mathbf{E}_i^- Z) & D_{22} + D_{21}(\mathbf{E}_iD_{12} + \mathbf{E}_i^- F) & \frac{\gamma^2}{2} I \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m] \tag{14}$$

where the symbol $*$ stands for symmetric block in matrix inequalities. Further, $z^{(l)}$ and $f^{(l)}$ denote the lth row of the matrices Z and F respectively. If $x(0) \in \mathcal{E}(P(\lambda(0)))$, then $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$ and the following inequality holds.

$$\|z\|_{l_\infty} \leq \gamma \tag{15}$$

where $P(\lambda(t)) = \sum_{i=1}^{2^m} \lambda_i(t)P_i,\ P_i = Q_i^{-1}$.

Proof: Since $G + G^T - Q_i > 0$ and $Q_i > 0$ from the elements $\{1, 1\}$ and $\{4, 4\}$ of Eq.(12), the matrix G is nonsingular. In the following, we first show that the condition (13) guarantees that $[x^T, w^T]^T \in \mathcal{L}([E, F])$ if $x \in \mathcal{E}(P(\lambda))$ and $w \in \mathcal{W}$, where $E = ZG^{-1}$. Since $Q_i > 0$, we have $(G - Q_i)^T Q_i^{-1}(G - Q_i) \geq 0$. Hence, the following inequality holds.

$$G^T Q_i^{-1} G \geq G + G^T - Q_i \tag{16}$$

Then, from Eq.(16) and $z^{(l)} = e^{(l)}G$, we can conclude that Eq.(13) implies

$$\begin{bmatrix} G^T Q_i^{-1} G & 0 & * \\ 0 & \frac{1}{\xi^2} I & * \\ e^{(l)}G & f^{(l)} & \frac{1}{2} \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m], \forall l \in [1, m] \tag{17}$$

where $e^{(l)}$ denotes the lth row of the matrix E. By multiplying Eq.(17) from the left by block-diag$[G^{-T}, I, 1]$ and from the right by block-diag$[G^{-1}, I, 1]$, we have

$$\begin{bmatrix} Q_i^{-1} & 0 & * \\ 0 & \frac{1}{\xi^2} I & * \\ e^{(l)} & f^{(l)} & \frac{1}{2} \end{bmatrix} \geq 0,\ \forall i \in [1, 2^m], \forall l \in [1, m] \tag{18}$$

Then, by substituting $Q_i^{-1} = P_i$ for Eq.(18), and multiplying each inequality by λ_i, and summing them up for $i = 1, \cdots, 2^m$, we obtain

$$\begin{bmatrix} P(\lambda) & 0 & * \\ 0 & \frac{1}{\xi^2} I & * \\ e^{(l)} & f^{(l)} & \frac{1}{2} \end{bmatrix} \geq 0,\ \forall l \in [1, m] \tag{19}$$

By applying Schur complement [3] to Eq.(19), we have

$$\begin{bmatrix} e^{(l)T} \\ f^{(l)T} \end{bmatrix} \begin{bmatrix} e^{(l)} & f^{(l)} \end{bmatrix} \leq \frac{1}{2} \begin{bmatrix} P(\lambda) & 0 \\ 0 & \frac{1}{\xi^2} I \end{bmatrix} \quad \forall l \in [1, m] \tag{20}$$

Eq.(20) implies that if $x \in \mathcal{E}(P(\lambda))$ and $w \in \mathcal{W}$ then $[x^T, w^T]^T \in \mathcal{L}([E, F])$.

Then, we show that if the condition (12) holds and $x(0) \in \mathcal{E}(P(\lambda(0)))$, then $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$. By substituting $Z = EG$ for Eq.(12), we have

$$\begin{bmatrix} (1-\alpha)(G + G^T - Q_i) & * & * \\ 0 & \frac{\alpha}{\xi^2} I & * \\ \mathcal{A}_i G & \mathcal{B}_i & Q_j \end{bmatrix} \geq 0 \quad \forall i, j \in [1, 2^m] \tag{21}$$

By taking account of Eq.(16), we can state that Eq.(21) implies

$$\begin{bmatrix} (1-\alpha)G^T Q_i^{-1} G & * & * \\ 0 & \frac{\alpha}{\xi^2} I & * \\ \mathcal{A}_i G & \mathcal{B}_i & Q_j \end{bmatrix} \geq 0 \quad \forall i, j \in [1, 2^m] \tag{22}$$

Then, by multiplying Eq.(22) from the left by block-diag$[G^{-T}, I, Q_j^{-1}]$ and from the right by block-diag$[G^{-1}, I, Q_j^{-1}]$, and substituting $Q_j^{-1} = P_j$, $Q_i^{-1} = P_i$ for the resulting inequality, we obtain

$$\begin{bmatrix} (1-\alpha)P_i & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * \\ P_j\mathcal{A}_i & P_j\mathcal{B}_i & P_j \end{bmatrix} \geq 0, \qquad \forall i, j \in [1, 2^m] \quad (23)$$

Then, by multiplying Eq.(23) by $\lambda_i(t)$, and summing them up for $i = 1, \cdots, 2^m$, and multiplying each inequality by $\lambda_j(t+1)$, and summing them up for $j = 1, \cdots, 2^m$, we have

$$\begin{bmatrix} (1-\alpha)P(\lambda(t)) & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * \\ P(\lambda(t+1))\mathcal{A}(\lambda(t)) & P(\lambda(t+1))\mathcal{B}(\lambda(t)) & P(\lambda(t+1)) \end{bmatrix} \geq 0 \quad (24)$$

Further, by multiplying Eq.(24) from both sides by diag$[I, I, P(\lambda(t+1))^{-1}]$, we have

$$\begin{bmatrix} (1-\alpha)P(\lambda(t)) & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * \\ \mathcal{A}(\lambda(t)) & \mathcal{B}(\lambda(t)) & P(\lambda(t+1))^{-1} \end{bmatrix} \geq 0 \quad (25)$$

By applying Schur complement to Eq.(25), we obtain

$$(1+\beta)\begin{bmatrix} \mathcal{A}(\lambda(t))^T \\ \mathcal{B}(\lambda(t))^T \end{bmatrix} P(\lambda(t+1)) \begin{bmatrix} \mathcal{A}(\lambda(t)) & \mathcal{B}(\lambda(t)) \end{bmatrix} \leq \begin{bmatrix} P(\lambda(t)) & 0 \\ 0 & \frac{\beta}{\xi^2}I \end{bmatrix} \quad (26)$$

where $\beta := \alpha/(1-\alpha)$. By multiplying Eq.(26) from the left by $[x(t)^T, w(t)^T]$ and from the right by $[x(t)^T, w(t)^T]^T$ and using Eq.(10), we obtain

$$\begin{aligned} &(1+\beta)x(t+1)^T P(\lambda(t+1))x(t+1) \\ &\leq \frac{\beta}{\xi^2}w(t)^T w(t) + x(t)^T P(\lambda(t))x(t) \end{aligned} \quad (27)$$

Eq.(27) implies the following.

$$\begin{aligned} &x(t)^T P(\lambda(t))x(t) \leq 1 \text{ and } w(t) \in \mathcal{W} \\ &\quad \Rightarrow x(t+1)^T P(\lambda(t+1))x(t+1) \leq 1 \end{aligned} \quad (28)$$

From Eq.(28) and $x(0) \in \mathcal{E}(P(\lambda(0)))$, we can conclude that $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$. By the way, from Lemma 1, the saturation function can be represented as the polytopic model (9) while $x \in \mathcal{L}(E, F)$. From Eq.(20) and $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$, we can conclude that the relation $x(t) \in \mathcal{L}(E, F), \forall t \geq 0$ holds.

Finally, we show that if the condition (14) holds and $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$, then the inequality (15) holds. From Eq.(16) and $z^{(l)} = e^{(l)}G$, we can conclude that Eq.(14) implies

$$\begin{bmatrix} G^T Q_i^{-1} G & * & * \\ 0 & \frac{1}{\xi^2}I & * \\ \mathcal{C}_i G & \mathcal{D}_i & \frac{\gamma^2}{2}I \end{bmatrix} \geq 0, \ \forall i \in [1, 2^m] \quad (29)$$

By multiplying Eq.(29) from the left by block-diag$[G^{-T}, I, 1]$ and from the right by block-diag$[G^{-1}, I, 1]$, we have

$$\begin{bmatrix} Q_i^{-1} & 0 & * \\ 0 & \frac{1}{\xi^2}I & * \\ \mathcal{C}_i & \mathcal{D}_i & \frac{\gamma^2}{2}I \end{bmatrix} \geq 0, \ \forall i \in [1, 2^m] \quad (30)$$

Then, by substituting $Q_i^{-1} = P_i$ for Eq.(30), and multiplying each inequality by λ_i, and summing them up for $i = 1, \cdots, 2^m$, we obtain

$$\begin{bmatrix} P(\lambda(t)) & 0 & * \\ 0 & \frac{1}{\xi^2}I & * \\ \mathcal{C}(\lambda(t)) & \mathcal{D}(\lambda(t)) & \frac{\gamma^2}{2}I \end{bmatrix} \geq 0 \quad (31)$$

By applying Schur complement [3] to Eq.(31), we have

$$\begin{bmatrix} \mathcal{C}(\lambda(t))^T \\ \mathcal{D}(\lambda(t))^T \end{bmatrix} \begin{bmatrix} \mathcal{C}(\lambda(t)) & \mathcal{D}(\lambda(t)) \end{bmatrix} \leq \frac{\gamma^2}{2} \begin{bmatrix} P(\lambda(t)) & 0 \\ 0 & \frac{1}{\xi^2}I \end{bmatrix} \quad (32)$$

Therefore, From Eq.(32) and $x(t) \in \mathcal{E}(P(\lambda(t))), \forall t \geq 0$, we can conclude that Eq.(15) holds. ■

Based on Theorem 1, the following optimization problems can be obtained.

Problem 1: For a given positive scalar ξ, $\min_{G,Z,Q_i>0,\alpha\in[0,1)} \gamma^2$ s.t. (12)–(14).

By solving Problem 1, we can compute an upper bound of $\|z\|_{l_\infty}$ of the system (5)–(7). However, unfortunately, there exist product terms among the decision variables Q_i, G and α in the matrix inequality condition (5). Therefore, Problem 1 is the BMI problem. However, if we fix the positive scalar α, all the conditions in the problem become the LMIs. Hence, we can obtain a smallest value of γ by performing the line search on the scalar $\alpha \in [0, 1)$.

Remark 1: The shape of the initial state set $\mathcal{E}(P(\lambda(0)))$ in Theorem 1 depends on the value of $\lambda(0)$. We can obtain an initial state set that does not depend on $\lambda(0)$ by minimizing trace$S$ under the constraint $\mathcal{E}(S) \subseteq \mathcal{E}(P(\lambda))$, where $S \in \mathcal{R}^{n\times n}$ is a positive definite matrix. This problem can easily be reduced to the LMI problem.

Remark 2: By introducing the matrix G, we obtain the conditions that do not have product terms between the Lyapunov matrix Q_i and the matrix $\mathcal{A}_i$. As a consequence, we can reduce the analysis conditions to the LMIs. This type of parameterization was introduced in [19] and was extended to the system with time varying parametric uncertainties in [6] and the nonlinear $\mathcal{H}_\infty$ control in [28].

Remark 3: It is possible to derive a condition for designing a state feedback controller based on Theorem 1. We assume that the matrix C_1 is a feedback gain to be designed. Then, by substituting $Y := C_1 G$ for Eqs.(12) and (14), we can obtain matrix inequality conditions for the design of the feedback gain C_1.

IV. Relationship with the Condition based on the Standard Sector Condition

In this section, we show relationship between Theorem 1 and an analysis condition based on the standard sector condition [21]. By applying the standard sector condition to the saturation nonlinearity, the following result can be derived.

Proposition 1: Consider the system (5)–(7). For a given positive definite matrix Q, assume that there exist diagonal matrices $R = \mathrm{diag}[r_1, \cdots, r_m] > 0,\ \Gamma > 0$ and positive scalars γ and $\alpha \in [0, 1)$ satisfying the following matrix inequalities.

$$\begin{bmatrix} (1-\alpha)Q & * & * & * \\ -(1-\alpha)RC_1Q & 2(1-\alpha)\Gamma & * & * \\ 0 & -(1-\alpha)D_{12}^T R & \frac{\alpha}{\xi^2}I & * \\ AQ + B_1C_1Q & -B_1\Gamma & B_1D_{12} + B_2 & Q \end{bmatrix} \geq 0 \quad (33)$$

$$\begin{bmatrix} Q & * & * \\ 0 & \frac{1}{\xi^2}I & * \\ (1-r_l)C_1^{(l)}Q & (1-r_l)D_{12}^{(l)} & \frac{1}{2} \end{bmatrix} \geq 0 \quad \forall l \in [1, m] \quad (34)$$

$$\begin{bmatrix} Q & * & * & * \\ -2RC_1Q & 4\Gamma & * & * \\ 0 & -2D_{12}^T R & \frac{1}{\xi^2}I & * \\ C_2Q + D_{21}C_1Q & -D_{21}\Gamma & D_{21}D_{12} + D_{22} & \frac{\gamma^2}{2}I \end{bmatrix} \geq 0 \quad (35)$$

If $x(0) \in \mathcal{E}(P)$, then $x(t) \in \mathcal{E}(P), P = Q^{-1}, \forall t \geq 0$ and the inequality $\|z\|_{l_\infty} \leq \gamma$ holds.

Proof: See Appendix I ∎

The l_∞-performance condition in Proposition 1 is the discrete-time version of the condition based on the circle criterion for the continuous time system [11], [18]. Based on Proposition 1, the following optimization problem can be obtained.

Problem 2: For a given positive scalar ξ, $\min_{R>0,Q>0,\Gamma>0,\alpha\in[0,1)} \gamma^2$ s.t. (33)–(35).

Since the conditions in Proposition 1 includes the product terms among the decision variables, this problem is the BMI problem. Note that, as opposed to Problem 1, Problem 2 is still the BMI problem even if the scalar α is fixed.

We introduce the following Lemma.

Lemma 2: For given matrices $P, Q_1, Q_2, R_1, R_2, R_3$ with suitable dimensions, the following holds.

$$P > 0 \text{ and } \begin{bmatrix} P & Q_1^T & Q_2^T \\ Q_1 & R_1 & R_2^T \\ Q_2 & R_2 & R_3 \end{bmatrix} \geq 0 \iff P > 0 \text{ and } \begin{bmatrix} P & Q_2^T & Q_1^T \\ Q_2 & R_3 & R_2 \\ Q_1 & R_2^T & R_1 \end{bmatrix} \geq 0$$

Then, we can state the following theorem.

Theorem 2: If there exist a solution that satisfies the conditions in Proposition 1, then there exist a solution that satisfies the conditions in Theorem 1.

Proof: Assume that the conditions in Proposition 1 holds. By applying Lemma 2 to Eq.(33), we obtain

$$\begin{bmatrix} (1-\alpha)Q & * & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * & * \\ AQ + B_1C_1Q & B_1D_{12} + B_2 & Q & * \\ -(1-\alpha)RC_1Q & -(1-\alpha)RD_{12} & -\Gamma B_1^T & 2(1-\alpha)\Gamma \end{bmatrix} \geq 0 \quad (36)$$

By multiplying Eq.(36) from the left by $\mathcal{T}_i$ and from the right by $\mathcal{T}_i^T$, we obtain

$$\begin{bmatrix} (1-\alpha)Q & * & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * & * \\ AQ + B_1C_1Q - B_1\mathbf{E}_i^- RC_1Q & B_1D_{12} + B_2 - B_1\mathbf{E}_i^- RD_{12} & Q & * \\ -(1-\alpha)RC_1Q & -(1-\alpha)RD_{12} & -\Gamma B_1^T + 2\Gamma\mathbf{E}_i^- RD_{12} & 2(1-\alpha)\Gamma \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m] \quad (37)$$

where

$$\mathcal{T}_i := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & B_1\mathbf{E}_i^-(1-\alpha)^{-1} \\ 0 & 0 & 0 & I \end{bmatrix} \quad (38)$$

To derive Eq.(37), we used the inequality $\mathbf{E}_i^-\mathbf{E}_i = 0$. Clearly, Eq.(37) implies the following.

$$\begin{bmatrix} (1-\alpha)Q & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * \\ AQ + B_1C_1Q - B_1\mathbf{E}_i^- RC_1Q & B_1D_{12} + B_2 - B_1\mathbf{E}_i^- RD_{12} & Q \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m] \quad (39)$$

By using the equality $\mathbf{E}_i + \mathbf{E}_i^- = I$, we obtain

$$\begin{aligned} &AQ + B_1C_1Q - B_1\mathbf{E}_i^- RC_1Q \\ &\quad = AQ + B_1\{\mathbf{E}_iC_1Q + \mathbf{E}_i^-(I-R)C_1Q\} \quad (40) \\ &B_2 + B_1D_{12} - B_1\mathbf{E}_i^- RD_{12} \\ &\quad = B_2 + B_1\{\mathbf{E}_iD_{12} + \mathbf{E}_i^-(I-R)D_{12}\} \quad (41) \end{aligned}$$

By substituting Eqs.(40) and (41) for Eq.(39), we obtain

$$\begin{bmatrix} (1-\alpha)Q & * & * \\ 0 & \frac{\alpha}{\xi^2}I & * \\ AQ + B_1\{\mathbf{E}_iC_1Q + \mathbf{E}_i^-(I-R)C_1Q\} & B_2 + B_1\{\mathbf{E}_iD_{12} + \mathbf{E}_i^-(I-R)D_{12}\} & Q \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m] \quad (42)$$

Eq.(42) implies that Eq.(12) holds with $Q_i = Q, \forall i \in [1, 2^m]$, $G = G^T = Q$, $Z = (I-R)C_1Q$ and $F = (I-R)D_{12}$.

Then we show that Eq.(35) implies Eq.(14). By applying Lemma 2 to Eq.(35), we obtain

$$\begin{bmatrix} Q & * & * & * \\ 0 & \frac{1}{\xi^2}I & * & * \\ \begin{matrix} C_2Q + D_{21}C_1Q \\ -2RC_1Q \end{matrix} & \begin{matrix} D_{21}D_{12} + D_{22} \\ -2RD_{12} \end{matrix} & \frac{\gamma^2}{2}I & * \\ & & -\Gamma D_{21}^T & 4\Gamma \end{bmatrix} \geq 0 \quad (43)$$

By multiplying Eq.(43) from the left by χ_i and from the right by χ_i^T, we obtain

$$\begin{bmatrix} Q & * & * \\ 0 & \frac{1}{\xi^2}I & * \\ C_2Q + D_{21}\{\mathbf{E}_iC_1Q + \mathbf{E}_i^-(I-R)C_1Q\} & D_{22} + D_{21}\{\mathbf{E}_iD_{12} + \mathbf{E}_i^-(I-R)D_{12}\} & \frac{\gamma^2}{2}I \end{bmatrix} \geq 0 \quad \forall i \in [1, 2^m] \quad (44)$$

where

$$\chi_i := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \frac{1}{2}D_{21}\mathbf{E}_i^- \\ 0 & 0 & 0 & I \end{bmatrix} \quad (45)$$

Note that we used $\mathbf{E}_i + \mathbf{E}_i^- = I$ and $\mathbf{E}_i^-\mathbf{E}_i = 0$ to derive Eq.(44). Clearly, Eq.(44) implies that Eq.(14) holds with $Q_i = Q, \forall i \in [1, 2^m]$, $G = G^T = Q$, $Z = (I-R)C_1Q$ and $F = (I-R)D_{12}$.

Furthermore, it is easy to verify that Eq.(34) implies Eq.(13). Therefore, we can conclude that Proposition 1 $\Rightarrow$ Theorem 1. ■

From Theorem 2, we can conclude that Theorem 1 yields less conservative results as compared with Proposition 1.

V. Numerical Example

Consider the system (5)-(8) with the following coefficient matrices.

$$A = \begin{bmatrix} 0.7387 & -0.1065 \\ 0.0750 & 0.8498 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.0341 \\ -0.1584 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1029 \\ 0.0013 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.5913 & 0.6436 \end{bmatrix}, C_2 = \begin{bmatrix} 1.0000 & 0 \end{bmatrix}$$
$$D_{12} = 1.5,\ D_{21} = 0,\ D_{22} = 1$$

Further, we assume that $\xi = 1$. We solved Problem 1 with $\alpha = 0.15$ and obtained $\gamma = 1.69$. Further, we solved Problem 2 with $\alpha = 0.14$ and $R = 0.56$ and obtained $\gamma = 1.75$. Therefore, Theorem 1 yields the less conservative result as compared with Proposition 1. Then we show the responses of $p(t)$ and $z(t)$ for $w(t) = \text{sgn}(\sin(0.4\pi t))$ and $x(0) = 0$ in Fig.1 and Fig.2. We can confirm that the maximum value of $|z(t)|$ is less than $\gamma = 1.69$.

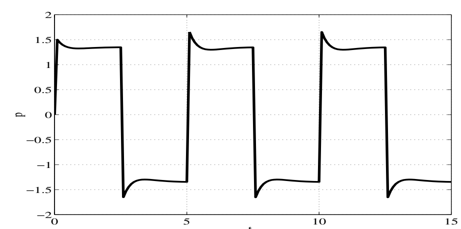

Fig. 1. Response of $p(t)$

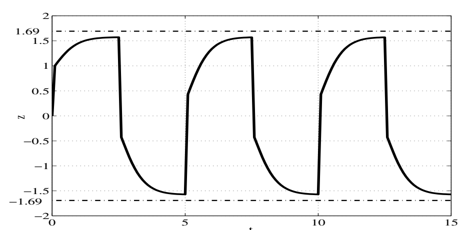

Fig. 2. Response of $z(t)$

VI. Conclusion

In this paper, we have shown a method of estimating l_∞-norm of an output signal of feedback systems with saturation nonlinearities. We summarize the paper as follows.

- We have derived a condition for estimating l_∞-norm of an output signal of feedback systems based on the polytopic representation. Although the obtained condition is represented as the BMI, the condition can be reduced to the LMI by fixing one scalar parameter. Therefore, we can carry out the computation by the line search on the scalar parameter.
- We have shown that the proposed analysis condition yields less conservative results as compared with the condition based on the standard sector condition.

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APPENDIX I
PROOF OF PROPOSITION 1

By using the dead-zone function defined by $\Psi(p(t)) := p(t) - \Phi(p(t))$, Eq.(5) can be transformed to

$$\begin{aligned} x(t+1) &= (A+B_1C_1)x(t) - B_1v(t) \\ &\quad +[B_1D_{12}+B_2]w(t) \quad (46) \\ p(t) &= C_1x(t) + D_{12}w(t) \quad (47) \\ z(t) &= [C_2+D_{21}C_1]x(t) - D_{21}v(t) \\ &\quad +[D_{21}D_{12}+D_{22}]w(t) \quad (48) \\ v(t) &= \Psi(p(t)) \quad (49) \end{aligned}$$

By the way, it can be shown that Eq.(34) implies

$$v^T\Gamma^{-1}[v-Rp] \le 0,\ \forall x \in \mathcal{E}(P),\ \forall w \in \mathcal{W} \quad (50)$$

where $\Gamma = \mathrm{diag}[\Gamma_1,\cdots,\Gamma_m] > 0,\ R = \mathrm{diag}[r_1,\cdots,r_m] > 0,\ r_i = 1 - 1/\eta_i$. By multiplying Eq.(33) from both sides by block-diag$[Q^{-1},\Gamma^{-1},I,I,I]$, and substituting $Q = P^{-1}$ for the inequality, and applying Schur complement to the $\{4,4\}$-block, and multiplying from the left by $[x^T,v^T,w^T]$ and from the right by $[x^T,v^T,w^T]^T$, and substituting Eq.(46), we obtain

$$\begin{aligned} &(1+\beta)x(t+1)^TPx(t+1) - \{x(t)^TPx(t) \\ &\quad + \frac{\beta}{\xi^2}w(t)^Tw(t)\} \le 2v(t)^T\Gamma^{-1}[v(t)-Rp(t)] \quad (51) \end{aligned}$$

Therefore, from Eqs.(50) and (51), we obtain

$$\begin{aligned} &(1+\beta)x(t+1)^TPx(t+1) \\ &\quad -\{x(t)^TPx(t) + \frac{\beta}{\xi^2}w(t)^Tw(t)\} \le 0 \\ &\quad\quad \forall x(t) \in \mathcal{E}(P), \forall w(t) \in \mathcal{W} \quad (52) \end{aligned}$$

Therefore, we can conclude that if $x(0) \in \mathcal{E}(P)$, then $x(t) \in \mathcal{E}(P),\ \forall t \ge 0$. It can also be verified that Eq.(35) implies

$$\begin{aligned} \frac{1}{\gamma^2}z(t)^Tz(t) &\le \frac{1}{2}\{x(t)^TPx(t) + \frac{1}{\xi^2}w(t)^Tw(t)\} \\ &\quad +2v(t)^T\Gamma^{-1}[v(t)-Rp(t)] \quad (53) \end{aligned}$$

Therefore, we can conclude that $\|z(t)\|_2 \le \gamma, \forall t \ge 0$.