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Positive μ-modification for Stable Adaptation in Dynamic Inversion Based Adaptive Control with Input Saturation

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***Abstract*— For a class of uncertain nonlinear dynamical systems in Brunovsky form a dynamic inversion (DI) based adaptive control framework is developed that provides stable adaptation in the presence of input constraints. The proposed design methodology, termed "positive μ-modification", protects the control law from actuator position saturation. Moreover, the design is Lyapunov based and ensures bounded tracking for initial conditions within the domain of attraction. An estimate of the latter is derived based on the input saturation magnitude and system parameters. Simulation of a benchmark example verifies the theoretical statements.**

I. INTRODUCTION

During the past decade control design in the presence of input saturation has attracted a vast amount of research effort (for chronological bibliography see [1]). This issue is especially challenging in adaptive systems, because continued adaptation during input saturation may easily lead to instability. In order to overcome the undesirable / destabilizing effects of control saturation during the adaptation process, an adaptive modification (proportional to control deficiency) to both the tracking error and the reference model dynamics was proposed by Monopoli in [2] but without any formal proof of stability. In the PCH method of Johnson and Calise a fixed gain adjustment (proportional to control deficiency) to the reference model was introduced [3]. Adaptive control with amplitude saturation for linear systems was addressed in [4]–[8]. Refs. [9], [10] consider both amplitude and rate saturation for nonlinear systems without explicit construction of the domain of attraction.

In this paper, we propose a DI based adaptive control framework for a class of nonlinear systems that yields bounded tracking in the presence of amplitude saturation. The novel design approach is termed "Positive μ-modification", or simply "μ-mod", and can be viewed as an extension of the results in [11]. The main features of the μ-mod design can be summarized: a) it enables to avoid control saturation overall through the choice of the design parameter μ, and b) numerically verifiable sufficient conditions are derived for its validity. The method is a direct extension of [11], which in turn was motivated by [5]. In fact, setting $\mu = 0$, or $\mu = \infty$, one can recover the results from [5]. We derive a lower bound on μ that completely protects control from the saturation phenomenon. The required value of μ depends only on the system parameters, initial errors and the maximum of the reference input. Using ideas from [5], we construct explicitly the domain of attraction. The benefit of the μ-modification based adaptive signal, in comparison to the method from [5], is in its ability to avoid the control saturation overall. This is essential when dealing with light structures in flight control applications, when operation at the limits can easily destabilize the structural modes. We prove that for stable systems the design ensures semiglobal bounded tracking, while for unstable systems we explicitly derive the domain of attraction. For further insights into the derived conditions, its application to scalar systems is analyzed separately.

This paper is organized as follows. Problem formulation is given in Section II. Section III defines the μ-mod based adaptive signal and discusses some of its properties. In Section IV, modified reference model dynamics is introduced. Stability properties of the μ-mod based adaptive control are analyzed in Section V. In Section VI, a simulation of a benchmark example is presented that verifies the theoretical statements and the benefits of the "μ-mod" based adaptive control design process. Conclusions, recommendations, and future research directions are given at the end of the paper.

II. PROBLEM FORMULATION

Consider the dynamics of an n-dimensional uncertain dynamical SISO system in Brunovsky form:

$$x^{(n)}(t) \quad = \quad bu(t) + d(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} \triangleq [x,\ \dot{x},\ \cdots,\ x^{(n-1)}]^T$, $b \geq b_0 > 0$ is unknown constant bounded away from zero, b_0 is known, $d(\boldsymbol{x})$ is an unknown continuous function. The control input $u \in \mathbb{R}$ is amplitude limited and is calculated using the following static actuator model:

$$\begin{aligned} u(t) &= u_{\max}\ \mathrm{sat}\left(\frac{u_c}{u_{\max}}\right) \\ &= \begin{cases} u_c(t), & |u_c(t)| \leq u_{\max} \\ u_{\max}\ \mathrm{sgn}\left(u_c(t)\right), & |u_c(t)| > u_{\max} \end{cases} \end{aligned} \tag{2}$$

Here $u_c(t)$ is the commanded control input, while $u_{\max} > 0$ defines the actuator amplitude saturation level.

Assumption 2.1: There exists $R > 0$ such that $\forall \boldsymbol{x} \in \mathcal{B}_R \triangleq \{\boldsymbol{x} : \|\boldsymbol{x}\| \leq R\}$:

$$b_0 u_{\max} \geq d_{\max} \triangleq \max_{\boldsymbol{x} \in \mathcal{B}_R} d(\boldsymbol{x}) \tag{3}$$

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Following [12], for all $\boldsymbol{x} \in \mathcal{B}_R \subset \mathbb{R}^n$ consider the following parameterization of the nonlinearity $d(\boldsymbol{x})$ using a linear combination of radial basis functions (RBF) $\phi_i(\boldsymbol{x})$:

$$d(\boldsymbol{x}) = \boldsymbol{W}^T\boldsymbol{\Phi}(\boldsymbol{x}) + \varepsilon(\boldsymbol{x}), \quad |\varepsilon(\boldsymbol{x})| < \varepsilon^*, \quad \|\boldsymbol{W}\| \leq W^* \quad (4)$$

where $\boldsymbol{W}$ is a vector of unknown constants, $\boldsymbol{\Phi}(\boldsymbol{x})$ is the vector of Gaussian basis functions $\phi_i(\boldsymbol{x})$, $|\phi_i(\boldsymbol{x})| \leq 1$, $\varepsilon(\boldsymbol{x})$ represents the uniformly bounded approximation error, while ε^*, W^* are known constants. Then $d_{\max} \leq \sum_{i=1}^{N} |W_i| + \varepsilon^*$, where N is the number of basis functions. Such approximation represents a linear in parameters neural network (NN) with RBFs in its inner layer. Let the *ideal* reference model be specified:

$$\begin{aligned} x_m^{(n)}(t) &= -a_n x_m^{(n-1)}(t) - \cdots - a_2\dot{x}_m(t) \\ &+ a_1(r(t) - x_m(t)), \end{aligned} \quad (5)$$

where $r(t) \in \mathbb{R}$ is a bounded reference input, while a_i are such that the corresponding characteristic polynomial is Hurwitz. Rewrite the system dynamics in (1) in the following way:

$$x^{(n)}(t) = bu_c(t) + b\Delta u(t) + d(\boldsymbol{x}) \quad (6)$$

where $\Delta u(t) = u(t) - u_c(t)$ is the control deficiency signal due to actuator constraint (2).

The control design problem, addressed in this paper, can be stated as follows:

Given reference model (5), define an adaptive control signal $u_c(t)$ and, if necessary, augment the input $r(t)$ to the reference model, so that the state $x(t)$ of the system (1) in the presence of input constraint (2) tracks the state $x_m(t)$ of the augmented reference model with bounded errors.

III. Positive μ-modification and Closed Loop System Dynamics

The main challenge in designing an adaptive controller for the system in (1), (2) is associated with the control deficiency signal $\Delta u(t) = u(t) - u_c(t)$ that appears in (6). Motivated by [5] and similar to [11], we propose yet another control design modification that protects the adaptive input signal from position saturation. To this end, choose a constant $0 < \delta < u_{\max}$ and define the conservative actuator constraint $u_{\max}^{\delta} = u_{\max} - \delta$. Then the control deficiency can be represented as:

$$\Delta u(t) = \Delta u_c(t) + \Delta_{\text{sat}}(t) \quad (7)$$

where

$$\begin{aligned} \Delta u_c(t) &= u_{\max}^{\delta}\text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) - u_c(t), \quad (8) \\ \Delta_{\text{sat}}(t) &= u_{\max}\text{sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u_{\max}^{\delta}\text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) \quad (9) \end{aligned}$$

DI based direct adaptive control with μ-modification is defined as:

$$\begin{aligned} u_c(t) &= u_{di}(t) + \mu\Delta u_c(t), \quad (10) \\ u_{di}(t) &= \frac{1}{\hat{b}(t)}\Big(- a_n x^{(n-1)}(t) - \cdots - a_2\dot{x}(t) \\ &+ a_1(r(t) - x(t)) - \hat{\boldsymbol{W}}(t)\boldsymbol{\Phi}(\boldsymbol{x}(t))\Big) \quad (11) \end{aligned}$$

where $u_{di}(t)$ denotes the DI based adaptive control defined in the absence of control saturation to track the *ideal* reference model (5), $\hat{\boldsymbol{W}}(t), \hat{b}(t)$ are adaptive parameters, provided $\hat{b}(t) \neq 0$, and μ is the design constant. The condition $\hat{b}(t) \neq 0$ will be ensured through the choice of adaptive laws by invoking the Projection operator [13].

Note that relation (10) defines the commanded control input $u_c(t)$ implicitly. Next we show that explicit solution of (10) can be found.

Lemma 3.1: If $\mu \geq 0$, then the solution to (10) is given by a *convex combination* of $u_{di}(t)$ and $u_{\max}^{\delta}\ \text{sat}\left(\frac{u_{di}(t)}{u_{\max}^{\delta}}\right)$ $\forall t > 0$:

$$\begin{aligned} u_c(t) &= \frac{1}{1+\mu}\left(u_{di}(t) + \mu u_{\max}^{\delta}\ \text{sat}\left(\frac{u_{di}(t)}{u_{\max}^{\delta}}\right)\right) \quad (12) \\ &= \begin{cases} u_{di}(t), & |u_{di}(t)| \leq u_{\max}^{\delta} \\ \frac{1}{1+\mu}\left(u_{di}(t) + \mu u_{\max}^{\delta}\right), & u_{di}(t) > u_{\max}^{\delta} \\ \frac{1}{1+\mu}\left(u_{di}(t) - \mu u_{\max}^{\delta}\right), & u_{di}(t) < -u_{\max}^{\delta} \end{cases} \end{aligned}$$

A proof can be found in [11].

Remark 3.1: The solution given by (12) is valid also for any $\mu \neq -1$, but in that case the *convexity* condition is violated. The significance of the latter will be apparent during the stability proof. Moreover, for $\delta = 0$ setting $\mu = 0$ one can recover the adaptive architecture of [5] when applied to an MRAC scheme for linear systems. On the other hand, if $u_{di}(t)$ is uniformly bounded, then as μ tends to infinity, $\lim u_c(t) = u_{\max}^{\delta}\text{sat}\left(\frac{u_{di}(t)}{u_{\max}^{\delta}}\right)$. Consequently, setting $\delta = 0$ and $\mu = \infty$ results in $u_c(t) = u_{\max}\text{sat}\left(\frac{u_{di}(t)}{u_{\max}}\right)$. Again, the latter yields exactly the same closed-loop dynamics as does the linear in parameters adaptive signal $u_{di}(t)$ in the MRAC scheme of [5].

Remark 3.2: Solving (10) for $\Delta u_c(t)$ and substituting $u_c(t)$ from (12), one obtains:

$$\Delta u_c(t) = \frac{1}{\mu}\left(u_c(t) - u_{di}(t)\right) = \frac{1}{1+\mu}\Delta u_{di}^{\delta}(t) \quad (13)$$

where $\Delta u_{di}^{\delta}(t) \triangleq u_{\max}^{\delta}\ \text{sat}\left(\frac{u_{di}(t)}{u_{\max}^{\delta}}\right) - u_{di}(t)$. Consequently, if $u_{di}(t)$ is bounded, then the control deficiency $\Delta u_c(t)$ is inversely proportional to μ: $\Delta u_c(t) = O(1/\mu)$.

Lemma 3.2: The following inequality is true for all $t > 0$:

$$u_c(t)\Delta u_c(t) \leq 0 \quad (14)$$

A proof can be found in [11].

Rewrite the system dynamics in (1) in the following way:

$$x^{(n)}(t) = (\hat{b}(t) - \Delta b(t))u(t) + \boldsymbol{W}^T\boldsymbol{\Phi}(\boldsymbol{x}(t)) + \varepsilon(\boldsymbol{x}) \quad (15)$$

where $\Delta b(t) = \hat{b}(t) - b$. Recalling that $u(t) = u_c(t) + \Delta u(t)$, substituting (7) and (10) into (15), yields the following closed-loop system dynamics:

$$\begin{aligned} x^{(n)}(t) &= -a_n x^{(n-1)}(t) - \cdots - a_2 \dot{x}(t) \\ &+ a_1(r(t) - x(t)) - \Delta \boldsymbol{W}(t)\boldsymbol{\Phi}(\boldsymbol{x}(t)) \\ &- \Delta b(t)u(t) + \hat{b}\Delta u_{di}(t) + \varepsilon(\boldsymbol{x}) \end{aligned} \quad (16)$$

where $\Delta \boldsymbol{W}(t) = \hat{\boldsymbol{W}}(t) - \boldsymbol{W}$, and

$$\Delta u_{di}(t) \triangleq u_{\max} \mathrm{sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u_{di}(t) \quad (17)$$

IV. Adaptive Reference Model and Error Dynamics

The system dynamics in (16) leads to consideration of the following *adaptive* reference model dynamics:

$$\begin{aligned} \dot{x}_m^{(n)}(t) &= -a_n x_m^{(n-1)}(t) - \cdots - a_2 \dot{x}_m(t) \\ &+ a_1(r(t) - x_m(t)) + \hat{b}\Delta u_{di}(t) \end{aligned} \quad (18)$$

The tracking error dynamics for the signal $e(t) = x(t) - x_m(t)$ can be written as:

$$\begin{aligned} &\dot{e}^{(n)}(t) = -a_n e^{(n-1)}(t) - \cdots - a_2 \dot{e}(t) - a_1 e(t) \\ &-\Delta \boldsymbol{W}(t)\boldsymbol{\Phi}(\boldsymbol{x}(t)) - \Delta b(t)u(t) + \varepsilon(\boldsymbol{x}) \end{aligned}$$

Introduce vector $\boldsymbol{e}(t) = [e(t), \ \cdots, \ e^{(n-1)}(t)]^T$. Then

$$\dot{\boldsymbol{e}}(t) = A\boldsymbol{e}(t) - \boldsymbol{b}(\Delta \boldsymbol{W}(t)\boldsymbol{\Phi}(\boldsymbol{x}(t)) - \varepsilon(\boldsymbol{x}) + \Delta b(t)u(t)) \quad (19)$$

where Hurwitz matrix A and vector $\boldsymbol{b}$ are

$$A = \begin{bmatrix} 0 & 1 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0\cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (20)$$

Consider the following adaptive laws

$$\begin{aligned} \dot{\hat{b}}(t) &= \gamma_b \mathrm{Proj}\left(\hat{b}, u(t)\boldsymbol{e}^T(t)P\boldsymbol{b}\right) \\ \dot{\hat{\boldsymbol{W}}}(t) &= \Gamma_W \mathrm{Proj}\left(\hat{\boldsymbol{W}}(t), \boldsymbol{\Phi}(\boldsymbol{x}(t))\boldsymbol{e}^T(t)P\boldsymbol{b}\right) \end{aligned} \quad (21)$$

where $\Gamma_W = \Gamma_W^T > 0$, $\gamma_b > 0$ are the adaptation gains, $\mathrm{Proj}(\cdot,\cdot)$ defines the projection operator [13], and $P = P^T > 0$ solves the Lyapunov equation for some positive definite $Q > 0$

$$A^T P + PA = -Q\,. \quad (22)$$

By definition, $\mathrm{Proj}(\cdot,\cdot)$ ensures boundedness of all parameters, and can be designed to provide $\hat{b}(t) > b_0 > 0\ \forall t > 0$ [13]. Consider the following candidate Lyapunov function

$$\begin{aligned} &V(\boldsymbol{e}(t), \Delta b(t), \Delta \boldsymbol{W}(t)) = \boldsymbol{e}^T(t)P\boldsymbol{e}(t) \\ &+\Delta \boldsymbol{W}^T(t)\Gamma_W^{-1}\Delta \boldsymbol{W}(t) + \gamma_b^{-1}\Delta b^2(t) \end{aligned} \quad (23)$$

Using the properties of the $\mathrm{Proj}(\cdot,\cdot)$ operator, the time derivative of the candidate Lyapunov function in (23) along the system trajectories (19), (21) can be upper bounded:

$$\begin{aligned} \dot{V}(t) &\leq -\boldsymbol{e}^T(t)Q\boldsymbol{e}(t) + 2\boldsymbol{e}^T P\boldsymbol{b}\varepsilon(\boldsymbol{x}) \\ &\leq -\lambda_{\min}(Q)\|\boldsymbol{e}\|^2 + 2\|\boldsymbol{e}\|\|P\boldsymbol{b}\|\varepsilon^* \\ &\leq -\|\boldsymbol{e}\|\left[\lambda_{\min}(Q)\|\boldsymbol{e}\| - 2\|P\boldsymbol{b}\|\varepsilon^*\right] \end{aligned} \quad (24)$$

Hence

$$\dot{V}(t) \leq 0 \qquad \text{if} \qquad \|\boldsymbol{e}\| \geq \frac{2\|P\boldsymbol{b}\|\varepsilon^*}{\lambda_{\min}(Q)} \quad (25)$$

Thus, if the initial condition $\boldsymbol{e}(0) \in \Omega_{\mathcal{B}_R}$, where $\Omega_{\mathcal{B}_R}$ is a level set of $V(\boldsymbol{e}(t), \Delta b(t), \Delta \boldsymbol{W}(t))$, such that for all $\boldsymbol{e} \in \Omega_{\mathcal{B}_R}$ one has $\boldsymbol{x} \in \mathcal{B}_R$, then the tracking error $\boldsymbol{e}(t)$ is ultimately bounded. At the same time, boundedness of adaptive parameters is ensured by the $\mathrm{Proj}(\cdot,\cdot)$ operator. Consequently, there exist $\Delta W^{\max}$, $\Delta b^{\max}$, such that $\|\Delta \boldsymbol{W}\| < \Delta W^{\max}$, $|\Delta b| < \Delta b^{\max}$, $\forall t > 0$, and $\exists\ \alpha$, such that $\Delta W^{\max} + W^* = \alpha \Delta b^{\max}$. However, due to the adaptive modification of the reference model, one still needs to show that both the system state and the state of the adaptive reference model are ultimately bounded. In the next section, we explicitly construct domains for the system initial conditions and initial parameter errors such that starting in these domains the system states remain bounded under the control action (12) with adaptive laws (21). Consequently, it will follow from (25) that all the signals remain bounded.

V. Stability Analysis

For the statement of our main result introduce the following notations: $\kappa = \frac{2|bu_{\max} - d_{\max}|}{\left|\lambda_{\min}(Q) - 2\|P\boldsymbol{b}\|\|k_x^*\|\right|}$, $\rho = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$ where $k_x^* = [-a_1 \ -a_2 \ \cdots \ -a_n]^T$.

Theorem 5.1: Let Assumption 2.1 hold with R satisfying

$$R > \kappa \|P\boldsymbol{b}\| \quad (26)$$

Further, for b in (1), $u_{\max}$ in (2) and P and Q in (22), assume that the maximum amplitude of the reference signal $r_{\max}$ is chosen such that:

$$r_{\max} < \frac{b_0 \lambda_{\min}(Q)\kappa}{2\rho a_1 b} - d_{\max}\,, \quad (27)$$

and for arbitrary $\delta > 0$ the design parameter μ is selected to satisfy the lower bound:

$$\begin{aligned} \mu &> \frac{\|k_x^*\|\|P\boldsymbol{b}\|\kappa \ + a_1 r_{\max} + (\Delta W^{\max} + W^*)N}{b_0 \delta} \\ &+ \frac{u_{\max}}{\delta} - 2 \end{aligned} \quad (28)$$

If the system initial condition and the initial value of the candidate Lyapunov function in (23) satisfy:

$$\boldsymbol{x}^T(0)P\boldsymbol{x}(0) < \lambda_{\min}(P)\kappa^2\|P\boldsymbol{b}\|^2 \quad (29)$$

$$\sqrt{V(0)} < \sqrt{\frac{1}{\gamma_b}}\frac{\lambda_{\min}(Q) - 2\rho\frac{a_1 b}{b_0}\frac{r_{\max}+d_{\max}}{\kappa}}{2\frac{\|P\boldsymbol{b}\|}{b_0}(\|k_x^*\| + b\alpha N)} \quad (30)$$

then

- the adaptive system in (19), (21) has bounded solutions $\forall r(t),\ |r(t)| \leq r_{\max}$,
- the tracking error $\boldsymbol{e}(t)$ is ultimately bounded, while

$$\boldsymbol{x}^T(t)P\boldsymbol{x}(t) < \lambda_{\min}(P)\kappa^2\|P\boldsymbol{b}\|^2\,,\ \forall t > 0\,,$$

- $|u_c(t)| \leq u_{\max}$, i.e. the position saturation of the commanded control signal $u_c(t)$ is overly prevented for all $t > 0$.

Proof. If $\Delta u(t) = 0$, then the adaptive reference model dynamics in (18) reduces to the one in (5), leading to the error dynamics in (19). Since (5) defines a stable reference model, then $x_m(t)$ is bounded, which together with (25), leads to boundedness of the system state $x(t)$.

If $\Delta u(t) \neq 0$, then $u(t) = u_{\max}\mathrm{sgn}(u_c(t))$. Write the system dynamics in (1) in the following way:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) - \boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t) + \boldsymbol{b}(bu_{\max}\mathrm{sgn}(u_c(t)) + d(\boldsymbol{x}))$$

Consider the following Lyapunov function candidate $W(\boldsymbol{x}(t)) = \boldsymbol{x}^T(t)P\boldsymbol{x}(t)$. Then

$$\begin{aligned}\dot{W}(\boldsymbol{x}(t)) &= -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &+2bu_{\max}|\boldsymbol{x}^T(t)P\boldsymbol{b}|\mathrm{sgn}(u_c(t))\mathrm{sgn}(\boldsymbol{x}^T(t)P\boldsymbol{b})\\ &+2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\end{aligned} \quad (31)$$

Consider two possibilities:

1) $\mathrm{sgn}(u_c(t)) = -\mathrm{sgn}(\boldsymbol{x}^T(t)P\boldsymbol{b})$.
2) $\mathrm{sgn}(u_c(t)) = \mathrm{sgn}(\boldsymbol{x}^T(t)P\boldsymbol{b})$.

If $\mathrm{sgn}(u_c(t)) = -\mathrm{sgn}(\boldsymbol{x}^T(t)P\boldsymbol{b})$, then it follows from (31) that

$$\begin{aligned}\dot{W} &= -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &- 2bu_{\max}|\boldsymbol{x}^T(t)P\boldsymbol{b}| + 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\\ &\leq -\lambda_{\min}(Q)\|\boldsymbol{x}(t)\|^2\\ &+ 2|\boldsymbol{x}^T(t)P\boldsymbol{b}|\left(d_{\max} + \|k_x^*\|\|\boldsymbol{x}(t)\| - bu_{\max}\right)\end{aligned} \quad (32)$$

If $\|k_x^*\|\|\boldsymbol{x}(t)\| + d_{\max} - bu_{\max} < 0$, which is equivalent to $\|\boldsymbol{x}(t)\| < |bu_{\max} - d_{\max}|/\|k_x^*\|$, we have $\dot{W} < 0$. If $\|k_x^*\|\|\boldsymbol{x}(t)\| + d_{\max} - bu_{\max} > 0$, which is equivalent to $\|\boldsymbol{x}(t)\| > |bu_{\max} - d_{\max}|/\|k_x^*\|$, the expression in (32) implies:

$$\begin{aligned}\dot{W} &\leq -\lambda_{\min}(Q)\|\boldsymbol{x}(t)\|^2 + 2\|P\boldsymbol{b}\|\|k_x^*\|\|\boldsymbol{x}(t)\|^2\\ &- 2\left|bu_{\max} - d_{\max}\right|\|\boldsymbol{x}(t)\|\|P\boldsymbol{b}\|\\ &\leq \left|\lambda_{\min}(Q) - 2\|P\boldsymbol{b}\|\|k_x^*\|\right|\|\boldsymbol{x}(t)\|^2\\ &- 2\left|bu_{\max} - d_{\max}\right|\|\boldsymbol{x}(t)\|\|P\boldsymbol{b}\|\end{aligned} \quad (33)$$

Therefore $\dot{W}(\boldsymbol{x}(t)) < 0$, if

$$\boldsymbol{x} \in \Omega_1 \triangleq \left\{\boldsymbol{x} \mid \|\boldsymbol{x}\| < \frac{2\left|bu_{\max} - d_{\max}\right|\|P\boldsymbol{b}\|}{\left|\lambda_{\min}(Q) - 2\|P\boldsymbol{b}\|\|k_x^*\|\right|}\right\} \quad (34)$$

Consider the largest set $\mathcal{B}_1$, enclosed in Ω_1, whose boundary forms a level set of the function $W(x(t))$: $\mathcal{B}_1 = \left\{\boldsymbol{x} \mid W(\boldsymbol{x}) \leq \lambda_{\min}(P)\kappa^2\|P\boldsymbol{b}\|^2\right\}$. It is obvious that for all initial conditions of $\boldsymbol{x}(t)$ from the set $\mathcal{B}_1$ we have $\dot{W}(\boldsymbol{x}(t)) < 0$, implying that the system states remain bounded.

In the second case, i.e. when $\mathrm{sgn}(u_c(t)) = \mathrm{sgn}(\boldsymbol{x}^T(t)P\boldsymbol{b})$, it follows from (31) that

$$\begin{aligned}\dot{W} &= -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &+ 2bu_{\max}|\boldsymbol{x}^T(t)P\boldsymbol{b}| + 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\\ &\leq -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &+ 2b|u_c(t)||\boldsymbol{x}^T(t)P\boldsymbol{b}| + 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\\ &= -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &+ 2bu_c(t)\boldsymbol{x}^T(t)P\boldsymbol{b} + 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\end{aligned}$$

Recall that $\mathrm{sgn}(u_c(t)) = -\mathrm{sgn}(\Delta u_c(t))$, and that $\mu > 0$, and substitute $u_c(t)$ from (10):

$$\begin{aligned}\dot{W} &\leq -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) - 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t)\\ &+ 2b(u_{di}(t) + \mu\Delta u_c(t))\boldsymbol{x}^T(t)P\boldsymbol{b}\\ &+ 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x}) \leq -\boldsymbol{x}^T(t)Q\boldsymbol{x}(t)\\ &- 2\boldsymbol{x}^T(t)P\boldsymbol{b}(k_x^*)^T\boldsymbol{x}(t) + 2\frac{b}{\hat{b}(t)}((k_x^*)^T\boldsymbol{x}(t) + a_1r(t)\\ &- \hat{\boldsymbol{W}}(t)\boldsymbol{\Phi}(\boldsymbol{x}(t)))\boldsymbol{x}^T(t)P\boldsymbol{b} + 2\boldsymbol{x}^T(t)P\boldsymbol{b}d(\boldsymbol{x})\end{aligned}$$

Further, notice that by definition, $\hat{\boldsymbol{W}} = \Delta\boldsymbol{W} + \boldsymbol{W}$, $|\phi_i(\boldsymbol{x})| \leq 1$, and that $\mathrm{Proj}(\cdot,\cdot)$ ensures that $\hat{b}(t) > b_0$. Therefore

$$\begin{aligned}\dot{W} &\leq -\Big(\lambda_{\min}(Q) - 2\frac{\Delta b^{\max}}{b_0}\|k_x^*\|\|P\boldsymbol{b}\|\\ &- 2\frac{b}{b_0}(\Delta W^{\max} + W^*)N\|P\boldsymbol{b}\|\Big)\|\boldsymbol{x}(t)\|^2\\ &+ 2\frac{a_1b}{b_0}\|P\boldsymbol{b}\|\|\boldsymbol{x}(t)\|(r_{\max} + d_{\max})\end{aligned} \quad (35)$$

Notice that since $V(\boldsymbol{e}(t), \Delta b(t), \Delta\boldsymbol{W}(t))$ is radially unbounded, and its derivative is negative $\dot{V}(t) < 0$, then the maximal values of all errors, including $\Delta b^{\max}, \Delta W^{\max}$, do not exceed the level set value of the Lyapunov function $V = V_0 = V(0)$. Therefore the assumed inequality (30) implies that

$$\Delta b^{\max} < \frac{\lambda_{\min}(Q) - 2\rho\frac{a_1b}{b_0}\frac{r_{\max}+d_{\max}}{\kappa}}{2\frac{\|P\boldsymbol{b}\|}{b_0}(\|k_x^*\| + b\alpha N)} \quad (36)$$

This in turn guarantees that $\beta \triangleq \lambda_{\min}(Q) - 2\frac{\Delta b^{\max}}{b_0}\|k_x^*\|\|P\boldsymbol{b}\| - 2\frac{b}{b_0}(\Delta W^{\max} + W^*)N\|P\boldsymbol{b}\| > 0$. Consequently, it follows from (35) that $\dot{W}(x(t)) < 0$, if

$$\|\boldsymbol{x}\| > \frac{2\frac{a_1b}{b_0}(r_{\max} + d_{\max})}{\beta}\|P\boldsymbol{b}\|$$

Define the ball:

$$\Omega_2 = \left\{\boldsymbol{x} \mid \|\boldsymbol{x}\| = \frac{2\frac{a_1b}{b_0}(r_{\max} + d_{\max})}{\beta}\|P\boldsymbol{b}\|\right\}$$

and the smallest set $\mathcal{B}_2$ that encloses Ω_2, the boundary of which is a level set of the Lyapunov function $W(x(t))$:

$$\mathcal{B}_2 = \left\{ \boldsymbol{x} \mid W \le \lambda_{\max}(P) \left[\frac{2\frac{a_1 b}{b_0}(r_{\max} + d_{\max})}{\beta} \|P\boldsymbol{b}\| \right]^2 \right\}$$

By rearranging the terms in (36), it follows that

$$2\sqrt{\lambda_{\max}(P)} \frac{\frac{a_1 b}{b_0}(r_{\max} + d_{\max})}{\beta} < \sqrt{\lambda_{\min}(P)}\ \kappa$$

and consequently $\mathcal{B}_2 \subset \mathcal{B}_1$, implying that the annulus region $\mathcal{B}_1 \setminus \mathcal{B}_2 \neq \emptyset$. Thus, our analysis of the closed-loop system dynamics reveals that when $\Delta u(t) \neq 0$, there always exists a *non-empty* annulus region

$$\begin{aligned} & 2\sqrt{\lambda_{\max}(P)} \frac{\frac{a_1 b}{b_0}(r_{\max} + d_{\max})}{\beta} \|P\boldsymbol{b}\| \\ < \quad & \|x\| < \sqrt{\lambda_{\min}(P)} \kappa \|P\boldsymbol{b}\| \end{aligned} \tag{37}$$

such that $\dot{W}(x(t)) < 0$ holds $\forall x$ satisfying (37). In other words, boundedness of all the signals is guaranteed as long as the system initial conditions satisfy (29) and the initial parameter errors comply with (30).

It remains only to show that the control signal will never incur saturation. To this end, notice that from (13) it follows that $\Delta u_c(t)$ can be upper bounded

$$\begin{aligned} |\Delta u_c(t)| &\le \frac{|\Delta u_{di}^{\delta}(t)|}{1+\mu} \\ &\le \frac{u_{\max}^{\delta} + \frac{1}{b_0}(\|k_x^*\|\|\boldsymbol{x}\| + a_1 r_{\max} + (\Delta W^{\max} + W^*)N)}{1+\mu} \end{aligned}$$

By definition $\Delta u_c(t) = u_{\max}^{\delta} \text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) - u_c(t)$. Hence

$$|\Delta u_c(t)| \ge |u_c(t)| - \left| u_{\max}^{\delta} \text{sat}\left(\frac{u_c(t)}{u_{\max}^{\delta}}\right) \right| \tag{38}$$

and consequently $|u_c(t)| \le u_{\max}^{\delta} + \frac{C}{1+\mu}$, where $C = u_{\max}^{\delta} + \frac{1}{b_0}(\|k_x^*\|\kappa\|P\boldsymbol{b}\| + a_1 r_{\max} + (\Delta W^{\max} + W^*)N)$. For a given $\delta > 0$, if one chooses $\mu > 0$ to satisfy

$$\frac{C}{1+\mu} < \delta \tag{39}$$

then $|u_c(t)| < u_{\max}$. Recalling that $u_{\max}^{\delta} = u_{\max} - \delta$, and substituting this into (39), one arrives at (28). The proof is complete.

Remark 5.1: Using explicit definition of $u_c(t)$ in (12), one can compute its time derivative for $|u_c(t)| > u_{\max}^{\delta}$:

$$\dot{u}_c(t)\Big|_{|u_c(t)| > u_{\max}^{\delta}} = \frac{\dot{u}_{di}(t)}{1+\mu} \tag{40}$$

Since $\dot{u}_c(t)$ is of order $1/(1+\mu)$, then the actuator limits can be enforced through the choice of μ, by controlling the time derivative $\dot{u}_c(t)$.

Remark 5.2: Notice that if one chooses $\delta \le \frac{u_{\max}}{2}$, then from (2) one can deduce a less conservative lower bound for μ than the one in (28). Indeed, it is easy to verify that if

$$\mu \ge \frac{\kappa\|k_x^*\|\|P\boldsymbol{b}\| + a_1 r_{\max} + (\Delta W^{\max} + W^*)N}{b_0 \delta} - \frac{u_{\max}}{\delta}$$

then $0 < |u_c(t)| \le u_{\max}$ for all $t > 0$.

Remark 5.3: Inequality in (27) ensures that the numerator in (30) is positive.

Remark 5.4: Theorem 5.1 implies that if the initial conditions of the state and parameter errors lie within certain bounds, then the adaptive system will have bounded solutions. The local nature of the result is due to the static actuator model constraints (2) imposed on the control input. For open loop stable systems, if $\mathcal{B}_R$ is an arbitrarily large ball within $\mathbb{R}^n$, the results are semiglobal.

Remark 5.5: If $\varepsilon(\boldsymbol{x}) = 0$, then similar to [11], one can show that $e(t) \to 0$ as $t \to \infty$.

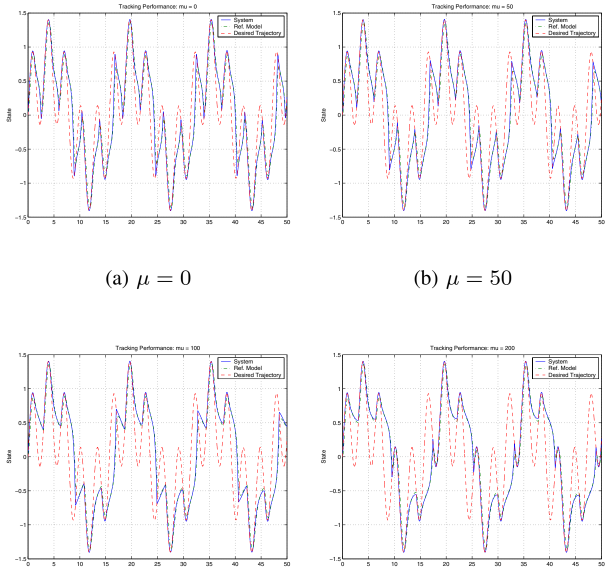
Remark 5.6: The condition in (30) can be viewed as an upper bound for α, which limits the choice of the adaptation gains Γ_W and γ_b.

VI. Simulations

In this section, the proposed μ-mod based design methodology is demonstrated using the first order system

$$\begin{aligned} \dot{x} &= 0.2x + 2u + 0.01x^3 - \exp(-10(x+0.5)^2) \\ &- \exp(-10(x-0.5)^2) + 0.5\sin(2x) \end{aligned}$$

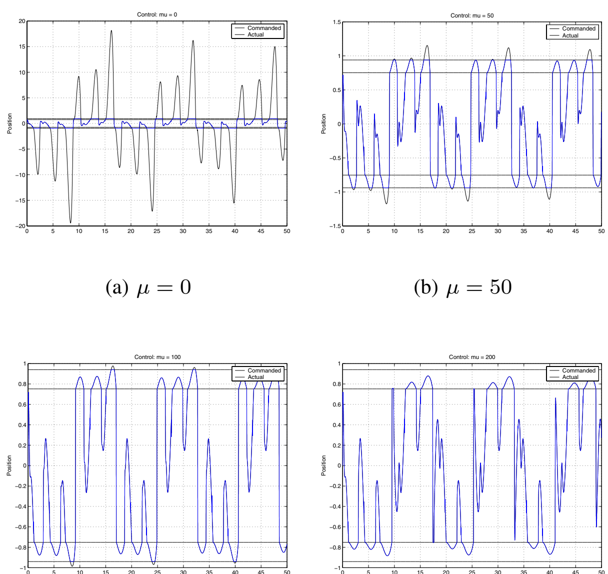
subject to the following actuator constraint $u_{\max} = 0.94$. A positive constant δ is set to 20% of the actuator position limit, that is: $\delta = 0.2u_{\max}$. The reference model without μ-modification is given as $\dot{x}_m = -6x_m + 6r$ along with the reference input $r = 0.7(\sin(2t) + \sin(0.4t))$. Both, the system and the reference model are initialized at zero. The adaptation rates are selected as $\gamma_a = 1, \gamma_b = 1$. Figs. 1(a)-1(d) demonstrate the closed-loop tracking performance for various values of μ. As expected, the plots indicate that large values of μ result in large changes to the reference model dynamics. At the same time, Figs. 2(a)-2(d) demonstrate that by choosing μ large enough the control deficiency is reduced, thus completely avoiding control saturation phenomenon. Consequently, the design constant μ can be viewed as a tuning "knob" that allows for a tradeoff between the adaptive changes to the reference model and a protection against saturating actuator position which is required for tracking the model.



(a) $\mu = 0$ (b) $\mu = 50$

(c) $\mu = 100$ (d) $\mu = 200$

Fig. 1. Tracking performance for various values of μ.



(a) $\mu = 0$ (b) $\mu = 50$

(c) $\mu = 100$ (d) $\mu = 200$

Fig. 2. Comparison of the control deficiency for various values of μ.

VII. Conclusions

A dynamic inversion based direct model reference adaptive control design methodology is developed for uncertain nonlinear systems in the presence of input constraints. The design is Lyapunov based and ensures bounded tracking for all initial conditions within the domain of attraction. The latter depends upon the system parameters and amplitude saturation level. The output feedback results can be obtained following the lines of [5]. Extension to nonlinear systems can be done following the lines of [9] for formulating the error dynamics. Current efforts are directed towards extension of the results to multivariable systems with magnitude and rate constrained control input.

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