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Stability of Unfalsified Adaptive Control Using Multiple Controllers

R. Wang and M. G. Safonov

Abstract— **This paper presents results on stability of adaptive control systems designed using unfalsified control methods. It is proved that stability can be guaranteed whenever the adaptive stabilization problem is feasible, i.e., whenever there is a stabilizing controller in its finite set of candidate controllers. Simulation results demonstrate the efficacy of the method.**

Index Terms— **Unfalsified adaptive control, Adaptive control, Safe adaptive control, Switching control, Unfalsification, Stability**

I. INTRODUCTION

Generally, an adaptive control system is defined by three essential elements: goals, information and a set of candidate controllers. An adaptive control algorithm for such systems is the scheme to select/choose/order/tune/switch among candidate controllers by using real-time and prior information to achieve specified goals. Among all these goals, to achieve stability is the minimum goal of an adaptive control system. Whenever an active controller can not stabilize the system, a 'good' algorithm should be able to 'know' the situation, abandon the active controller, and switch to a stabilizing controller if there is one in the candidate controller set. If there is a stabilizing controller for the system in the candidate controller set at any time, i.e., if stability is feasible, a good adaptive control algorithm should be able to stabilize the system without further assumptions on the plant.

Unfortunately, good adaptive control algorithms are rare. With few exceptions, modern adaptive methods have continued to rely on additional plant modeling assumptions, compromising the robust performance properties that adaptive control is intended to enhance. See, for example, [1] and [2] for overviews. Recent efforts have been made to relax plant assumptions. A pioneering breakthrough in this direction was done by Martensson[3] in mid 80s. Martensson showed that plant model assumptions are not required, only feasibility. Specifically, he showed how to achieve adaptive goals without plant assumptions by simply using a pre-routed switching among the candidate controllers until one controller was found which could achieve the control objective. Other pre-routed based switching scheme can be found in [4], [5], [6], [7] and [8]. However, pre-routed switching schemes generally give poor transient response and switching to a stabilizing controller takes long time, specifically when the number of candidate controllers is large.

To alleviate this problem, Zhivoglyadov *et.al.* ([9], [10], [11]) proposed a *localization approach* where a fast algorithm was introduced to prune 'bad' candidate controllers, and therefore a 'good' set of candidate controller was quickly localized.

Another approach to speed up the switching process was proposed by Morse([12], [13]) to improve the transient response. The main idea of this approach is to apply an 'optimal' candidate controller based on certain on-line plant model estimation rather than directly sequentially eliminating controllers. One of the problems with it is that the notion of 'optimal' controller may be very ambiguous. To clarify this, a new monitoring signal using ν-gap metric, δ_ν, was introduced in [14]. The plant model with the smallest monitoring signal is viewed as the 'closest' plant model to the true plant. Thus, its corresponding controller is chosen as the 'optimal' controller. Unfortunately, the ν-gap metric may be unsuitable for evaluating the closeness of systems having uncertain poles and zeros on or near the imaginary axis[15]. Even when this metric is appropriate for closeness evaluation of some systems, if the assumption that there is at least one plant model in the candidate model set sufficiently close to the actual plant, is violated, such two-step model validation approach may still fail to switch to a stabilizing controller when there is one in the candidate controller set([16], [17], [18]). The reason is that this stabilizing controller may correspond to the relatively 'further' plant model and thus will never be chosen as the 'optimal' controller. A related early result was the hysteresis switching lemma of Morse, Mayne and Goodwin [7], which established that a cost-function approach can be used to 'prove' adaptive system stability and convergence subject to standard assumptions on the plant. But, unlike Martensson [3], these papers failed to recognize the possibility of achieving adaptive stabilization subject only to feasibility of the adaptive control problem, without further plant modeling assumptions.

One newly proposed switching adaptive control approach, named *unfalsified adaptive control*, can do direct validation of candidate controllers very fast without making any assumptions on the plant, by using experiment data only, and thus avoid the problems mentioned above. A useful notion in this method, *fictitious reference signal*[16], which facilitates validation of controllers from open-loop

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experimental data, or even from closed-loop data acquired while another controller is in the feedback loop. This leads to fast validation and relatively better transient response compared with Martensson's work[3] while avoiding the robustness pitfalls associated with other recent methods that rely on unnecessary plant assumptions.

In a recent paper[17], stability and convergence of adaptive control systems is re-examined from the perspective of the hysteresis switching lemma of Morse, Mayne and Goodwin[7] in order to address the above problems in [14]. More explicitly, given plant data from time 0 to ∞, assume that the candidate controller set contains at least one robustly stabilizing and robustly performing controller. It is proved in [17] that if the hysteresis algorithm of Morse, Mayne and Goodwin[7] is employed with a certain type of data-driven cost function which

1) is monotonically non-decreasing in time and uniformly bounded above for all conceivable plant data, and
2) has a plant-independent property called 'cost-detectability',

then unfalsified adaptive algorithms can consistently and reliably identify controllers that quickly and reliably achieve unfalsified stability and performance specifications based on cost-minimization. Thus, one important question arises: Is the unfalsified stability of an adaptive control system equivalent to its stability?

In this paper, the above question is answered. By establishing the relationship between fictitious reference signals[16] and true reference signals, the stability of an unfalsified adaptive control system with finitely many switches is theoretically proved as long as there is a stabilizing controller in the candidate controller set.

The paper is organized as follows. In Section II, the formulation of a safe adaptive stabilization problem is given. In Section III, an unfalsified adaptive control algorithm is produced to solve this problem. Stability of this algorithm is proved in Section IV and simulation result is in Section V. Conclusion follows in Section VI.

II. PROBLEM FORMULATION

A. General Definitions

Definition 1: (L_{2e} signals). We define the truncation over a time interval (a, b) as

$$x_{(a,b)}(t) = \begin{cases} x(t), & \text{if } t \in [a, b] \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

and x_τ denotes the time truncation over the interval $(0, \tau)$

$$x_\tau(t) = \begin{cases} x(t), & \text{if } t \in [0, \tau] \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

We say $x \in L_{2e}$ if $\| x \|$ exists where

$$\| x \| = \sqrt{\int_0^\infty \|x(t)\|^2 \, dt} \quad (3)$$

$\square$

Definition 2: (Stable[19], [20], Finite-gain stable). A system, $\Sigma : L_{2e} \to L_{2e}$, is said to be *stable* if for any $v \in L_{2e}, v \neq 0$,

$$\sup_{\tau \in (0,\infty)} \frac{\| (\Sigma v)_\tau \|}{\| v_\tau \|} < \infty; \quad (4)$$

Otherwise, it is said to be *unstable*. Additionally, if

$$\sup_{v \in L_{2e},\, \tau \in R_+} \frac{\| (\Sigma v)_\tau \|}{\| v_\tau \|} < \infty, \quad (5)$$

it is said to be *finite-gain stable*; Otherwise, it is said to be *finite-gain unstable*. $\square$

Definition 3: (Incrementally stable). A system, $\Sigma : L_{2e} \to L_{2e}$, is said to be *incrementally stable* if for any $v, w \in L_{2e}, v - w \neq 0$,

$$\sup_{\tau \in (0,\infty)} \frac{\| (\Sigma v - \Sigma w)_\tau \|}{\| (v - w)_\tau \|} < \infty; \quad (6)$$

Otherwise, it is said to be *not-incrementally-stable*. Additionally, if

$$\sup_{v,w \in L_{2e}, \tau \in R_+} \frac{\| (\Sigma v - \Sigma w)_\tau \|}{\| (v - w)_\tau \|} < \infty, \quad (7)$$

it is said to be *finite-gain incrementally stable*; Otherwise, it is said to be *finite-gain incrementally unstable*. $\square$

Definition 4: (Minimum-phase system). A system, $\Sigma : L_{2e} \to L_{2e}$, is called a *minimum-phase system* if its inverse system, $\Sigma^{-1} : L_{2e} \to L_{2e}$, exists and this inverse system is causal and incrementally stable. $\square$

For example, for a SISO LTI system, minimum phase means that it has no zeros in RHP and is bi-proper.

Definition 5: (Stabilizing controller). A *stabilizing controller* is a controller with which a system is stable; otherwise, the controller is called a *destabilizing controller*. $\square$

B. safe adaptive stabilization Problem

An *adaptive control system* is a control system with an adaptive controller. An *adaptive controller* is a controller with adjustable parameters/structures and a mechanism for adjusting the parameters/structures[21]. A set composed by time-invariant controllers with any of these possible parameters/structures is called *candidate controller set*. It is necessary to adapt because plant is unknown to us. A plant is a *completely unknown plant*, if we have no prior knowledge about what is the structure/parameters of the plant and what is the relationship between the plant and its environment, which is composed by candidate controllers, noises and disturbances. A block diagram of an adaptive control system is shown in Fig.1.

Generally, we say adaptive stabilization of an adaptive control system is *feasible* if a stabilizing controller is available in the candidate controller set, even though which controllers are stabilizing is not known *à priori*.

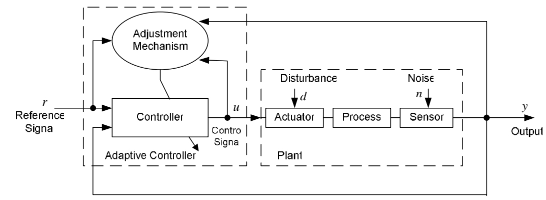

Fig. 1. Adaptive control system

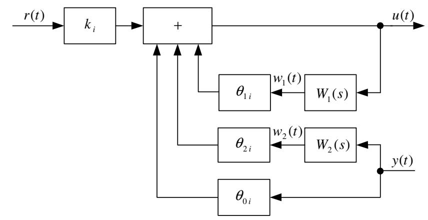

Fig. 2. A controller with a minimum-phase subsystem from r to u

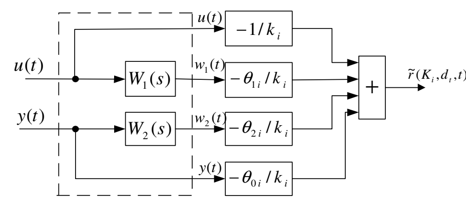

Fig. 3. Fictitious reference signal of the controller in Fig.2

Definition 6: (Safe adaptive control). Given a completely unknown plant and a candidate controller set, a *safe adaptive control* law is one that never fails to stabilize whenever adaptive stabilization is feasible. □

It should be emphasized that a safe adaptive system is more than just robustly stable for plant uncertainty in a given uncertainty set. A safe adaptive system is stable whenever a stabilizing controller exists in its candidate controller set, without any regard to prior knowledge of plant uncertainty.

Problem (Safe adaptive stabilization problem). In an adaptive control system, given a completely unknown plant and a candidate controller set, find an adaptive control algorithm to stabilize the plant whenever stabilization is feasible. □

To solve this problem, the following unfalsified adaptive control using multiple controllers is proposed.

III. UNFALSIFIED MULTIPLE CONTROLLER ADAPTIVE CONTROL (MCAC)

Before introducing an unfalsified adaptive control algorithm to solve the safe adaptive stabilization problem, several definitions in unfalsified control approach are reviewed and several new definitions are introduced in subsection III-A.

A. Definitions in unfalsified control

Let $\mathcal{D} \triangleq \mathcal{U} \times \mathcal{Y}$ denote the set of all possible plant measurement data $d = (u, y)$ over the time 0 to ∞, and denote by d_τ the time-truncation of d. Thus, d_τ represents past experimental plant data up to current time τ. Given past data d_τ, we denote by D_τ the set of signals in $\mathcal{R} \times \mathcal{Y} \times \mathcal{U}$ that interpolate (i.e., are consistent with) d_τ:

$$D_\tau \triangleq D(d_\tau) = \left\{ \ (r, y, u) \ \middle| \ (y_\tau, u_\tau) = d_\tau \ \right\}. \qquad (8)$$

The set of candidate controllers K is denoted $\mathbb{K}$. Let $V(K, d_\tau, \tau)$ The scalar valued function $V(K, d_\tau, \tau)$ is a *cost function*. It is used to evaluate candidate controllers K based on past data d_τ in [7] and is also closely related to the cost functions employed in unfalsified control methods [22].

Definition 7: (Falsified/Unfalsified controller, Unfalsified cost level γ, Unfalsified controller set $\mathbb{K}_{unf}$. Given a scalar valued cost function $V(K, d_\tau, \tau)$, a set of controllers $\mathbb{K}$ and a scalar $\gamma \in \mathbb{R}$, we say that a controller $K \in \mathbb{K}$ is *falsified* with respect to cost level γ by past measurement information d_τ if $V(K, d_\tau, \tau) > \gamma$. Otherwise the control law K is said to be *unfalsified*. The least value of γ for which K is unfalsified is the *unfalsified cost level* of K. The set of unfalsified controllers having an unfalsified cost level of γ or less is called *unfalsified controller set* $\mathbb{K}_{unf}(\gamma)$. □

Remark 1: By definition of 'controller falsification', at any time τ, $\{K_i \mid V(K_i, d_\tau, \tau) < \gamma_1, K_i \in \mathbb{K}\} \subset \{K_i \mid V(K_i, d_\tau, \tau) < \gamma_2, K_i \in \mathbb{K}\}$ if $\gamma_1 < \gamma_2$.

Definition 8: (Fictitious reference signal). Given plant data D and a candidate controller K, the *fictitious reference signal* for this controller is a hypothetical signal that would have produced exactly the same data d had the controller been in the feedback loop with the completely unknown plant during the entire time period over which d were collected. The fictitious reference signal of controller K with plant data d at time t is denoted as $\widetilde{r}(K, d_t, t)$ or as $\widetilde{r}$ when no confusion will be aroused. □

Remark 2: . Fictitious reference signals are not the true signals([16], [23]), hence the name fictitious. If a controller has a minimum-phase subsystem from one of its inputs r to output u, its fictitious reference signals are much easier to be determined. For example, a controller, K_i, with the structure in Fig. 2 is such a controller. Its fictitious reference signal would be

$$\widetilde{r}(K_i, d_t, t) = \frac{1}{k_i}[u(t) - \theta_{1i} W_1(s) u(t) - \theta_{2i} W_2 y(t) - \theta_{0i} y(t)], \qquad (9)$$

which is shown in Fig.3. Note that, by definition, if K_i is not in the feedback loop from the beginning during which data set d_t was collected, its fictitious reference signal may be different from the actual reference signal.

Remark 3: . Given data $d_\tau = (u_0, y_0)$ and a controller K having graph K, the fictitious reference signals are the elements $r_0 \in$ that satisfy $(r_0, y_0, u_0) \in D_\tau \cap \mathsf{K}$. As noted in [22], fictitious reference signal allows unfalsified control performance goals of the form $J(r, y, u, \tau) \leq \gamma$ to be

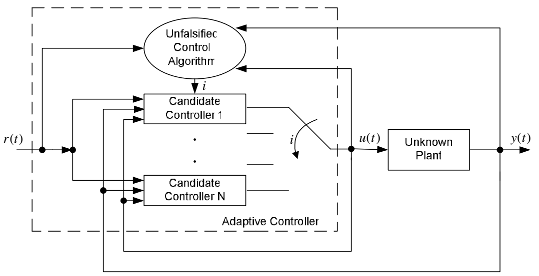

Fig. 4. Unfalsified adaptive control system using multiple controllers

expressed in a form suitable for use in conjunction with the convergence lemma of [7]:

$$V(K, d_\tau, \tau) = J(\tilde{r}(K, d_\tau, \tau), d_\tau, \tau) \tag{10}$$

where $d_\tau \triangleq (u_\tau, y_\tau)$.

Theorem 1: [16] A control law $K \in \mathbb{K}$ is unfalsified by past measurement information D_τ if, and only if, for each triple $(r_0, y_0, u_0) \in D_\tau \cap \mathsf{K}$, there exists at least one pair (u_1, y_1) such that $(r_0, y_1, u_1) \in D_\tau \cap \mathsf{K} \cap \mathsf{T}_{\mathsf{spec}}$. $\square$
Here, $T_{spec} \subset \mathcal{R} \times \mathcal{Y} \times \mathcal{U}$ is a given performance specification set.

Definition 9: (Unfalsified stability). Given candidate controller set $\mathbb{K}$, data d and a system in Fig. 1, we say the system does not have the property of *unfalsified stability* if the hypothesis that at least one candidate controller in $\mathbb{K}$ can stabilize this system, is falsified, *i.e.* , for every controller K, $K \in \mathbb{K}$, $\exists\, \widetilde{r}(K, d) \in L_{2e}$ and $\widetilde{r}(K, d) \neq 0$ s.t.

$$\limsup_{l \to \infty\ \tau \in (l, \infty)} \frac{\| [y, u]_\tau \|}{\| \widetilde{r}_\tau(K, d) \|} = \infty. \tag{11}$$

Otherwise, the system is said to have the property of *unfalsified stability*. $\square$

Remark 4: The hypothesis that a controller can stabilize a system can be falsified even when the controller is not in the feedback loop since fictitious reference signal can be computed when that controller is not in the feedback loop.

B. Multiple Controller Adaptive Control

Consider the general adaptive control system in Fig. 1 as a direct adaptive control system. It has an inputs (r, x_0, d, n), outputs (y, u). The signal r is the reference signal, d plant disturbance, n measurement noise, and y and u the measured outputs of the system and the adaptive controller respectively. The plant, which includes the disturbance/noise signals (d, n), is supposed to be unknown to us. The adaptive controller in such a system is realized by switching among candidate controllers under the supervision of the adaptation control algorithm. To solve the safe adaptive control problem, we will use the 'hysteresis algorithm' of [7] as described in [22], and with 'cost-detectable' cost function of [17].

In the following algorithm, we consider a deterministic system $\Sigma : L_{2e} \to L_{2e}$, whose structure is given in Fig. 4 with input $r(t)$ and measurable output $[u(t), y(t)]$. In this system, we assume if the norm of an input signal of any components in this system at some instant is zero, the output is zero too. For simplicity, noise $n(t)$, disturbance $d(t)$ and initial conditions x_0 in Fig. 1 are assumed to be zeros in Fig. 4. We are given a finite set of candidate controllers $\mathbb{K} = \{K_i\}, i = 1, 2, \ldots, N$. As we know, a controller is a system from its input (r, y) to its output u. In this paper, we only consider stable controllers with a minimum-phase subsystem from its r to u. Suppose the unfalsified controller set at each time τ is denoted by $\mathbb{K}_{unf}(\tau)$. Denote $\widehat{K}(\tau)$ as the time varying online controller. At each time instant, say τ, if the current controller's cost exceeds the minimal cost by more than a pre-specific small number ϵ, the task is to identify and switch to the optimal controller $K^\star(\tau)$, *i.e.* .,

$$K^\star(\tau) = arg \min_{K_i \in \mathbb{K}} V(K_i, d_\tau, \tau) \text{ and } \widehat{K}(\tau) = K^\star(\tau), \tag{12}$$

where $V(K_i, d_\tau, \tau)$ is a given cost function. The steps of the algorithm are:

Algorithm 1. (ϵ-Hysteresis Algorithm [7])

1) Initialize: Let $t = 0$, $\tau = 0$; choose $\epsilon > 0$.
   Let $\hat{K}(0) = K_0, K_0 \in \mathbb{K}$, be the first controller in the loop.
2) $\tau \leftarrow \tau + 1$.
   If $V(\hat{K}(\tau - 1), d_\tau, \tau) > \min_{K \in \mathbb{K}} V(K, d_\tau, \tau) + \epsilon$ then $\hat{K}(\tau) \leftarrow \arg\min_{K \in \mathbb{K}} V(K, d_\tau, \tau)$,
   else $\hat{K}(\tau) \leftarrow \hat{K}(\tau - 1)$
3) go to 2.

Remark 5: If the cost function is chosen so that $V(K, d_\tau, \tau)$ is monotone non-decreasing in τ for all $K \in \mathbb{K}$, then for each $\gamma \in \mathbb{R}$ the unfalsified set $\mathbb{K}_{unf}(\gamma, \tau)$ shrinks monotonically as τ increases; that is, if $t_1 < t_2$, $\mathbb{K}_{unf}(t_1) \subset \mathbb{K}_{unf}(t_2)$.

IV. RESULTS ON STABILITY

Now, let us consider the case where the cost function for controller K_i at time $\tau \in R_+$ is given by

$$V(K_i, d_\tau, \tau) \triangleq \max_{t \le \tau} \begin{cases} \frac{\|\widetilde{e}_t(K_i, d_t)\| + \lambda \|u_t\|}{\|\widetilde{r}_t(K_i, d_t)\|}, & \text{if } \| \widetilde{r}_t(K_i, d_t) \| \neq 0 \\ 0, & \text{if } \| \widetilde{r}_t(K_i, d_t) \| = 0 \end{cases} \tag{13}$$

where λ is some nonnegative constant, $\widetilde{r}_t(K_i, d_t)$ the fictitious reference signal and $\widetilde{e}_t(K_i, d_t)$ the fictitious error of the i-th controller, defined as:

$$\widetilde{e}_t(K_i, d_t) \triangleq W_m \widetilde{r}_t(K_i, d_t) - y. \tag{14}$$

Here, W_m is a reference model (or, in the jargon of robust control theory, W_m is a 'weighting function'); it is chosen to penalize control error signal e. The cost function (13) is chosen because it ensures 'cost detectability' [17]; it is good for ensuring safe adaptive control as will be shown below.

The following lemmas and theorem require the following assumptions on the controller and cost function. There are no assumptions on the plant:

1) Each controller $K \in \mathbb{K}$ is of the form shown in Fig. 5

$$u = K_u e; \quad e = r - K_y y \tag{15}$$

where K_u is minimum phase and K_y is stable.

2) The cost function be given by (13).

Lemma 1: (*Convergence* [17]) If the problem is feasible, then there are finitely many switches among candidate controllers before switching stops, and the cost $V(K, d_\tau, \tau)$ is bounded as $\tau \to \infty$.

Proof. By construction, the cost function has the properties that 1) it is bounded if and only if stability is unfalsified and 2) is monotone in τ for each K. The result follows immediately from [17, Prop. 1]. □

By Lemma 1, controller switching stops eventually. Let t_f be the time when K^f is finally switched into the loop for the last time and let K^f be the final controller connected to the feedback loop.

Lemma 2: then,

$$\lim_{\tau\to\infty} \| [\widetilde{r}(K^f, d) - r]_{(\tau,\infty)} \| = 0. \tag{16}$$

Proof. Please refer to Appendix VII-A. □

Lemma 3: (*Boundedness of the ratio of fictitious reference signal to reference signal*) Consider an unfalsified adaptive control system. Assume that each candidate controller is stable and has a minimum-phase subsystem from r to u. For any r, if there are finitely many switches among candidate controllers before switching stops and K^f is the final controller connected to the feedback loop, then

$$\sup_{\tau\in(0,\infty)} \frac{\| \widetilde{r}_\tau(K^f, d) \|}{\| r_\tau \|} < \infty. \tag{17}$$

Proof. Please refer to Appendix VII-B. □

Theorem 2: (*Stability Theorem*) Consider the system in Fig.4, using unfalsified adaptive control Algorithm 1 and cost function (13). If there is at least one stabilizing controller in the finite candidate controller set, the unfalsified adaptive control system is stable.

Proof. To prove the system is stable, by definition of stability, it is enough to show that for some $\beta_2 < \infty$

$$\| y_\tau \| + \lambda \| u_\tau \| < \beta_2 \| r_\tau \|, \text{ for } \forall\, \tau \in R_+. \tag{18}$$

From Lemma 1 we have that stability is unfalsified, so (18) holds with r_τ replaced by $\widetilde{r}_\tau(K^f, d)$; i.e., there exists $\beta_3 < \infty$ such that

$$\| y_\tau \| + \lambda \| u_\tau \| < \beta_3 \| \widetilde{r}_\tau(K^f, d) \|, \text{ for } \forall\ \tau \in R_+. \tag{19}$$

The result then follows from Lemma 3. So, the unfalsified adaptive control system is stable. □

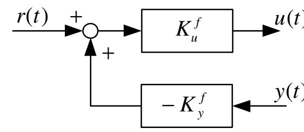

Fig. 5. A general structure of controller K^f

V. SIMULATION

It is important in designing cost functions for adaptive systems that careful attention be paid to the issue of cost detectability. For example as we showed in [18], model mismatch instability occurs despite feasibility of the adaptive problem using fixed multiple plant models adaptive algorithm of [24], which uses the cost function

$$V(K, d_t, t) = \alpha e^2(K, d_t)(t) + \beta \int_0^t exp(-\lambda(t-\tau)) e^2(K, d_\tau)(\tau)\, d\tau \tag{20}$$

where $e(K, d_t) = \hat{y}(K, d_t) - y$ is the identification error associated with an *assumed* i-th plant model for which the i-th controller $K_i \in \mathbb{L}$ would be optimal if there was no residual model mismatch. Here, $\hat{y}(K, d_t)(t)$ is the output of the i-th candidate plant model at time t when the past input is the measured plant input data u_t. As shown by us in [18], a danger with the method in [24] is that the cost function (20) fails the cost detectability requirement of [17, Proposition 1]. If it happens that there is substantial model mismatch and none of the assumed candidate plant models is sufficiently close to the unknown true plant, simulation results show that adaptive control law of [24] can incorrectly discard a stabilizing controller and instead lock on to a destabilizing controller. In contrast, with the 'cost-detectable' cost-function (13) the stabilizing controller is correctly identified by cost minimization and safe adaptive control is robustly achieved, as predicted by Theorem 2 and as shown by the simulation results in [18].

VI. CONCLUSION

In this paper, we have reported results on stability of unfalsified adaptive control methods based on the Morse-Mayne-Goodwin hysteresis algorithm [7]. For the new class of unfalsified control cost functions with the cost-detectability property defined in [17], the hysteresis algorithm is proved yield *safe adaptive control* that is guaranteed stable without plant model assumptions, subject only to the feasibility requirement that there exists at least one stabilizing controller amongst the candidate controllers. Simulation results demonstrate that the proposed multi-controller-adaptive control (MCAC) laws are not only safe as predicted by Theorem 2, but also quick.

VII. APPENDIX

A. Proof of lemma 2

The final controller K^f has a general structure in Fig. 5, where K_u^f is the subsystem from r to u, and $-K_y^f$

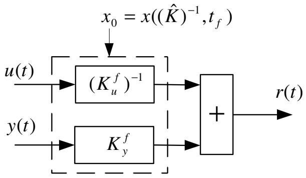

Fig. 6. Fictitious reference signal generating system $\widehat{\Sigma}$

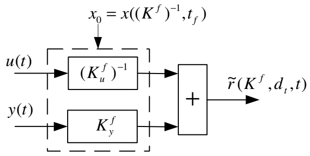

Fig. 7. Fictitious reference signal generating system Σ^f

is the subsystem from y to r. For $t > t_f$, the processes generating r and $\widetilde{r}(K^f, d_t)$ are shown in Figs. 6 and 7, respectively. By hypothesis, K_u^f is minimum phase, so $(K_u^f)^{-1}$ is incrementally stable. By hypothesis K_y is also stable. Since the only difference after time t_f between the process generating r and that generating $\widetilde{r}(K^f, d_t)$ arise from differences the initial state of the K_u^f blocks at time t_f, the result follows immediately. $\square$

B. Proof of lemma 3

Since when $\| r_\tau \| = 0$, then$\| [u, y]_\tau \| = 0$, and thus $\| \widetilde{r}_\tau \| = 0$. So, define

$$\frac{\| \widetilde{r}_\tau \|}{\| r_\tau \|} = 0, \text{when} \| r \| = 0. \tag{21}$$

By lemma 2, for any $\delta > 0$ there exists some finite t_b, $t_b > t_f$, s.t.,

$$| \| \widetilde{r}_{(t_b,\tau)} \| - \| r_{(t_b,\tau)} \| | < \| (\widetilde{r} - r)_{(t_b,\tau)} \| < \delta. \tag{22}$$

So,

$$\| \widetilde{r}_{(t_b,\tau)} \|^2 < \left(\delta + \| r_{(t_b,\tau)} \|\right)^2 = \delta^2 + \| r_{(t_b,\tau)} \|^2 + 2\delta \| r_{(t_b,\tau)} \| \tag{23}$$

Because $\widetilde{r}, r \in L_{2e}$, there exists some finite $M > 0$ s.t.

$$\| \widetilde{r}_\tau \| \leq \| r_\tau \| \cdot M, \tau \in (0, t_b). \tag{24}$$

Therefore, by (23) and (24), for any $\tau > t_b$, $\exists$ finite $\delta' > 0$, s.t.,

$$\begin{aligned} \frac{\| \widetilde{r}_\tau \|^2}{\| r_\tau \|^2} &= \frac{\| \widetilde{r}_{t_b} \|^2 + \| \widetilde{r}_{(t_b,\tau)} \|^2}{\| r_\tau \|^2} \\ &< \frac{M^2 \| r_{t_b} \|^2 + \delta^2 + \| r_{(t_b,\tau)} \|^2 + 2\delta \| r_{(t_b,\tau)} \|}{\| r_\tau \|^2} \\ &< \delta'. \end{aligned} \tag{25}$$

The last inequality holds because the denominator $\| r_\tau \|^2$ is monotone increasing with respect to τ, while M and δ are finite, $\frac{\|r_{t_b}\|^2}{\|r_\tau\|^2} < 1$, and $\frac{\|r_{(t_b,\tau)}\|^2}{\|r_\tau\|^2} < 1$. Thus,

$$\sup_{\tau \in (0,\infty)} \frac{\| \widetilde{r}_\tau(K^f, d) \|}{\| r_\tau \|} < \infty, \text{for any } r. \tag{26}$$

So (17) is established.

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