2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

On Model Reduction via Empirical Balanced Truncation

Douglas A. Lawrence, James H. Myatt, and R. Chris Camphouse

Abstract— **Empirical balanced truncation is considered as an approach for deriving reduced-order models of large-scale nonlinear systems that are of interest in the design of feedback control systems. Empirical balanced truncation is related to the widely-applied Proper Orthogonal Decomposition (POD) methodology and yet may be better suited for closed-loop control because order reduction is based on the system's state-to-output interaction along with its input-to-state interaction, not just the latter. Refinements to the scheme originally proposed in the literature are presented leading to reduced data requirements that may become significant for applications such as aerodynamic flow control. Towards that end, the $1-$dimensional Burgers' equation is used to validate the basic ideas, implementation details, and applicability to closed-loop control system design.**

I. INTRODUCTION

There has been a recent surge in basic and applied research on reduced-order modeling and control of distributed parameter systems in general and aerodynamic flow control problems in particular. Much of this effort can be classified according to 'design-then-reduce' and 'reduce-then-design' philosophies. In the former approach, a high-order controller is first designed on the basis of a large-scale plant model and is subsequently approximated by a reduced-order controller. In the latter approach, order reduction is first performed on the large-scale plant model and then a reduced-order controller is designed on the basis of the reduced-order plant model. In most of the work in either category, the Karhunen-Loève decomposition, also known as Proper Orthogonal Decomposition (POD), is the key tool for performing model reduction on either plant or controller.

The design-then-reduce approach is motivated by the fact that the ultimate goal in designing a reduced-order controller is the accurate approximation of closed-loop behavior. For this reason, it is potentially ill-advised to discard plant information as the first step in the design process, as argued in [1]. Consequently, the feedback loop must first be closed using the high-order plant and controller, with controller order reduction then performed with respect to a measure of closed-loop approximation quality. In the context of distributed parameter systems, a design-then-reduce approach has been successfully applied to a variety of systems described by partial differential equations ([5], [6], [7], [9], [10]). In this work, Proper Orthogonal Decomposition is applied to the functional gains of an infinite dimensional control law to yield a reduced-order controller.

The first author was supported under the 2004 NRC Summer Faculty Fellowship Program.

D.A. Lawrence is with the School of EECS, Ohio University, Athens, OH 45701, USA dal@ohio.edu

J.H. Myatt and R.C. Camphouse are with the Control Design & Analysis Branch, Air Vehicles Directorate, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433, USA

A reduce-then-design strategy is applicable in cases where it is known that the internal dynamics of a large-scale system are either well-approximated by a small number of modes or have negligible impact on the behavior from actuated input to measured output. Also, model reduction may be a necessary first step in cases where the construction of high-order controllers is computationally intractable. Numerous flow control studies have been performed using Proper Orthogonal Decomposition to first generate reduced-order flow field models. A number of works by various researchers using POD have indicated that such techniques can make use of experimentally derived data to generate reduced-order flow field models that are directly amenable to control design ([2], [13], [14], [15], [16], [18], [20], [21], [24], [25], [26]). POD has also been used successfully generate reduced-order models for other distributed parameter systems ([3], [23]). Another interesting approach bases model reduction on a time scale separation property of the system dynamics whereby the "slow" dominant modes correspond to a low-dimensional positively invariant manifold called an inertial manifold ([4], [8], [12], [11]), and methods for the computation of approximate inertial manifolds are presented.

A model reduction technique bearing some resemblance to POD employs balanced truncation ([17]) to extract a reduced-order model that accurately reproduces the large-scale system's interaction between its inputs (actuators) and measurements (sensors). The approach is based upon the work of Moore ([19]) except that empirically-derived controllability and observability gramians are used in place of their analytically-derived counterparts. As proposed in [17], separate sets of experiments/simulations are required to calculate the respective gramian. First, the state response to impulsive or otherwise sufficiently rich excitations for zero initial conditions is used to construct an emprical controllability gramian. Second, the output response for zero input across a large set of initial states leads to the construction of an empirical observability gramian. In contrast, POD as implemented in the references cited above does not consider the interaction between the system state and those sensed variables that are available for feedback in a closed-loop control system. Consequently, order reduction based on measurements that only characterize the input-to-state interaction may inadvertently fail to capture dynamic input-output behavior that is critical from the standpoint of closing the loop between sensors and actuators.

This paper considers empirical balanced truncation applied to the problem of reduced-order modeling for closed-loop control. Refinements to the originally proposed scheme are presented that impose a significantly lighter requirement on the data needed and the manner in which it is collected. The approach is applied to the $1-$dimensional Burgers' equation, a partial differential equation in one spatial dimension that possesses features comparable to the Navier-Stokes equations governing fluid flow and yet lends itself to simulation and computation in a desktop computing environment.

The remainder of the paper is organized as follows. Section II introduces the Burgers' equation example. Section III presents the requisite background material for linear systems upon which the aforementioned refinements are based. Section IV considers empirical balanced truncation applied to nonlinear systems. Section V discusses the application of this technique to the Burgers' equation example. Finally, concluding remarks are offered in Section VI.

II. MOTIVATING EXAMPLE

We consider the $1-$dimensional Burgers' equation on the spatial domain $\Omega = [0, 1]$

$$w_t(t,x) = \kappa w_{xx}(t,x) - w(t,x)w_x(t,x) + u(t,x) \qquad (1)$$

with initial condition $w(0,x) = w_0(x)$. The distributed control term is formulated as

$$u(t,x) = u_1(t)b_1(x) + u_2(t)b_2(x)$$

and we also include a $2-$dimensional measurement given by

$$y_i(t) = \int_\Omega c_i(x)w(t,x)dx \quad i = 1,2$$

In what follows, we set $\kappa = 0.01$ and take

$$b_1(x) = c_1(x) = \sin^2(\pi x) \qquad b_2(x) = c_2(x) = \sin^2(2\pi x)$$

In terms of a finite set of basis functions $\{\varphi(x), i = 1, \ldots, n\}$, the solution to (1) can be approximated via

$$w(t,x) \approx \sum_{i=1}^{n} w_i(t)\varphi_i(x)$$

In this example, we use $n = 99$ triangular basis functions corresponding to a uniform discretization of Ω with a spatial increment of $\Delta x = 0.01$.

A Galerkin-type projection of (1) onto this basis yields an $n-$dimensional ordinary differential equation (ODE) of the form

$$\begin{aligned} M\dot{w}(t) &= Kw(t) + N(w(t)) + Lu(t) \qquad w(0) = w_0 \\ y(t) &= Cw(t) \end{aligned}$$

in which the coefficient matrices have entries

$$m_{ij} = \int_\Omega \varphi_i(x)\varphi_j(x)dx \qquad k_{ij} = -\kappa \int_\Omega \varphi_i'(x)\varphi_j'(x)dx$$
$$\ell_{ij} = \int_\Omega \varphi_i(x)b_j(x)dx \qquad c_{ij} = \int_\Omega c_i(x)\varphi_j(x)dx$$

and the nonlinear term is specified by quadratic component functions

$$n_k(w) = w^T N^k w \qquad n_{ij}^k = \int_\Omega \varphi_i(x)\varphi_j'(x)\varphi_k(x)dx$$

The mass matrix M is invertible and so the preceding ODE can be reformulated as

$$\begin{aligned} \dot{w}(t) &= Aw(t) + G(w(t)) + Bu(t) \qquad w(0) = w_0 \\ y(t) &= Cw(t) \end{aligned} \qquad (2)$$

This finite-dimensional approximation to (1) will constitute the high-order model to which empirical balanced truncation will be applied in the sequel.

III. EMPIRICAL BALANCED TRUNCATION FOR LINEAR SYSTEMS

It is well-known that a stable, minimal, linear discrete-time $m-$input, $p-$output, $n-$dimensional state equation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

with reachability gramian W and observability gramian M can be transformed via $x(k) = Tz(k)$ into one for which the transformed gramians become equal and diagonal. That is,

$$\hat{W} = T^{-1}WT^{-T} = \Sigma \qquad \hat{M} = T^T MT = \Sigma$$

in which Σ is a diagonal matrix displaying the system's Hankel singular values. Towards developing a procedure for empirical balancing, we first consider an approximate balancing problem involving the $q-$step reachability gramian

$$W_q = R_q R_q^T \quad \text{where} \quad R_q = \begin{bmatrix} B & AB & \cdots & A^{q-1}B \end{bmatrix}$$

and $\ell-$step observability gramian

$$M_\ell = O_\ell^T O_\ell \quad \text{where} \quad O_\ell = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\ell-1} \end{bmatrix}$$

for positive integers q and ℓ to be specified shortly. Note that

$$\lim_{q\to\infty} W_q = W \quad \text{and} \quad \lim_{\ell\to\infty} M_\ell = M.$$

The goal now is to construct a nonsingular coordinate transformation matrix T yielding

$$\hat{W}_q = T^{-1}W_qT^{-T} = \Sigma_1 = T^T M_\ell T = \hat{M}_\ell$$

in which the diagonal matrix Σ_1 displays the nonzero singular values of the $\ell p \times qm$ Hankel matrix

$$\begin{aligned} \mathcal{H}_{\ell q} &= O_\ell R_q \\ &= \begin{bmatrix} CB & CAB & \cdots & CA^{q-1}B \\ CAB & CA^2B & \cdots & CA^qB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\ell-1}B & CA^\ell B & \cdots & CA^{\ell+q-2}B) \end{bmatrix} \end{aligned}$$

for integers q and ℓ chosen so that

$$\operatorname{rank}\mathcal{H}_{\ell q} = \operatorname{rank}\mathcal{H}_{\ell+1,q+j} = n \quad \forall j \geq 1$$

which, following a standard result [22], is always possible.

In terms of the singular value decomposition (SVD) of $\mathcal{H}_{\ell q}$

$$\mathcal{H}_{\ell q} = U\Sigma V^T = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right] \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} V_1^T \\ V_2^T \end{array}\right]$$

we construct T according to

$$T = R_q V_1 \Sigma_1^{-1/2}.$$

It is straightforward to verify that $T^{-1} = \Sigma_1^{-1/2} U_1^T O_\ell$ from which

$$\begin{aligned} \hat{W}_q &= T^{-1} W_q T^{-T} \\ &= \left(\Sigma_1^{-1/2} U_1^T O_\ell\right)\left(R_q R_q^T\right)\left(O_\ell^T U_1 \Sigma^{-1/2}\right) \\ &= \Sigma_1^{-1/2} U_1^T \mathcal{H}_{\ell q} \mathcal{H}_{\ell q}^T U_1 \Sigma_1^{-1/2} \\ &= \Sigma_1 \end{aligned}$$

and

$$\begin{aligned} \hat{M}_\ell &= T^T M_\ell T \\ &= \left(\Sigma_1^{-1/2} V_1^T R_q^T\right) M_\ell \left(R_q V_1 \Sigma_1^{-1/2}\right) \\ &= \Sigma_1^{-1/2} V_1^T \mathcal{H}_{\ell q}^T \mathcal{H}_{\ell q} V_1 \Sigma_1^{-1/2} \\ &= \Sigma_1 \end{aligned}$$

as desired.

Balanced truncation can subsequently be performed in the usual way by discarding states in the approximately balanced realization that are both weakly reachable and weakly observable as dictated by the relative sizes of the Hankel singular values. Suppose that with $\Sigma_1 = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\}$,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0,$$

and that for some r, $\sigma_r \gg \sigma_{r+1}$. As a result Σ_1 can be partitioned according to

$$\Sigma_1 = \left[\begin{array}{cc} \Sigma_r & 0 \\ 0 & \Sigma_{n-r} \end{array}\right]$$

in which

$$\Sigma_r = \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\} \quad \Sigma_{n-r} = \operatorname{diag}\{\sigma_{r+1}, \ldots, \sigma_n\}$$

along with a conformable column-wise partitioning of U_1 and V_1

$$U_1 = \left[\begin{array}{cc} U_r & U_{n-r} \end{array}\right] \qquad V_1 = \left[\begin{array}{cc} V_r & V_{n-r} \end{array}\right].$$

This leads to the construction of an *immersion/projection* pair

$$T_r = R_q V_r \Sigma_r^{-1/2} \qquad T_r^\dagger = \Sigma_r^{-1/2} U_r^T O_\ell$$

satisfying $T^\dagger T = I$ from which a reduced-order $r-$dimensional state equation is specified by the coefficient matrices

$$A_r = T_r^\dagger A T_r \qquad B_r = T_r^\dagger B \qquad C_r = C T_r.$$

The projection is also given by $T_r^\dagger = \Sigma_r^{-1/2}(R_q V_r)^\dagger$. Thus, the immersion/projection pair and hence the reduced-order model can be computed directly from the reachability matrix R_q and the SVD of the Hankel matrix $\mathcal{H}_{\ell q}$ in contrast to constructions involving the reachability and observability gramians or their finite step approximations. In the context of empirical balanced truncation, this becomes significant with respect to simulation/experimentation and data collection requirements. Rather than separately estimating the reachability gramian from the state response for a sufficiently rich input signal and then estimating the observability gramian based on the unforced output response measured across a large set of initial states, the above constructions require estimates of the *input-state* Markov parameters

$$A^k B, \quad k = 0, \ldots, q-1$$

and *input-output* Markov parameters

$$CA^k B, \quad k = 0, \ldots, \ell + q - 2$$

that can be computed from a single simulation/experiment in which a sufficiently rich input signal is applied and the state and output responses are collected. For example, the Discrete Fourier Transform (DFT) can be used to map the time domain data into spectral densities from which frequency response estimates can be calculated using reliable signal processing techniques. The Markov parameters can then be obtained by applying the Inverse Discrete Fourier Transform (IDFT) to the frequency response estimates.

IV. EMPIRICAL MODEL REDUCTION FOR NONLINEAR SYSTEMS

In this section, we apply the linear constructions of the preceding section to nonlinear systems. As in [17], the rationale for doing so is that linear subspace approximations to exact submanifolds associated with nonlinear reachability and (un)observabilty require only standard matrix manipulations utilizing simulation/experimental data. The computational advantages of the scheme presented here carry over directly to the nonlinear setting.

For the purpose of assessing the utility of this reduced-order modeling approach for closed-loop control, a nonlinear controller design is presented. The controller is observer-based with observer and feedback gains constructed using linear LQG theory based on the linearization of the nonlinear reduced-order model about a nominal equilibrium. The observer contains a copy of the nonlinear reduced-order model for improved regulation performance.

A. Reduced-Order Modeling

Consider an $n-$dimensional nonlinear system described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{3}$$

and assumed to have an equilibrium at $(x,u,y) = (0,0,0)$ so that $f(0,0) = 0$ and $h(0) = 0$. The reduced-order model is derived from (3) via the construction of an immersion/projection pair

$$x = T_r x_r \qquad x_r = T_r^\dagger x$$

following the procedure of the preceding section and resulting in

$$\begin{aligned} \dot{x}_r(t) &= T_r^\dagger f(T_r x_r(t), u(t)) \\ y_r(t) &= h(T_r x_r(t)) \end{aligned} \tag{4}$$

The linearization of (3) about the equilibrium $(x,u,y) = (0,0,0)$ can be represented by the triple (A,B,C) in which

$$A = \frac{\partial f}{\partial x}(0,0) \quad B = \frac{\partial f}{\partial u}(0,0) \quad C = \frac{\partial h}{\partial x}(0) \tag{5}$$

The reduced-order model (4), for convenience written as

$$\begin{aligned} \dot{x}_r(t) &= f_r(x_r(t), u(t)) \\ y_r(t) &= h_r(x_r(t)) \end{aligned} \tag{6}$$

has the linearization about $(x_r,u,y) = (0,0,0)$ represented by

$$A_r = \frac{\partial f_r}{\partial x_r}(0,0) \quad B_r = \frac{\partial f_r}{\partial u}(0,0) \quad C_r = \frac{\partial h_r}{\partial x_r}(0) \tag{7}$$

These two linearizations are related via

$$A_r = T_r^\dagger A T_r \qquad B_r = T_r^\dagger B \qquad C_r = C T_r.$$

B. Controller Design

For simplicity, we consider reduced-order controllers with the following observer-based structure

$$\begin{aligned} \dot{x}_c(t) &= f_r(x_c(t), -F_r x_c(t)) + L_r\left(y(t) - h_r(x_c(t))\right) \\ u(t) &= -F_r x_c(t) \end{aligned} \tag{8}$$

in which the gain matrices F_r and L_r are designed using standard tools for linear LQG synthesis based on the reduced-order linearization (7). Local exponential stability of the feedback interconnection of the reduced-order nonlinear model (6) and the reduced-order controller (8) holds provided the $2r \times 2r$ matrix

$$\begin{bmatrix} A_r & -B_r F_r \\ L_r C_r & A_r - B_r F_r - L_r C_r \end{bmatrix}$$

has all eigenvalues confined to the open left half of the complex plane. As we are ultimately interested in local exponential stability of the feedback interconnection of the large-scale nonlinear system (3) and the reduced-order controller (8), a comparable eigenvalue condition must hold for the $(n+r) \times (n+r)$ matrix

$$\begin{bmatrix} A & -B F_r \\ L_r C & A_r - B_r F_r - L_r C_r \end{bmatrix} \tag{9}$$

which involves the full-order linearization (5).

V. RESULTS

A 1024 point sequence of normally distributed 2×1 random vectors was passed through a zero-order hold device with sample period $\Delta t = 0.01s$ resulting in a random input signal with a bandwidth of 50Hz. This, in turn, was applied to the large-scale model (2) and the state and output responses were sampled every $\Delta t = 0.01s$. It is important to note, in support of the previous claim that the methodology presented in this paper offers significantly reduced data requirements as compared with the originally proposed scheme [17], this single simulation yields the data required to generate a reduced-order model.

An empirical Hankel matrix $\mathcal{H}_{\ell q}$ was constructed for parameter values $\ell = 200$ and $q = 300$. Based upon the Hankel singular values, plotted in Fig. 1, the dimension of the reduced-order model was chosen to be $r = 9$. The associated mode shapes are shown in Fig. 2 and the $9^{\text{th}}-$order model has the form

$$\begin{aligned} \dot{w}_r(t) &= A_r w_r(t) + G_r(w_r(t)) + B_r u(t) \\ y_r(t) &= C_r w_r(t) \end{aligned} \tag{10}$$

in which the nonlinear term has quadratic dependence on the reduced-order state.

For validation purposes, the full and reduced-order model responses were compared for frequency sweep inputs. Over a 2 second simulation interval, the frequency of Input 1 ranged from 0.1 to 10 Hz while the frequency of Input 2 ranged from 0.1 to 1 Hz. The output responses of the full and reduced-order model are shown in Fig. 3. The full-order state response is shown in Fig. 4 and the reduced-order state response mapped into the full-order state space via the immersion T is shown in Fig. 5. These responses indicate that the reduced-order model reproduces the full-order responses with reasonable accuracy over a range of input frequencies.

Following the discussion in Section IV-B, a reduced-order controller of the form

$$\begin{aligned} \dot{w}_c(t) &= \big(A_r - B_r F_r\big) w_c(t) + G_r(w_c(t)) \\ &\qquad + L_r\big(y(t) - C_r w_c(t)\big) \\ u(t) &= -F_r w_r(t) \end{aligned} \tag{11}$$

was constructed. Note that the quadratic nonlinearity is duplicated in the controller as called for by (8). A similar strategy was adopted in [5] specifically for the $1-$dimensional Burgers' equation, albeit in a design-then-reduce context. The gains F_r and L_r were constructed using standard LQG techniques for linear time-invariant systems and it was verified that the 108×108 matrix (9) was Hurwitz.

Simulations were conducted for the feedback interconnection of the full-order nonlinear system (2) and the reduced-order nonlinear controller (11). The initial state of the full-order model was chosen as

$$w_i(0) = \sin^2(2\pi i \Delta x) \quad i = 1, \ldots, 99$$

and the initial state of the controller was set to a $9-$dimensional zero vector. The output response of the full-order system is shown in Fig. 6 and the state-response of the full-order system is shown in Fig. 7. These plots depict good regulation performance as predicted by that of the linear LQG controller designed on the basis of the linearization of the reduced-order model (10).

VI. CONCLUDING REMARKS

Empirical balanced truncation has been considered as an approach for deriving reduced-order models of large-scale nonlinear systems that are of interest in a 'reduce-then-design' context. Like Proper Orthogonal Decomposition (POD), empirical balanced truncation is a data-based approach that can be implemented via standard matrix computations. However, since order reduction is based on the system's input-output behavior rather than solely on its input-state interaction, empirical balanced truncation is better suited for closed-loop control.

Refinements to the scheme originally proposed in [17] have been presented that lead to reduced data requirements that may become significant for applications such as aerodynamic flow control. Essentially, the balancing transformation is constructed from input-state and input-output Markov parameters that can be identified from state and output measurements collected in a single experiment/simulation. A particular advantage of this approach is that, since an empirical observability gramian is not required, the need to preset the system's initial state over a set that spans the full-order state-space is eliminated.

The approach has been applied with favorable results to the $1-$dimensional Burgers' equation, a partial differential equation in one spatial dimension that possesses features comparable to the Navier-Stokes equations governing fluid flow. The application to more realistic aerodynamic flow control problems is a topic of on-going investigation.

REFERENCES

[1] ANDERSON, B. D. O., "Controller Design: Moving from Theory to Practice (1992) Bode Prize Lecture)," *IEEE Control Systems Magazine*, pp. 16 – 25, August, 1993.

[2] ARIAN, E., FAHL, M., AND SACHS, E. W., "Trust-region Proper Orthogonal Decomposition for Flow Control", ICASE Report 2000-25, ICASE, NASA Langley Research Center, Hampton, VA, 2000.

[3] ARMAOU, A. AND CHRISTOFIDES, P. D., "Dynamic Optimization of Dissipative PDE Systems using Empirical Eigenfunctions," *Proceedings of the 2002 American Control Conference*, pp. 1040 – 1048, Anchorage, AK, 2002.

[4] ARMAOU, A. AND CHRISTOFIDES, P. D., "Dynamic Optimization of Dissipative PDE Systems using Nonlinear Order Reduction," *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 2310 – 2316, Las Vegas, NV, 2002.

[5] ATWELL, A., BORGGAARD, J. T., AND KING, B. B., "Reduced Order Controllers for Burgers' Equation with a Nonlinear Observer," *International Journal of Applied Mathematics and Computational Sciences*, Vol. 11, No. 6, pp. 1311 – 1330, 2001.

[6] ATWELL, J. A., "Proper Orthogonal Decomposition for Reduced Order Control of Partial Differential Equations", Ph.D. Dissertation, Virginia Polytechnic Institute and State University, 2000.

[7] ATWELL, J. A., AND KING, B. B., "Proper Orthogonal Decomposition for Reduced Basis Feedback Controllers for Parabolic Equations", ICAM Report 99-01-01, Virginia Polytechnic Institute and State University, 1999.

[8] BAKER, J. AND CHRISTOFIDES, P. D., "Finite-Dimensional Approximation and Control of Nonlinear Parabolic PDE Systems," *International Journal of Control*, Vol. 73, No. 5, pp. 439 – 456, 2000.

[9] BURNS, J. A., AND KING, B. B., "On the Design of Feedback Controllers for a Convecting Fluid Flow via Reduced Order Modeling", *1999 IEEE International Conference on Control Applications*, pp. 1157 – 1162, Hawaii, 1999.

[10] CAMPHOUSE, R. C. AND MYATT, J. H., "Feedback Control for a Two-Dimensional Burgers' Equation System Model," *Proceedings of the 2nd Flow Control Conference*, AIAA-2004-2411, Portland, OR, 2004.

[11] CHRISTOFIDES, P. D., *Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Applications to Transport-Reaction Processes*, Birkhauser, Boston, 2001.

[12] CHRISTOFIDES, P. D. AND DAOUTIDIS, P., "Finite-Dimensional Control of Parabolic PDE Systems using Approximate Inertial Manifolds," *Journal of Mathematical Analysis and Applications*, Vol. 216, No. 2, 398 – 420, 1997.

[13] GILLIES, E. A., "Low-dimensional Control of the Circular Cylinder Wake", *Journal of Fluid Mechanics*, pp. 157 – 178, 1998.

[14] GRAHAM, W. R., PERAIRE, J., AND TANG, K. Y., "Optimal Control of Vortex Shedding Using Low Order Models. Part I: Open-Loop Model Development", *International Journal for Numerical Methods in Engineering*, Vol. 44, No. 7, pp. 945 – 972, 1999.

[15] GRAHAM, W. R., PERAIRE, J., AND TANG, K. Y., "Optimal Control of Vortex Shedding Using Low Order Models. Part II: Model-Based Control", *International Journal for Numerical Methods in Engineering*, Vol. 44, No. 7, pp. 973 – 990, 1999.

[16] HALL, J. K., AND ADDINGTON, G. A., "Flow Control on a Reduced-Order Two-Dimensional Cylinder Model", *Atmospheric Flight Mechanics Conference*, AIAA, August, 2001.

[17] LALL, S., MARSDEN, J., AND GLAVAŠKI, S., "A Subspace Approach to Balanced Truncation for Model Reduction of Nonlinear Systems," *International Journal of Robust and Nonlinear Control*, Vol. 12, pp. 519 – 535, 2002.

[18] LUMLEY, J., "Coherent Structures in Turbulence," In *Transition and Turbulence*, edited by R. E. Meyer, Academic Press, Mathematics Research Center Symposia and Advanced Seminar Series, 1981.

[19] MOORE, B. C., "Principle Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Transactions on Automatic Control*, Vol. 26, No. 1, pp. 17 – 32, 1981.

[20] RAVINDRAN, S. S., "Proper Orthogonal Decomposition in Optimal Control of Fluids", NASA Technical Memorandum TM-1999-209113, 1999.

[21] REDINIOTIS, O. K., KO, J., YUE, X., AND KURDILA, A. J., "Synthetic Jets, their Reduced Order Modeling and Applications to Flow Control," *37th Aerospace Sciences Meeting & Exhibit*, AIAA 99-1000, Reno, NV, 1999.

[22] RUGH, W. J., *Linear System Theory*, Prentice-Hall, Upper Saddle River, NJ, 1996.

[23] SHVARTSMAN, S. Y., THEODOROPOULOS, C., RICO-MARTINEZ, R., KEVREKIDIS, I. G., TITI, E. S., MOUNTZIARIS, T. J., "Order Reduction for Nonlinear Dynamic Models of Distributed Reacting Systems," *Journal of Process Control*, Vol. 10, pp. 177 – 184, 2000.

[24] SINGH, S. N., MYATT, J. H., ADDINGTON, G. A., BANDA, S. S., AND HALL, J. K., "Optimal Feedback Control of Vortex Shedding Using Proper Orthogonal Decomposition Models", *Journal of Fluids Engineering*, Vol. 123, September, 2001.

[25] SINGH, S. N., MYATT, J. H., ADDINGTON, G. A., BANDA, S. S., AND HALL, J. K., "Adaptive Feedback Linearizing Control of Proper Orthogonal Decomposition Nonlinear Flow Models", *Proceedings of the American Control Conference*, Arlington, VA, June, 2001.

[26] SIROVICH, L., "Turbulence and the Dynamics of Coherent Structures: Part I: Coherent Structures," *Quarterly of Applied Mathematics*, Vol. 45, pp. 561 – 571, 1987.

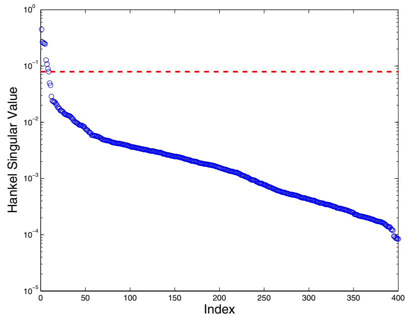

Fig. 1 Hankel singular values.

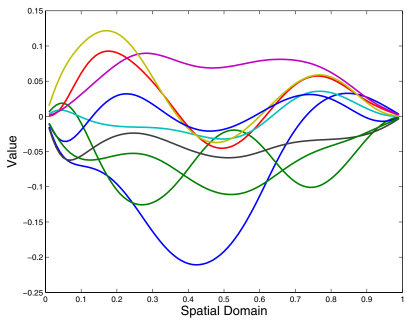

Fig. 2 Reduced-order mode shapes.

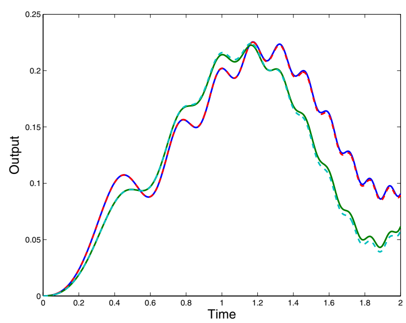

Fig. 3 Full-order system outputs (Solid) vs. reduced-order model outputs (Dashed).

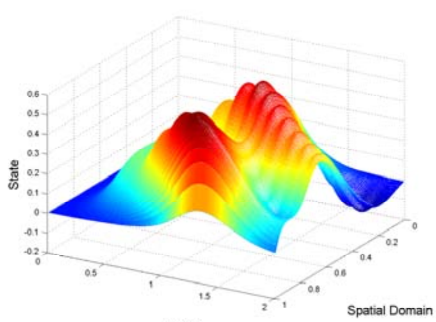

Fig. 4 Full-order system state.

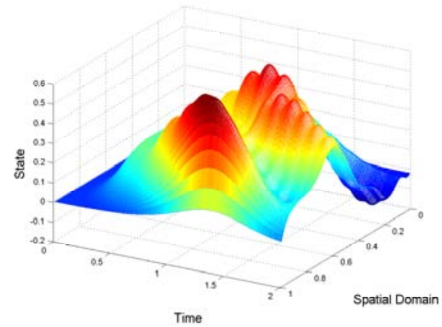

Fig. 5 Reduced-order model state with immersion.

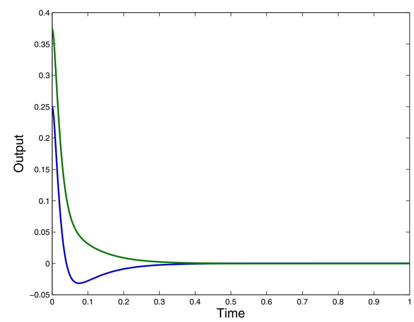

Fig. 6 Full-order system outputs with reduced-order controller.

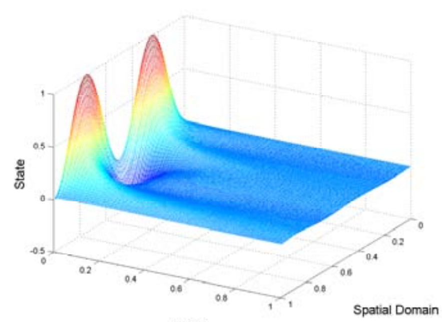

Fig. 7 Full-order system state with reduced-order controller.