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On The Design of A Reduced Order H_∞ Filter for Systems with Unknown Delay

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Abstract—In this note we investigate the filtering problem for linear time-delay systems. Unlike most of the existing results, we consider systems with an unknown and constant delay. From the Razumikhin stability theorem and some judicious mathematical manipulations, a reduced order H_∞ filter able to ensure the asymptotic stability and the pre-specified H_∞ performances of the filtering process is designed. Also the maximal allowable delay preserving the asymptotic stability is given. A numerical example is provided to show the effectiveness of the proposed technique.

Index Terms—H_∞ Filtering, Time-delay systems, Unknown delay, Rasumikhin stability analysis, linear matrix inequality.

I. Introduction

The filtering problem for systems with time delay has been source of attraction and extensively studied in the past few years. Indeed, a large number of relevant control design methods such as state feedback stabilization need the knowledge of the entire system state. Unfortunately, for technical and economical reasons, this is not the case in several practical situations. To deal with, tremendous research activities on filtering design have been reported in the literature and assure satisfactory estimation performances in the presence or not of uncertainties and/or external perturbations. We refer the reader to recent advances in this field (for example , [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]) and the references therein. The basic tools used mainly in these works were the Kalman filter and the H_∞ filtering theory. However most of the reported results within this framework of study concern full order observers design with few extensions to the reduced-order case (see [11]), whereas the advantage of the use of the latter is clear at different heading especially from computational requirements point of view. Moreover, it is worth noting that almost the totality of the investigations in the filtering processes concern the class of delayed systems with a known constant or varying delay, whereas little attention has been paid to the problem of filtering of systems with an unknown delay. Indeed dealing with such problem leads to further complications compared to the classical filtering problem, for instance the form of the filter that must not contain the delay since it is unknown, which complicate the task. To cope with this problem, straightforward approaches have been nicely developed in [12] and [13]. That in [12] consists in using a delay-independent filter, that leads as noticed by the authors themselves to conservative assumptions. In [13] a robust delay-dependent H_∞ filtering design has been proposed, based on the descriptor model transformation, with less conservative conditions.

In this paper, we present an alternative and complementary approach to those proposed in the aforementioned papers, based on the Razumikhin stability Theorem. It is worth noting that the systems considered in this note are in the general form in which the delay is constant and unknown. Sufficient conditions given in terms of linear matrix inequalities LMIs and scalar constraints are derived to guarantee the asymptotic stability of the estimation errors, and that the induced $\mathcal{L}_2$-norm of the system relating the exogenous signals to the estimation errors is less than a prescribed level. Furthermore the maximal allowable delay keeping the asymptotic stability of the filtering errors and the system states is given. The interest of the proposed approach lies in its novelty and on the fact that it can be easily extended to deal with the robust filtering problem. The organization of this paper is as follows. Section 2 gives the problem statement and some lemmas which will be used in the course of the paper. Section 3 presents the main results which provide sufficient conditions to achieve the asymptotic stability and the disturbance attenuation level in the filtering process. Furthermore an explicit expression of the upper bound of the unknown time delay is derived while preserving the asymptotic stability property. Section 4 gives an illustrative example to illustrate our results. Finally, the conclusion is given in section 5.

II. Problem statement

The linear continuous time-delay system considered here is described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_w w(t) && (1.a)\\ x(t) &= \phi(t), \qquad t \in [-h, 0] && (1.b)\\ y(t) &= Cx(t) + C_d x(t-\tau) + C_w w(t) && (1.c)\\ z(t) &= Lx(t) && (1.d) \end{aligned}$$

where $x(t) \in I\!R^n$ is the state, $w(t) \in \mathcal{L}_2^q[0,\infty)$ is the exogenous disturbance signal, $y(t) \in I\!R^m$ is the measurement and $z(t) \in I\!R^r$ is the signal to be estimated, with $r \leq n$.

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A, A_d, C, C_d, B_w, C_w, L are known constant matrices with appropriate dimensions, the time delay τ is unknown and constant and satisfy $\tau \in [0, h]$, where h is a positive constant scalar, $\phi(t)$ is a continuous vector function.
For the filtering purpose, we consider the following reduced-order filter:

$$\dot{\hat{z}}(t) = N\hat{z}(t) + Dy(t) \qquad (2)$$

where $\hat{z}$ is the estimate of z, and the constant matrices N and D are the filter matrices to be determined. Let $e(t) = z(t) - \hat{z}(t)$ be the filtering error vector. The problem is to find a filter of the form (2), that guarantees the asymptotic stability of the estimation errors, and ensures for a prescribed value of γ, that the following performance index is negative

$$J(w) = \int_0^\infty (e^T e - \gamma^2 w^T w)\, dt, \qquad \forall w(t) \in \mathcal{L}_2^q]0, \infty)$$

It follows from (1) and (2) that

$$\begin{aligned}\dot{e}(t) &= Ne(t) + (LA - DC - NL)x(t) \\ &\quad +(LA_d - DC_d)x(t-\tau) + (LB_w - DC_w)w(t).\end{aligned} \qquad (3)$$

Then using the Leibniz-Newton formula and some integral manipulations, equations (3) and (1.a) can be rewritten as follows

$$\begin{aligned}\dot{e}(t) = Ne(t) &+ \big(L(A + A_d) - D(C + C_d) - NL\big)x(t) \\ &- \int_{t-h}^{t} \big(LA_dA - DC_dA\big)x(s)\, ds \\ &+ \int_{t-h}^{t-\tau} (LA_dA - DC_dA)x(s)\, ds \\ &- \int_{t-\tau}^{t} (LA_dA_d - DC_dA_d)x(s-\tau)\, ds \\ &- \int_{t-\tau}^{t} (LA_dB_w - DC_dB_w)w(s)\, ds \\ &+ (LB_w - DC_w)w(t),\end{aligned} \qquad (4)$$

$$\begin{aligned}\dot{x}(t) = (A + A_d)x(t) &- \int_{t-h}^{t} A_dAx(s)\, ds \\ &+ \int_{t-h}^{t-\tau} A_dAx(s)\, ds - \int_{t-\tau}^{t} A_dA_dx(s-\tau)\, ds \\ &- \int_{t-\tau}^{t} A_dB_ww(s)\, ds + B_ww(t).\end{aligned} \qquad (5)$$

Thus we deduce from (4) and (5) the following augmented system:

$$\dot{x}_a(t) = \widehat{A}x_a(t) + \int_{t-h}^{t} \widehat{A}_dx_a(s)\, ds + F - \int_{t-\tau}^{t} \widehat{B}_ww(s)\, ds + \widetilde{B}_ww(t) \qquad (6.a)$$

$$z_a(t) = C_ax_a(t) \qquad (6.b)$$

where

$$x_a(t) = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix}, \qquad (7.a)$$

$$\widehat{A} = \begin{bmatrix} N & L(A + A_d) - D(C + C_d) - NL \\ 0 & A + A_d \end{bmatrix}, \qquad (7.b)$$

$$\widehat{A}_d = \begin{bmatrix} 0 & DC_dA - LA_dA \\ 0 & -A_dA \end{bmatrix}, \qquad (7.c)$$

$$\widehat{B}_w = \begin{bmatrix} LA_dB_w - DC_dB_w \\ A_dB_w \end{bmatrix}, \qquad (7.d)$$

$$\widetilde{B}_w = \begin{bmatrix} LB_w - DC_w \\ B_w \end{bmatrix}, \qquad (7.e)$$

$$C_a = \begin{bmatrix} I_r & 0 \end{bmatrix}, \qquad (7.f)$$

$$F = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix}, \qquad (7.g)$$

$$\begin{aligned}\Delta f_1 = &\int_{t-h}^{t-\tau} (LA_dA - DC_dA)x(s)\, ds \\ &- \int_{t-\tau}^{t} (LA_dA_d - DC_dA_d)x(s-\tau)\, ds,\end{aligned} \qquad (7.h)$$

$$\Delta f_2 = \int_{t-h}^{t-\tau} A_dAx(s)\, ds - \int_{t-\tau}^{t} A_dA_dx(s-\tau)\, ds. \qquad (7.i)$$

Let $x_{a_t}(s) = x_a(t+s)$, $\forall s \in [-2h, 0]$ and $\|x_{a_t}\|_r = sup_{-2h\le s\le 0}\|x_a(t+s)\|$.
The filtering problem then becomes one of finding the filter parameters, such that the asymptotic stability and $\mathcal{L}_2$ norm γ attenuation level of the system (6) are guaranteed.

III. MAIN RESULTS

In this section, the filtering problem as has been described above will be discussed and solved. Sufficient conditions to achieve this goal are presented in the following theorem in terms of linear matrix inequalities and scalar constraints.

Theorem III.1 *For a prescribed $\gamma > 0$, and any constant but unknown time delay, $0 \le \tau \le h$, where h is a known positive scalar, if there exist positive definite symmetric matrices $P_1 \in \mathbb{R}^{r\times r}$, $P_2 \in \mathbb{R}^{n\times n}$, and matrices $X \in \mathbb{R}^{r\times m}$, $Y \in \mathbb{R}^{r\times r}$, S_{12}, T_{12}, $U_{12} \in \mathbb{R}^{r\times n}$, and symmetric matrices S_{11}, T_{11}, $U_{11} \in \mathbb{R}^{r\times r}$, and S_{22}, T_{22}, $U_{22} \in \mathbb{R}^{n\times n}$ that satisfy the following four conditions*

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ \Phi_{21} & \Phi_{22} & * & * \\ 0 & \Phi_{32} & -h^{-1}S_{11} & * \\ 0 & -P_2A_dA & -h^{-1}S_{12}^T & -h^{-1}S_{22} \end{bmatrix} < 0, \qquad (8)$$

$$\begin{bmatrix} -\gamma^2 I_q & * & * & * & * \\ \Psi_{21} & -h^{-1}T_{11} & * & * & * \\ P_2A_dB_w & -h^{-1}T_{12}^T & -h^{-1}T_{22} & * & * \\ \Psi_{41} & 0 & 0 & -U_{11} & * \\ P_2B_w & 0 & 0 & -U_{12}^T & -U_{22} \end{bmatrix} < 0, \qquad (9)$$

$$\lambda_{min}(P) \neq h\lambda_{max}(W^TS^{-1}W), \qquad (10)$$

$$\lambda_{min}(Q) - 2h\alpha\lambda_{max}(P)(\|LA_dA - DC_dA\| + \|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) > 0, \quad (11)$$

where

$$Q > 0, \quad (12)$$

with

$$\begin{aligned}
Q &= -\Big(V^T + V + h(W^TS^{-1}W + S + T) \\
&\quad + C_a^TC_a + U\Big), && (13)\\
\Phi_{11} &= Y^T + Y + h(S_{11} + T_{11}) + U_{11} + I_r, && (14)\\
\Phi_{21} &= (A + A_d)^TL^TP_1 - (C + C_d)^TX^T - L^TY^T \\
&\quad + h(S_{12}^T + T_{12}^T) + U_{12}^T, && (15)\\
\Phi_{22} &= (A + A_d)^TP_2 + P_2(A + A_d) + h(S_{22} + T_{22}) \\
&\quad + U_{22}, && (16)\\
\Phi_{32} &= XC_dA - P_1LA_dA && (17)\\
\Psi_{21} &= P_1LA_dB_w - XC_dB_w && (18)\\
\Psi_{41} &= P_1LB_w - XC_w && (19)\\
\alpha &= \Big|\frac{\lambda_{max}(P)}{\lambda_{min}(P) - h\lambda_{max}(W^TS^{-1}W)}\Big|^{1/2}, && (20)\\
P &= \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, && (21)\\
V &= \begin{bmatrix} Y & P_1L(A + A_d) - X(C + C_d) - YL \\ 0 & P_2(A + A_2) \end{bmatrix}, && (22)\\
W &= \begin{bmatrix} 0 & XC_dA - P_1LA_dA \\ 0 & -P_2A_dA \end{bmatrix}, && (23)
\end{aligned}$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix}. \quad (24)$$

Then system (6) is asymptotically stable and $\|H_{z_aw}(s)\|_\infty \leq \gamma$. Moreover the matrices N and D of the filter (2) are given by

$$\begin{aligned}
N &= P_1^{-1}Y, && (25.a)\\
D &= P_1^{-1}X. && (25.b)
\end{aligned}$$

Remark III.2 *Thanks to the Schur Complement lemma, the existence of Q satisfying* (12) *and* (13) *is guaranteed if the LMI* (8) *holds.*

Proof of Theorem 3.1: For more clarity, the proof will be presented into two parts. In the first one we deal with the asymptotic stability problem of system (6), while in the second part, the H_∞ performance will be tackled. At the outset we assume that all the conditions of theorem III.1 are satisfied and without loss of generality, let $\phi(t) = \hat{z}(t) = w(t) = 0, \quad \forall t \in [-h, 0]$, thus we have $x_a(t) = w(t) = 0, \quad \forall t \in [-h, 0]$. It is worth noting that for the asymptotic stability analysis, the Razumikhin stability theorem (see [14]) will be used. We adopt the following Lyapunov-Rasumikhin function candidate:

$$V(t, x_a(t)) = V_1(t, x_a(t)) + V_2(t, x_a(t)), \quad (26)$$

$$V_1(t, x_a(t)) = x_a^T(t)Px_a(t) \quad (27)$$

$$V_2(t, x_a(t)) = \int_{-h}^{0}\int_{t+s}^{t} x_a^T(\theta)\widehat{A}_d^TPS^{-1}P\widehat{A}_dx_a(\theta)\,\mathrm{d}\theta\,\mathrm{d}s, \quad (28)$$

where P and S are given in (21) and (24). It is obvious that there exist η_1 and η_2 such that

$$\eta_1\|x_a\|^2 \leq V(t, x_a(t)) \leq \eta_2\|x_{a_t}\|_r^2,$$

for instance, take $\eta_1 = \lambda_{min}(P)$ and $\eta_2 = \lambda_{max}(P) + h\lambda_{max}(\widehat{A}_d^TPS^{-1}P\widehat{A}_d)$.
On the other hand, differentiating (26) with respect to t along the trajectories of (6.a), we obtain:

$$\begin{aligned}
\dot{V}(t, x_a(t)) &= x_a^T(t)(\widehat{A}^TP + P\widehat{A} + h\widehat{A}_d^TPS^{-1}P\widehat{A}_d)x_a(t) \\
&+ 2\int_{t-h}^{t} x_a^T(t)P\widehat{A}_dx_a(s)\,ds + 2x_a^T(t)PF && (29)\\
&- \int_{t-h}^{t} x_a^T(s)\widehat{A}_d^TPS^{-1}P\widehat{A}_dx_a(s)\,ds && (30)\\
&- 2\int_{t-\tau}^{t} x_a^T(t)P\widehat{B}_ww(s)\,ds + 2x_a^T(t)P\widetilde{B}_ww(t), && (31)
\end{aligned}$$

and it is readily checked that the following inequality

$$2x_a^T(t)P\widehat{A}_dx_a(s) \leq x_a^T(t)Sx_a(t) + x_a^T(s)\widehat{A}_d^TPS^{-1}P\widehat{A}_dx_a(s), \quad (32)$$

holds. On the other hand, using (7.g), (7.h) and (7.i), it is straightforward to see that

$$2x_a^T(t)PF \leq 2h\lambda_{max}(P)(\|LA_dA - DC_dA\| + \|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|)\|x_a\|\|x_{a_t}\|_r. \quad (33)$$

On the other hand, for the linearization purpose, remark that in view of (7.b), (7.c), (21) and (25) we have

$$\begin{aligned}
P\widehat{A} &= V && (34)\\
P\widehat{A}_d &= W && (35)
\end{aligned}$$

where V and W are given in (22) and (23). Therefore by combining (31), (32), (33), (34) and (35), we deduce the following bound of $\dot{V}(t, x_a(t))$:

$$\begin{aligned}
\dot{V}(t, x_a(t)) &\leq x_a^T(t)(V^T + V + hW^TS^{-1}W + hS)x_a(t) \\
&+ 2h\lambda_{max}(P)(\|LA_dA - DC_dA\| \\
&+ \|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|)\|x_a\|\|x_{a_t}\|_r \\
&- 2\int_{t-\tau}^{t} x_a^T(t)P\widehat{B}_ww(s)\,ds + 2x_a^T(t)P\widetilde{B}_ww(t). && (36)
\end{aligned}$$

Remark III.3 *Our aim can be summarized as follows, find $q > 1$ and $\mu > 0$ such that*

$$\dot{V}(t, x_a(t)) \leq -\mu\|x_a(t)\|^2 \quad \textit{if} \quad V(\psi, x_a(\psi)) < qV(t, x_a(t)), \quad \forall\psi \in [t - 2h, t],$$

indeed, according to Razumikhin theorem see [14] for more detail, the uniform asymptotic stability of the equilibrium state $x_a(t) \equiv 0$ is then guaranteed.

Assume that $V(\psi, x_a(\psi)) < qV(t, x_a(t)), \;\; \forall \psi \in [t-2h, t]$, then by using standard norm inequalities, and under the following condition

$$\lambda_{min}(P) \neq qh\lambda_{max}(W^T S^{-1} W), \tag{37}$$

one can deduce that

$$\|x_{a_t}\|_r \leq \beta(q)\|x_a\|, \tag{38}$$

where

$$\beta(q) = \Big| \frac{q\lambda_{max}(P)}{\lambda_{min}(P) - qh\lambda_{max}(W^T S^{-1} W)} \Big|^{1/2}.$$

where $|.|$ denotes the absolute value operator. On the other hand, due to (13), we deduce the existence of the matrix

$$R = Q + C_a^T C_a + hT + U > 0, \tag{39}$$

such that

$$-R = V^T + V + hW^T S^{-1} W + hS. \tag{40}$$

Then combining inequalities (36), (38) and equality (40), the inequality (36) implies that

$$\begin{array}{l} \dot{V}(t, x_a(t)) \quad \leq \\ -(\lambda_{min}(R) - 2h\beta(q)\lambda_{max}(P)(\|LA_dA - DC_dA\| \\ +\|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|))\|x_a\|^2 \\ -2\int_{t-\tau}^{t} x_a^T(t)P\widehat{B}_w w(s)\, ds + 2x_a^T(t)P\widetilde{B}_w w(t), \end{array} \tag{41}$$

and then from the above inequality and under condition (37), we can deduce that for $w(t) = 0$, the asymptotic stability of the augmented system (6) is guaranteed if the following condition holds

$$\begin{array}{l} f(q) = \lambda_{min}(R) - 2h\beta(q)\lambda_{max}(P)(\|LA_dA - DC_dA\| \\ +\|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) > 0. \end{array} \tag{42}$$

However by using the two following facts: first due to condition (10), $f(q)$ is continue at the point $q = 1$. On the other hand in view of (39) and condition (11), we can deduce that

$$\begin{array}{l} f(1) = \lambda_{min}(R) - 2h\alpha\lambda_{max}(P)(\|LA_dA - DC_dA\| \\ +\|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) > 0, \end{array} \tag{43}$$

where α is defined in (20). Then the existence of $\xi > 0$, as small as it can be, is guaranteed such that for $q = (1 + \xi)^2$ the condition (42) holds. Furthermore it is noticed that for ξ close to zero, the conditions (10) and (37) are equivalent. In other words sufficient conditions to satisfy (42) are conditions (43) and (10). However sufficient condition to satisfy (43) is that condition (11) holds. Then to summarize for $w(t) = 0$, if $V(\psi, x_a(\psi)) < qV(t, x_a(t)), \;\; \forall \psi \in [t - 2h, t]$, where $q = (1 + \xi)^2$ and $\xi > 0$ close to zero, then conditions (10) and (11) guarantee the existence of

$$\begin{array}{l} \mu = \lambda_{min}(Q) - 2h\alpha\lambda_{max}(P)(\|LA_dA - DC_dA\| \\ +\|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) > 0 \end{array}$$

such that $\dot{V}(t, x_a(t)) \leq -\mu\|x_a(t)\|^2$. On the other hand it is noticed that a necessary condition to satisfy (11) is that (12) holds, which in view of Remark III.2 is guaranteed if LMI (8) is satisfied. Therefore the asymptotic stability of system (6) is guaranteed by conditions (8), (10) and (11). This completes the first part of the proof. ▲

In the second part, the purpose is to check that $\|H_{z_a w}(s)\|_\infty \leq \gamma$, $\forall w(t) \in \mathcal{L}_2^q[0, \infty)$, where γ is the pre-specified upper bound, is guaranteed under the criteria given in Theorem III.1. In this situation, the filtering error system is said to have a guaranteed γ level of disturbance attenuation. This problem can be formulated also as follows, find a criterion that will ensure that the following performance index is negative

$$\int_0^\infty (z_a^T(t)z_a(t) - \gamma^2 w^T(t)w(t))\, dt, \quad \forall w(t) \in \mathcal{L}_2^q[0, \infty] \tag{44}$$

Let

$$\widehat{V}(t, w(t)) = \int_{-\tau}^{0}\int_{t+s}^{t} w^T(\theta)\widehat{B}_w^T P T^{-1} P\widehat{B}_w w(\theta)\, \mathrm{d}\theta\, \mathrm{d}s, \tag{45}$$

where T is defined in (24). Since $x_a(t) = w(t) = 0, \;\; \forall t \in [-h, 0]$, we have $V(t, x_a(t)) = \widehat{V}(t, w(t)) = 0, \;\; \forall t \in [-h, 0]$, and it is obvious to see that $V(\delta, x_a(\delta)) > 0$ and $\widehat{V}(\delta, w(\delta)) > 0$. Then we can deduce that

$$\begin{array}{l} \int_0^\delta (z_a^T(t)z_a(t) - \gamma^2 w^T(t)w(t))\, dt \\ \quad \leq \int_0^\delta (x_a^T(t)C_a^T C_a x_a(t) - \gamma^2 w^T(t)w(t) \\ \qquad +\dot{V}(t, x_a(t)) + \dot{\widehat{V}}(t, w(t)))\, dt. \end{array} \tag{46}$$

Therefore combining (36), (38) and the two following inequalities

$$\begin{array}{l} -2x_a^T(t)P\widehat{B}_w w(s) \\ \quad \leq x_a^T(t)Tx_a(t) + w^T(s)\widehat{B}_w^T P T^{-1} P\widehat{B}_w w(s), \end{array} \tag{47}$$

$$\begin{array}{l} 2x_a^T(t)P\widetilde{B}_w w(t) \\ \quad \leq x_a^T(t)Ux_a(t) + w^T(t)\widetilde{B}_w^T P U^{-1} P\widetilde{B}_w w(t), \end{array} \tag{48}$$

where T and U are defined in (24), and using some standard algebraic manipulations, we deduce the following inequality

$$\begin{array}{l} \int_0^\delta (z_a^T(t)z_a(t) - \gamma^2 w^T(t)w(t))\, dt \leq \\ \int_0^\delta -(\lambda_{min}(Q) - 2h\beta(q)\lambda_{max}(P)(\|LA_dA - DC_dA\| + \\ \|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|))\|x_a\|^2 + \\ w^T(t)(h\widehat{B}_w^T P T^{-1} P\widehat{B}_w + \widetilde{B}_w^T P U^{-1} P\widetilde{B}_w - \gamma^2 I_q)w(t))\, dt \end{array} \tag{49}$$

where Q is defined in (13).

Therefore, the performance index (44) is negative if the following conditions hold

$$\begin{array}{l} \lambda_{min}(Q) - 2h\beta(q)\lambda_{max}(P)(\|LA_dA - DC_dA\| \\ +\|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) > 0, \end{array} \tag{50}$$

$$h\widehat{B}_w^T P T^{-1} P\widehat{B}_w + \widetilde{B}_w^T P U^{-1} P\widetilde{B}_w - \gamma^2 I_q < 0. \tag{51}$$

Now, following the same reasoning given in the first part of the proof, condition (11) implies (50). On the other hand,

substitution of $\widehat{B}_w$, $\widetilde{B}_w$ and P, as defined in (7.d), (7.e) and (21) respectively, and D as defined in (25.b), in the inequality (51) leads after applying the Schur complement lemma to LMI (9).
This completes the proof of Theorem III.1. ■

Remark III.4 *The proposed approach is based on the fact that we isolate and norm bounding the further terms, and thereafter we check if the minimal eigenvalue of the remaining term is great than the obtained bound. Then the proposed method may be applicable even in the case of systems with uncertain parameters, just by including the uncertain terms to the further terms.*

Remark III.5 *It is worth noting that theorem III.1 has two conditions* (10) *and* (11) *that must be verified a posteriori. This is why there is some systems for which our method can not be applied. However, in the case where the conditions mentioned above fail, we try to perturb the variables of the* ($LMIs$) (8) *and* (9) *by restricting their field of evolution, in order to force the solver of the* LMI *to search others feasible solutions.*

Hereafter we give the upper bound of the unknown delays without affecting the stability of the filtering process.

Property III.6 *The asymptotic stability of the augmented systems* (6) *is preserved for all* $\tau \leq \tau_{max}$ *where*

$$\tau_{max} = \frac{-\Phi^2 \lambda_{max}(W^T S^{-1} W) + \sqrt{AA}}{8\lambda_{max}(P)}, \quad (52)$$

with

$$\Phi = \frac{\lambda_{min}(Q)}{BB}. \quad (53)$$

$$AA = \Phi^4 \lambda_{max}^2(W^T S^{-1} W) + 16\Phi^2 \lambda_{min}(P)\lambda_{max}(P) \quad (54)$$

$$BB = \lambda_{max}(P)(\|LA_dA - DC_dA\| + \|LA_dA_d - DC_dA_d\| + \|A_dA\| + \|A_dA_d\|) \quad (55)$$

Proof: From the inequality (11), and by using some technical arrangements, we obtain

$$\frac{4h^2 \lambda_{max}(P)}{|\lambda_{min}(P) - h\lambda_{max}(W^T S^{-1} W)|} < \Phi^2, \quad (56)$$

which leads after development to the two following conditions

$$\left\{ \begin{array}{l} 4h^2 \lambda_{max}(P) + h\Phi^2 \lambda_{max}(W^T S^{-1} W) \\ \qquad -\Phi^2 \lambda_{min}(P) < 0 \\ 4h^2 \lambda_{max}(P) - h\Phi^2 \lambda_{max}(W^T S^{-1} W) \\ \qquad +\Phi^2 \lambda_{min}(P) < 0 \end{array} \right. . \quad (57)$$

Hence we have two inequalities of second order with respect to h. Thus the study of the interval of evolution of h leads to the result given in the property, namely all the solutions of (56) are less than τ_{max} as defined in (52)–(53). Therefore τ_{max} represents the upper bound of arbitrary unknown delays, while preserving the asymptotic stability of the filtering process. ■

IV. Illustrative Example

In this section, we provide an example to illustrate the performances of the obtained results.
Consider the following system with an unknown time delay described as

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -5 & 0 \\ 1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(t-\tau) \\ &\quad + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 5 & 1 \end{bmatrix} x(t-\tau) \\ &\quad + \begin{bmatrix} 1 & 1 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

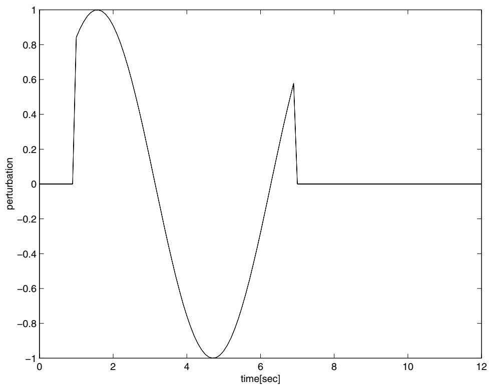
We focus on the design of an H_∞ filter of structure (2) such that the filtering process remains asymptotically stable, and the transfer function from the exogenous disturbances to the estimation errors does not exceed a certain pre-specified H_∞-norm upper bound value, namely in this example we try to ensure that $\|H_{z_a w}(s)\| \leq \gamma_{min} = 0.43$, $\forall w(t) \in \mathcal{L}_2^q[0,\infty)$. Solving the $LMIs$ (8), (9) under the constraints (10) and (11), we obtain the following solutions

$$P_1 = 0.2031, \quad P_2 = \begin{bmatrix} 0.1428 & 0.0188 \\ 0.0188 & 0.1970 \end{bmatrix},$$
$$S_{11} = 0.2369, \quad S_{12} = \begin{bmatrix} -0.0181 & -0.0028 \end{bmatrix},$$
$$S_{22} = \begin{bmatrix} 0.2884 & -0.0117 \\ -0.0117 & 0.2698 \end{bmatrix}, \quad T_{11} = 0.2368,$$
$$T_{12} = \begin{bmatrix} -0.0190 & -0.0024 \end{bmatrix},$$
$$T_{22} = \begin{bmatrix} 0.2854 & -0.0088 \\ -0.0088 & 0.2645 \end{bmatrix},$$
$$U_{11} = 0.4569, \quad U_{12} = \begin{bmatrix} 0.0918 & 0.0206 \end{bmatrix},$$
$$U_{22} = \begin{bmatrix} 0.4565 & 0.0102 \\ 0.0102 & 0.4545 \end{bmatrix},$$
$$X = 0.0036, \quad Y = -1.0278.$$

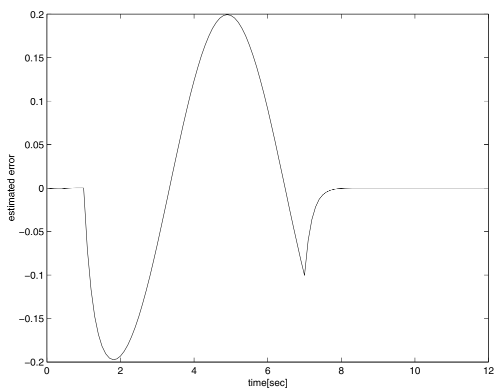
Thus we deduce the parameters of our filter (2)

$$N = -5.0597, \quad D = 0.0178.$$

The maximal allowable delay obtained after solving our conditions is $\tau_{max} = 0.638$, and using the result of the property (III.6) we obtain the same result $\tau_{max} = 0.63816$. To check the pre-specified attenuation level, we calculate and obtain $\|e\|/\|w\| = 0.13829 < \gamma$, thus the H_∞ upper bound constraint is satisfied. For the simulation purpose, the choice of the unknown delay was arbitrary but less than $\tau_{max} = 0.638$. For instance for $\tau = 0.4$, the obtained results are depicted in figure 1 and 2. It is worth noting that the choice of the perturbation as is depicted in Figure 1 was arbitrary in the $\mathcal{L}_2^q[0,\infty)$ space. The estimation error between the true trajectories and the estimated trajectories of the system under study are depicted in figure 2.



(a) The Perturbation



(b) The estimated error

V. Conclusion

In this note the filtering problem for linear systems with an unknown delay in the states and outputs has been tackled. The delay is assumed to be unknown. This represents a broad class of delayed systems that we encounter in several practical situations. To deal with, a reduced order H_∞ filter has been introduced. Both asymptotic stability and disturbance attenuation problems have been discussed in detail. Sufficient conditions have been derived to solve these problems, and an upper bound of arbitrary time delays without affecting the asymptotic stability property has been derived. Finally the effectiveness and usefulness of the proposed approach have been shown trough a numerical example.

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