2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Loop Shaping for Robust Performance Using Rbode Plot

Lu Xia, William Messner

***Abstract*— H_∞ and μ-synthesis have been widely used to design controllers to achieve robust performance. However, these automated tools often do not provide much insight into the relationship between the open loop frequency response and the performance in the presence of model uncertainties. This paper presents a method for translating the robust performance criterion into contours on the open loop Bode plot of a compensated SISO system so that robust controllers can be directly synthesized with classical loop shaping. The design concept is to shape the open loop response to avoid intersections between the contours and magnitude and phase graphs. In contrast to the asymptotic criteria in the literature, these contours incorporate both phase and magnitude information, and provide precise bounds in the vicinity of the 0 dB crossover frequency. An application of this concept to the design of the controller for an Lateral Tape Motion compensation system shows its utility.**

I. INTRODUCTION

Automated design tools H_∞ and μ-synthesis have been widely used to assure both stability and performance in closed loop systems subject to plant uncertainties [1]. However, these techniques do not provide much insight into the relationship between the open loop system response and the closed-loop performance. Designs resulting from H_∞ or μ-synthesis may be more conservative than necessary because the frequency ranges over which the open loop does not meet the conditions for assuring robustness, and the extent to which the open loop exceeds these conditions may not be clear. Thus the designer's capability for tuning the open loop by selection of weighting functions to meet these conditions is limited. Loop shaping methodologies [2][3] address performance and uncertainties by providing asymptotic bounds on the magnitude response at low and/or high frequencies. However the relationship between open loop response and closed loop performance and robustness remains unclear near the open loop 0dB crossover frequency, i.e. the mid-frequency range. This paper develops modifications of the Bode plots by adding contours representing robust performance bounds to the open loop magnitude and phase plots. These contours provide the insight between the open loop frequency response and closed loop robust performance that has heretofore been lacking.

Consider the feedback system in Fig. 1. The problem of interest is to synthesize a controller $C(s)$ for the uncertain plant $\tilde{P}(s)$ so that the closed-loop system achieves specified performance in presence of disturbance d and noise n.

The authors are with the Data Storage Systems Center, Carnegie Mellon University, Pittsburgh, PA 15213 lxia,bmessner@andrew.cmu.edu

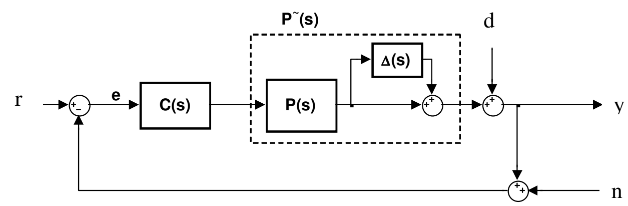


Fig. 1. Block diagram with multiplicative model uncertainty and reference, disturbance, and noise inputs

The block diagram shows a nominal plant $P(s)$ that is disturbed by multiplicative uncertainty $\Delta(s)$ which is an unknown but stable transfer function. The relationship between the three transfer functions is

$$\tilde{P}(s) = P(s)(1 + \Delta(s)), \tag{1}$$

where $|\Delta(j\omega)| < |W_u(j\omega)|, \ \forall\omega$. The weighting function $W_u(s)$ is a proper, stable and minimum phase transfer function satisfying

$$\left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \le |W_u(j\omega)|, \ \forall\omega, \tag{2}$$

where the magnitude response of $W_u(j\omega)$ is an upper bound of the magnitude of the model uncertainty $\Delta(s)$.

By convention, the nominal open loop transfer function, sensitivity function, and complementary sensitivity function are

$$\begin{aligned} L(s) &\equiv C(s)P(s) \\ S(s) &\equiv \frac{1}{1+L(s)} \\ T(s) &\equiv \frac{L(s)}{1+L(s)}. \end{aligned} \tag{3}$$

The desired performance of the system in this case is specified through a weighting function $W_s(s)$ on the sensitivity function $S(s)$ such that

$$|S(j\omega)| < |W_s(j\omega)|^{-1}, \forall\omega \tag{4}$$

If the nominal loop function $L(s)$ is stable, a necessary and sufficient condition for a controller to achieve robust stability and robust performance is [3]

$$\| \ |W_u(s)T(s)| + |W_s(s)S(s)| \ \|_\infty < 1. \tag{5}$$

In most applications, system characteristics are harder to capture at higher frequency, and the uncertainty weight $W_u(s)$ takes the form of a high pass filter. On the other hand, satisfactory tracking performance requires significant disturbance rejection at low frequency disturbances, and so $W_s(s)$ has the form of a low pass filter. These facts imply asymptotic magnitude bounds for the open loop gain $|L(j\omega)|$ for different frequency ranges. Over these "low" and "high" frequency ranges, one weighting function dominates the bound (5) on $L(j\omega)$. Fig. 2 shows an example of such asymptotic bounds.

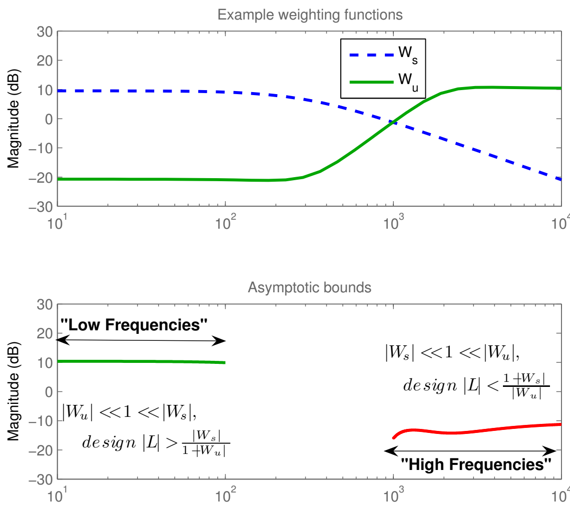


Fig. 2. Example of weighting functions $W_u(s)$ and $W_s(s)$, and the corresponding asymptotic bounds

These bounds provide good design guides for robust controller design at low or high frequencies. However, for intermediate frequencies, the relationship of the open-loop frequency to robust performance remains unclear. It is often necessary to check the closed condition (5) after the loop shaping and then to readjust the controller iteratively.

This paper presents the idea of translating robust performance criterion into contours on the open-loop Bode plot of the nominal system. We call this modified Bode Plot the "Rbode" Plot where the "R" refers to "robust". With the Rbode plot, the classical loop shaping can directly synthesize a robust controller by avoiding intersections with the contours. The Rbode concept is very similar to the Sbode plot in [4] and the fsbode in [5]. The difference is that the contours of the Sbode and fsbode plots represent the specific sensitivity magnitudes, while the Rbode contours partition the conventional Bode plots into regions that do and do not meet specific robust performance criteria.

The paper is organized as follows: Section II derives the Rbode plot contour functions and discusses the conditions on the two weighting functions when robust controller exists. Section III shows a design example for active tape steering system using Rbode plot. Concluding remarks appear in Section IV.

II. DERIVATION OF RBODE PLOTS

Define the robust level R as a function of the two weighting functions and the open loop frequency response:

$$R(j\omega, W_u, W_s, L) \equiv |W_u(j\omega)T(j\omega)| + |W_s(j\omega)S(j\omega)|$$

$$= \frac{|W_u(j\omega)L(j\omega)| + |W_s(j\omega)|}{|1+L(j\omega)|} \quad (6)$$

Inequality (5) is equivalent to $R(j\omega, W_u, W_s, L) < 1$, for all frequencies ω, which is the necessary and sufficient condition that closed loop achieves robust performance.

The idea behind robust contours is to determine the mapping from open-loop frequency response to robust performance level $R(j\omega)$ by incorporating the open-loop phase response, which is missing in (5). The derivation that follows suppresses parameter $j\omega$ for notational convenience. The derivation begins with the relationship

$$\begin{aligned} |1+L|^2 &= (1+|L|cos(arg(L)))^2 + (|L|sin(arg(L)))^2 \\ &= 1 + 2|L|cos(arg(L)) + |L|^2 \end{aligned} \quad (7)$$

and

$$R(W_u, W_s, L) = \frac{|W_u L| + |W_s|}{\sqrt{1+2|L|cos(arg(L))+|L|^2}} \quad (8)$$

The Rbode contours represent relations between the open-loop magnitude and phase and the weighting functions W_u and W_s.

A. Magnitude response contours

To derive the magnitude contours, square both sides of $R(W_u, W_s, L) < 1$, and then substitute the relation (8). Bringing all function to the left hand side results in the following inequality:

$$\begin{aligned} (1-|W_u|^2)|L|^2 &+ 2(cos(arg(L)) - |W_u||W_s|)|L| \\ &+ 1 - |W_s|^2 > 0 \end{aligned} \quad (9)$$

Solving inequality 9 determines the bounds on the open loop magnitude $|L|$ as a function of the open loop phase $arg(L)$ and the weighting functions. Define roots of the left side of (9)

$$\begin{aligned} \lambda_1 &= \frac{|W_u||W_s| - cos(arg(L)) - \sqrt{D}}{1-|W_u|^2} \\ \lambda_2 &= \frac{|W_u||W_s| - cos(arg(L)) + \sqrt{D}}{1-|W_u|^2} \end{aligned}$$

$$\begin{aligned} where\ D = &(cos(arg(L)) - |W_u||W_s|)^2 \\ &-(1-|W_u|^2)(1-|W_s|^2) \end{aligned} \quad (10)$$

Note that λ_1 and λ_2 are functions of $j\omega$. $|L|$ can satisfy inequality (9) depending on the nature of λ_1 and λ_2 and the sign of the lead coefficient $1-|W_u|^2$. Consider the following cases:

1) Frequencies where $1-|W_u|^2 > 0$, i.e. $|W_u| < 1$. These are the frequencies where there is high confidence in the model.
   a) When $D < 0$, λ_1 and λ_2 are both complex, and any positive value of $|L|$ satisfies the inequality (9).
   b) When $\lambda_2 \geq \lambda_1 \geq 0$, $|L|$ satisfies inequality (9) when

$$|L| < \lambda_1,\ or\ |L| > \lambda_2 \quad (11)$$

c) When $\lambda_2 \geq 0 \geq \lambda_1$, $|L|$ satisfies inequality (9) when

$$|L| > \lambda_2 \tag{12}$$

Note this case only happens where $|W_s| > 1$ and $|W_u| < 1$, which means disturbance rejection is required at frequencies where there is high confidence in the model.

d) When $0 > \lambda_2 \geq \lambda_1$, any $|L| > 0$ satisfies inequality (9).

2) Frequencies where $1 - |W_u|^2 > 0$, i.e. $|W_u| > 1$. These are the frequencies where there is low confidence in the model $P(s)$:

a) When $|W_s| > 1$, no possible values of $|L|$ satisfies the inequality (9) since all three terms on the left hand side of (9) are always negative. This implies requirement for disturbance rejection at frequencies where there is low confidence in the model. There is no compensator design that can achieve robust performance for such a combination of weighting functions.

b) When $|W_s| \leq 1$, and $\lambda_1 > 0 \geq \lambda_2$ since $\lambda_1\lambda_2 = \frac{1-|W_s|^2}{1-|W_u|^2} \leq 0$. $|L|$ satisfies inequality (9) when

$$0 \leq |L| < \lambda_1 \tag{13}$$

3) Frequencies where $1 - |W_u|^2 = 0$

a) When $cos(arg(L)) - |W_u||W_s| > 0$, $|L|$ satisfies (9) when

$$|L| > \lambda \equiv \frac{|W_s|^2 - 1}{2(cos(arg(L) - |W_u W_s|)} \tag{14}$$

b) When $cos(arg(L) - |W_u||W_s| < 0$, $|L|$ satisfies (9) when

$$0 < |L| < \lambda \equiv \frac{|W_s|^2 - 1}{2(cos(arg(L) - |W_u||W_s|)} \tag{15}$$

c) When $cos(arg(L)) - |W_u W_s| = 0$, $|L|$ always satisfies (9) when $1 - |W_s|^2 > 0$, and never satisfies (9) when $1 - |W_s|^2 \leq 0$.

These cases lead to the definition of the magnitude bounds. Define the lower bound $M_l(j\omega)$ and upper bound $M_u(j\omega)$ of Rbode contours as follows:

$$M_l(j\omega) = \begin{cases} \lambda_2(j\omega) & \text{for case } 1.b \\ \lambda_2(j\omega) & \text{for case } 1.c \\ \lambda(j\omega) & \text{for case } 3.a \text{ and } \lambda > 0 \end{cases} \tag{16}$$

$$M_u(j\omega) = \begin{cases} \lambda_1(j\omega) & \text{for case } 1.b \\ \lambda_1(j\omega) & \text{for case } 2.b \\ \lambda(j\omega) & \text{for case } 3.b \text{ and } \lambda > 0 \end{cases} \tag{17}$$

Note that M_u and M_l are functions only of $arg(L)$ and of the two weighting functions, both of which are fixed in the design process. Multiplying controller $C(s)$ by a positive constant does not alter the contours. By graphing M_u and M_l on the same axes as the open loop Bode magnitude plot one can determine whether the closed loop system will achieve robust performance. The closed loop will achieve robust performance if and only if

1) For those frequencies where both M_u and M_l are defined,
   a) The graph of $|L|$ lies between M_u and M_l when $M_u > M_l$
   b) The graph of $|L|$ lies below M_u or above M_l when $M_u < M_l$;

   and
2) For frequencies where only M_u exists, the graph of $|L|$ lies below M_u; and
3) For frequencies where only M_l exists, the graph of $|L|$ lies above M_l; and
4) For frequencies where neither M_u nor M_l is defined, any positive value of $|L|$ satisfies (9).

Any intersection between $|L|$ and M_u or M_l indicates that $|L|$ *does not* satisfy inequality (9). Fig. 6 shows an example of M_u and M_l where the system does not achieve robust performance. Note the intersections between the contours and $|L|$ around 1000 rad/s. Fig. 8 shows an example of M_u and M_l where $|L|$ does satisfy (9).

B. Phase response contours

Development of the phase contours is easier. Solving inequality (9) for $cos(arg(L))$ gives

$$cos[arg(L)] > \frac{|W_s|^2 - 1 + 2|W_s W_u||L| + (|W_u|^2 - 1)|L|^2}{2|L|} \tag{18}$$

Consider the following cases for inequality (18) where RHS stands for *right hand side of inequality* (18):

1) Frequencies where $RHS < -1$. Any real phase value can satisfy (18);
2) Frequencies where $RHS < 1$. The open loop phase can satisfy (18) if there exist an integer k such that

$$\phi_{u,k} \equiv 360^\circ k - \phi > arg(L) > \phi_{l,k} \equiv 360^\circ k + \phi, \tag{19}$$

where

$$\phi = arccos\left(\frac{(|W_s| + |W_u||L|)^2 - 1 - |L|^2}{2|L|}\right) \tag{20}$$

3) $RHS > 1$, no real phase value can satisfy (18).

As with the magnitude contours, graphing the functions ϕ_u and ϕ_l for $k = 0, \pm1, \pm2, \cdots$ on the Bode phase plot of the open loop $L(j\omega)$ can determine whether the closed loop system achieves robust performance. The closed loop system achieves robust performance if and only if

1) At frequencies where $|RHS| < 1$, there exists at least one integer k such that $\phi_{u,k} > arg(L(j\omega)) > \phi_{l,k}$.
2) There are no frequencies at which $RHS \geq 1$.

Any intersection between the phase contour and the open loop phase indicates that $arg(L)$ does not satisfy (18). Fig. 6 shows an example where the phase response intersects the phase contour at 40 rads/s and approximately 1000 rad/s. Note that these intersection occur at the same frequencies as intersection on the magnitude plot. Fig. 8 shows an example where the phase response does satisfy robust performance.

Furthermore, if $R(j\omega, W_u, W_s, L) < \alpha,\ \forall\omega$, where $\alpha > 0$, we say that the robust performance is achieved at level α. For robust performance of level α,

$$R(j\omega, W_u, W_s, L) < \alpha \Leftrightarrow \frac{1}{\alpha}R(j\omega, W_u, W_s, L) < 1$$
$$\Leftrightarrow R(j\omega, \frac{1}{\alpha}W_u, \frac{1}{\alpha}W_s, L) < 1 \qquad (21)$$

That is, robust performance of level α with weighting functions $W_u(s)$ and $W_s(s)$ is equivalent to robust performance of level 1 with weighting functions $\frac{1}{\alpha}W_u(s)$ and $\frac{1}{\alpha}W_s(s)$. With this relationship, it's straightforward to plot multiple contours on the Rbode plots to show the robustness level of the design.

III. APPLICATION

This section demonstrates how to use the Rbode plots in the loop shaping process to synthesize a robust controller.

A. System description and modeling

Tape drives are subject to lateral tape motion (LTM) caused by tape stacking imperfections eccentricity of the supply reel. LTM causes damage to tape edges when tape strikes flanges on the guides or on the reels themselves, and LTM has become a limiting factor in the development of high density, high performance tape drive. Researchers at Carnegie Mellon University have proposed to actively steer one or more tape guides on the tape transport path to compensate for LTM. To accurately simulate the tape steering dynamics, a model containing over 200 states is required. This model is unwieldy for controller design. To facilitate the design process a simplified model is extracted by matching the frequency response of the lower order model to data from the high order model. The missing higher-order dynamics are characterized as modeling uncertainties.

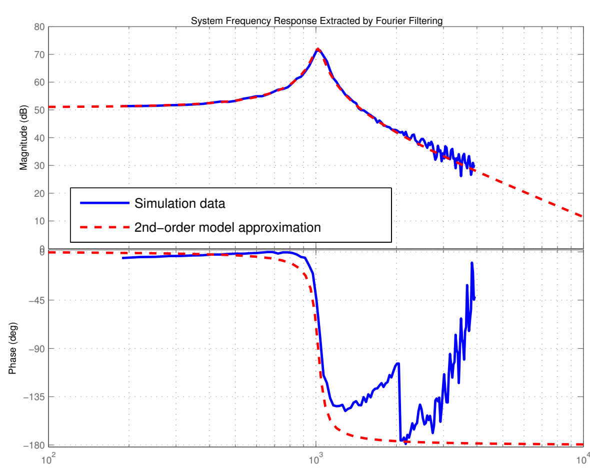


Fig. 3. Simulation data and nominal model

Fig. 3 shows the simulated frequency response of from guide tilting angle to LTM from the high order model. A second-order model

$$P(s) = \frac{3.693e8}{s^2 + 90.52s + 1.04e6} \qquad (22)$$

with resonance frequency $\omega_n = 1020 rads/s$ and damping ratio $\zeta = 0.044$ matches the higher order model magnitude response quite well below 2000 rad/s, and the phase match is also good for frequencies up to around 1000 rad/s.

The second-order transfer function

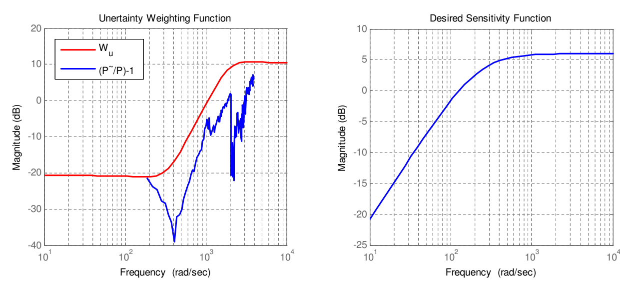
$$W_u(s) = 3.2882\frac{s^2 + 384.6s + 1.07e5}{s^2 + 2300s + 3.826e6} \qquad (23)$$

captures the mismatch between the simulation data and the lower-order model(Fig. 4(a)) and is used as the uncertainty weighting function. On the other hand, low damping ratios are normally associated with high uncertainties. The weight $W_u(s)$ also serves as an upper bound of model uncertainties when the damping ratio varies between 0.024 to 0.18 or when the resonance frequency varies from $985 rads/s$ to $1060 rads/s$.

Fig. 4(b) shows the desired sensitivity weighting function,

$$W_s(s) = 0.5 + \frac{110}{s} \qquad (24)$$

as the most problematic disturbances occur at relatively low frequencies.



(a) Uncertainty $W_u(s)$ (b) Desired S(s)= $1/W_s(s)$

Fig. 4. Weighting functions

B. Use Rbode to synthesize a robust controller

Fig. 5 shows the initial Rbode plot of the plant with no compensation. The open-loop magnitude response intersects the lower bound contour (dashed curve) near 1000 rads/s, indicating that simply closing the loop without compensation will achieve robust performance nor robust stability. The upper bound contour extends to positive infinity, which implies that the compensation must reduce the high frequency gain to fall below the upper bound contours(solid curve).

The contours and the compensated open-loop are not independent. Dynamic compensation (i.e. compensation more than a constant gain) will change both the magnitude and phase of the open loop, and the contours as well.

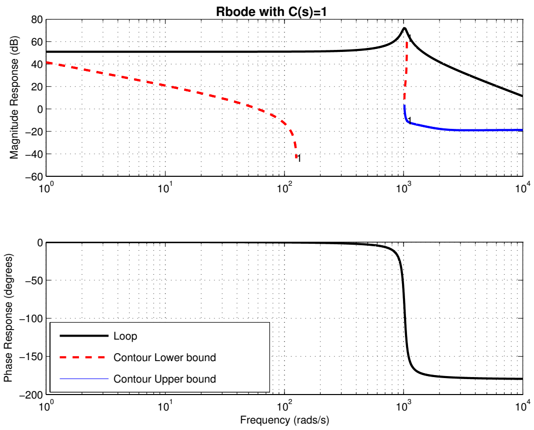


Fig. 5. Rbode when $C(s) = 1$

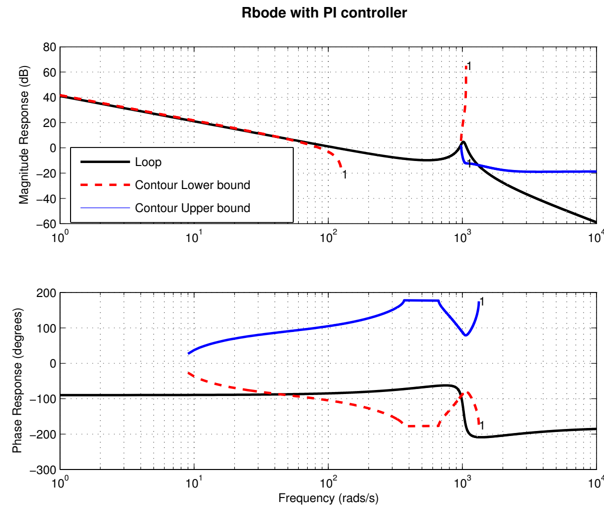


Fig. 6. Adding PI to roll off gain at high frequencies

Therefore the design process is iterative, but the iteration does not require computation of the closed loop response. The insights of shaping the open loop magnitude response to make it lie above or below appropriate contours are still helpful. The contours change very little over certain frequency ranges, and the design process must progressively address smaller and smaller frequency ranges over which the open loop violates the upper or lower contour limits. The rest of the example demonstrates this process.

Employing PI compensator

$$C_{pi} = \frac{1/1050s + 1}{s} \tag{25}$$

leads to Fig. 6. The compensated system lies very slightly below the lower bound curve at frequencies less than 40 rads/s. Simply increasing the gain would make the design achieve the robust performance specification at low frequencies. More problematic is the violation of the upper bound contour near the resonance peak. The designer can alter the open loop by considering either the phase contours, which results in a phase stabilized design, or the magnitude contours, which implies the use of notch filtering. The magnitude contours and notch filtering are more intuitive. Applying a 26dB notch at 1020 rads/s

$$C_{notch} = \frac{s^2 + 102.2s + 1.04e6}{s^2 + 2040s + 1.04e6} \tag{26}$$

results in Fig. 7.

Now there is plenty of room to increase the gain to move the magnitude response above the klow frequency lower bound contour while remaining below the high frequency upper bound contour. Increasing the gain by 5dB gives Fig. 8. And the controller is:

$$C = 1.7783 \times C_{pi} \times C_{notch} \tag{27}$$

One may employ the Nyquist plot to visualize the robust performance of the closed loop by rewriting (5) as

$$|\,|W_u| + |W_s L|\,| < |1 + L|. \tag{28}$$

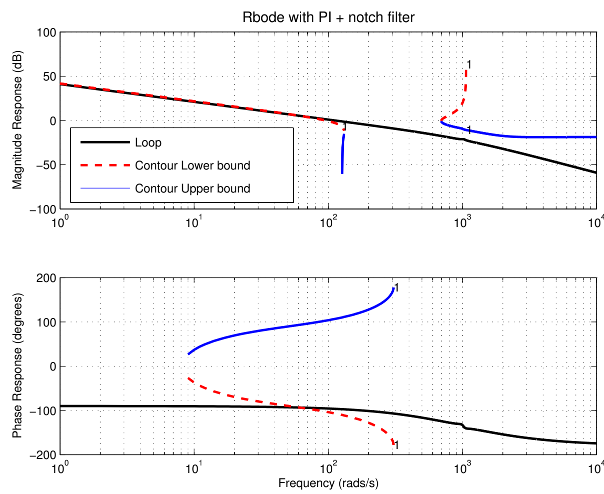


Fig. 7. Rbode with PI + notch filter

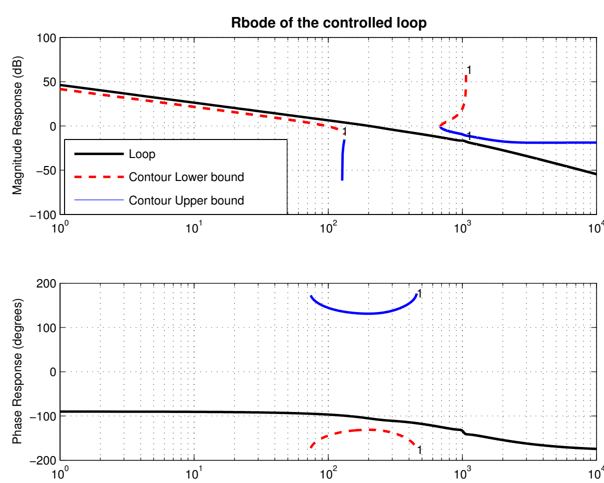


Fig. 8. Adjust gain to finalize the controller design

Inequality (28) implies that the distance from $(-1, 0)$ to the open loop response $L(j\omega)$ must exceeds $|W_u(j\omega)| + |W_s(j\omega)L(j\omega)|$ for every $j\omega$. Plotting the left hand side of (28) as circles centered $L(j\omega)$ with radius $|W_u(j\omega)| + |W_s(j\omega)L(j\omega)|$ yields the the Nyquist plot of Fig. 9.

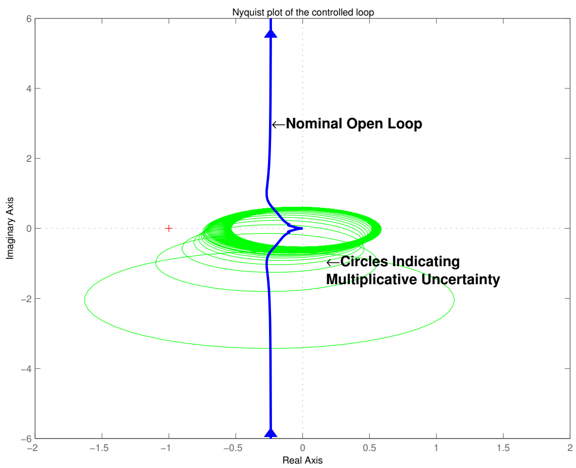


Fig. 9. Nyquist plot of the controlled loop function

Each circle in the robust nyquist plot corresponds to one specific frequency point ω, and is centered at $L(j\omega)$ with radius $|W_u(j\omega)| + |W_s(j\omega)||L(j\omega)|$. The robust performance criteria is satisfied for this frequency if and only if (-1,0) point stays outside of every circle for every ω. The series of circles check the criteria for the entire frequency range. Fig. 9 confirms that design acheives robust performance in closed loop.

Fig. 10 shows the sensitivity function of the nominal loop, as well as the sensitivities of the loop when the nomical plant model is perturbed by variations in the damping ratio or the resonance frequency. Both disturbed plants are within the uncertainty class specified by the uncertainty weighting function $W_u(s)$ and the design successfully ensures sensitivity performance in the face of the model disturbances.

IV. CONCLUSION

This paper developed the idea of modifying conventional bode plots to assist robust controller design through classical loop shaping. The Rbode plots show the precise relationship between open-loop frequency response and closed-loop robust performance, and thus provide a better guideline for robust controller design using loop shaping. The design example showed how to use the Rbode plots to manually synthesize a controller to achieve robust performance.

REFERENCES

[1] G.J. Balas, J. Doyle, K. Glover, A. packard, and R.Smith, μ-Analysis and synthesis Toolbox User's guide. The Mathworks Inc., second edition, 1995.

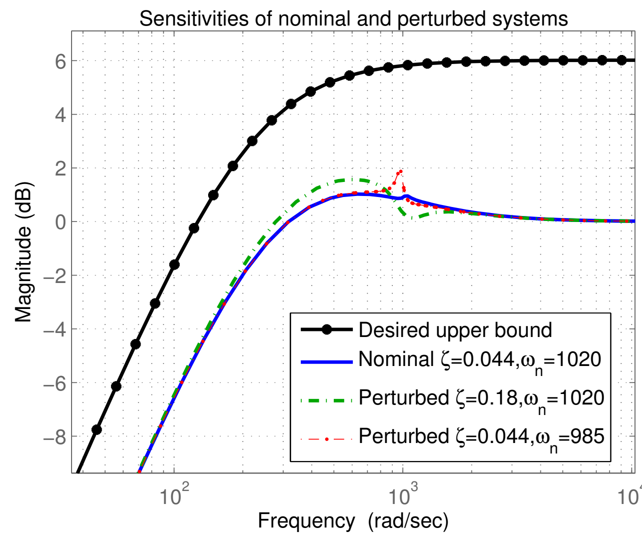


Fig. 10. Comparison between the desired sensitivity and the sensitivity of the nominal as well as perturbed plants

[2] D. McFarlane and K. Glover, A loop shaping design procedure using H_∞ synthesis. *IEEE Transactions on Automatic Control*, 37(6),1992.

[3] J. Doyle, B. Francis and A. Tannenbaum, *Feedback Control Theory*. Macmillan Publishing Company, 1992.

[4] W. Messner, Some Advances in Loop Shaping with Applications to Disk Drives,*Proceedings of the 2002 American Control Conference*,37(2), 2001.

[5] S.C.Smith and W. Messner, Loop shaping with closed-loop magnitude contours on the Bode plot,*IEEE Transactions on Autmatic Control*, Anchorage, AK, May 2002.