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Delay-dependent Robust Stability Criterion and Robust Stabilization for Uncertain Singular Time-delay Systems

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Abstract— **In this paper, the delay-dependent robust stability criterion and robust stabilization for singular time-delay systems with norm-bounded parametric uncertainty is investigated. First, in terms of the Lyapunov technique and linear matrix inequalities (LMIs), a delay-dependent stability criterion for the nominal singular time-delay system is established, which guarantees the system to be regular, impulse free and asymptotically stable. Then, based on the concept of generalized quadratic stabilization, the problem of robust stabilization for the uncertain singular time-delay systems is solved via a state feedback, which guarantees the closed-loop system to be regular, impulse free and asymptotically stable for all admissible uncertainties. The explicit expression for the desired robustly stabilizing controller is given by using the LMIs and the cone complementarity linearization iterative algorithm.**

Index Terms — **singular time-delay systems, norm-bounded parametric uncertainty, robust stabilization, delay-dependent criteria, linear matrix inequality (LMI).**

I. INTRODUCTION

Singular time-delay systems, which are also referred to as generalized differential-difference equations or generalized functional differential equations, have some properties that need not be considered in regular systems, such as non-regularity and solutions with impulses, and so on. So the problem of analysis and synthesis for singular time-delay systems is much more complicated than that of regular ones. A sufficient condition for stability of singular time-delay systems in terms of linear matrix inequalities (LMIs) is derived in [1] and [2] independently. Based on this condition, the robust stabilization problem via state feedback controller [1] and observer-based dynamic output feedback and compensator [3], and the guaranteed cost control problem [2] are discussed, respectively. These above results are all delay-independent, so they are quite conservative especially when the delay is comparatively small. To the best of our knowledge, it seems that there are no previous results on delay-dependent stability and stabilization for singular time-delay systems. This has motivated our research.

In this paper, the problem of delay-dependent robust stability criterion and robust stabilization for singular time-delay systems with norm-bounded parametric uncertainty is investigated. We consider the case of single constant time-delay, the value of which is not required to be known. Delay-dependent stability criterion for the nominal singular time-delay system is established in terms of Lyapunov technique and LMIs. And then a cone complementarity linearization iterative algorithm via minor modification is proposed to design a state feedback controller such that the resultant closed-loop system is regular, impulse free and asymptotically stable for all admissible uncertainties. Since the stability criterion is delay-dependent, it is less conservative than that obtained in [1] and [2].

The paper is organized as follows: Section II briefly presents the problem statement and some basic definitions and preliminary lemmas. The delay-dependent stability criteria and the algorithm to design robustly stabilizing controller are given in Section III and final conclusions end the paper.

Notations: $C_{n,\tau} := C([-\tau,0],R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into R^n, $x_t := x(t+\theta),\theta\in[-\tau,0]$ denotes the function family defined on $[-\tau,0]$ which is generated by n-dimensional real vector valued continuous function $x(t),t\in[-\tau,+\infty)$. Obviously, $x_t\in C_{n,\tau}$. The following norms will be used, $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm, $\|\phi\|_c := \sup_{-\tau\le t\le 0}\|\phi(t)\|$ stands for the norm of a function $\phi\in C_{n,\tau}$.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a class of uncertain singular time-delay system

$$\begin{cases} E\dot{x}(t)=(A+\triangle A)x(t)+(A_\tau+\triangle A_\tau)x(t-\tau) \\ \qquad\quad +(B+\triangle B)u(t), \\ x(t)=\phi(t), \quad t\in[-\tau,0] \end{cases} \tag{1}$$

where $x(t)\in R^n$, $u(t)\in R^q$ are the state and control input, respectively. E, A, A_τ and B are known real constant matrices with appropriate dimensions and $0 <$rank $E = p< n$. τ is an unknown constant delay and satisfies $0<\tau\le\tau_m$. $\phi(t)\in C_{n,\tau}$ is a compatible vector valued initial function. $\triangle A, \triangle A_\tau$ and $\triangle B$ are time-invariant matrices representing norm-bounded parametric uncertainties which are of the following forms:

$$\begin{bmatrix} \triangle A & \triangle A_\tau & \triangle B \end{bmatrix} = DF\begin{bmatrix} E_1 & E_\tau & E_2 \end{bmatrix} \tag{2a}$$

$$F^TF\le I_j, \quad F\in R^{i\times j} \tag{2b}$$

where, $D\in R^{n\times i}$, $E_1\in R^{j\times n}$, $E_\tau\in R^{j\times n}$, $E_2\in R^{j\times q}$ are known real constant matrices and F is an uncertain real constant matrix. $\triangle A,\triangle A_\tau$ and $\triangle B$ are said to be

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admissible if (2) is satisfied.

First of all, we will give some definitions and lemmas about the nominal unforced singular time-delay system of (1):

$$\left\{ \begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + A_\tau x(t-\tau), \\ x(t) & = & \phi(t), \quad t \in [-\tau, 0] \end{array} \right. \tag{3}$$

For this purpose, the following notations are needed:

$S_0 := \{\phi(t) \mid \phi(t) \in C_{n,\tau}, \phi(t)$ is the compatible initial function of system (3) $\}$;

$S := \{\phi(t) \mid \phi(t) \in S_0$, and there exists a unique continuous solution of system (3) on $[0, +\infty)$ for $\phi(t)\}$;

$B(0,\delta) := \{\phi(t) \mid \phi(t) \in C_{n,\tau}, \| \phi \|_c \leq \delta, \delta > 0\}$.

Definition 1 [4]:

1) The pair (E, A) is said to be regular if det$(sE - A) \not\equiv 0$;

2) The pair (E, A) is said to be impulse free if degree$\{$det$(sE - A)\}$=rank$E$.

Lemma 1 [1]: If the pair (E, A) is regular and impulse free, then for all compatible initial function $\phi(t)$, there exists a unique continuous solution on $[0, +\infty)$ to system (3). That is, if the pair (E, A) is regular and impulse free, then $S = S_0$.

Remark 1: Suppose the pair (E, A) is regular and impulse free, then the system (3) is r. s. e. (restricted system equivalence) [4] to the following system:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = A_1 x_1(t) + A_{\tau 11} x_1(t-\tau) + A_{\tau 12} x_2(t-\tau), \\ 0 = x_2(t) + A_{\tau 21} x_1(t-\tau) + A_{\tau 22} x_2(t-\tau), \\ \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right] = \left[\begin{array}{c} \phi_1(t) \\ \phi_2(t) \end{array} \right], \quad t \in [-\tau, 0]. \end{array} \right. \tag{4}$$

In this case, the compatible initial condition is

$$0 = \phi_2(0) + A_{\tau 21}\phi_1(-\tau) + A_{\tau 22}\phi_2(-\tau). \tag{5}$$

Definition 2: The singular time-delay system (3) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.

Definition 3:

1) The zero solution of system (3) is said to be stable, if for any $\epsilon > 0$, there exists a scalar $\delta(\epsilon) > 0$ such that for any compatible initial function $\phi \in B(0, \delta(\epsilon)) \cap S$, the solution $x(t)$ of system (3) satisfies $\| x(t) \| \leq \epsilon, t \geq 0$.

2) The zero solution of system (3) is said to be asymptotically stable, if the zero solution of system (3) is stable, and furthermore, there is a $b_0 > 0$ such that the initial function $\phi \in B(0, b_0) \cap S$ implies that $\lim_{t\to\infty} x(t) = 0$.

Definition 4: The system (3) is said to be stable (asymptotically stable), if its zero solution is stable (asymptotically stable).

Lemma 2 [5]: For matrix $Q \geq 0$, if there is a zero element q_i on the main diagonal line of Q, then the column and row which q_i lies on are both zero.

III. MAIN RESULTS

The objective of this paper is to design a state feedback controller

$$u(t) = Kx(t) \tag{6}$$

such that the closed-loop system constructed by (1) and (6):

$$\left\{ \begin{array}{rcl} E\dot{x}(t) & = & (A_c + \triangle A_c)x(t) + (A_\tau + \triangle A_\tau)x(t-\tau), \\ x(t) & = & \phi(t), \quad t \in [-\tau, 0] \end{array} \right. \tag{7}$$

is robustly stable, i. e., the closed-loop system (7) is regular, impulse free and asymptotically stable for all admissible uncertainties (2), where $A_c = A + BK, \triangle A_c = DFE_c, E_c := E_1 + E_2 K$.

In this section, first of all, we present the delay-dependent criterion guaranteeing system (3) to be regular, impulse free and asymptotically stable, which will play a key role in solving the aforementioned problem.

Theorem 1: The singular time-delay system (3) is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0 < \tau \leq \tau_m$, if there exist matrices $Q > 0, X \geq 0, Z > 0$ and P, Y satisfying

$$PE = E^T P^T \geq 0, \tag{8a}$$

$$\left[\begin{array}{cc} \Gamma & PA_\tau - Y + \tau_m A^T Z A_\tau \\ * & -Q + \tau_m A_\tau^T Z A_\tau \end{array} \right] < 0, \tag{8b}$$

$$\left[\begin{array}{cc} X & Y \\ * & E^T Z E \end{array} \right] \geq 0, \tag{8c}$$

where $\Gamma = A^T P^T + PA + Q + \tau_m X + Y + Y^T + \tau_m A^T Z A$.

Proof: 1. Prove the system (3) is regular and impulse free, i. e., the pair (E, A) is regular and impulse free.

From $0 <$rank$E = p < n$, there exist nonsingular matrices M, N such that

$$\bar{E} = MEN = \left[\begin{array}{cc} I_p & 0 \\ 0 & 0 \end{array} \right]. \tag{9}$$

Denote

$$\bar{A} := MAN = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \tag{10a}$$

$$\bar{A}_\tau := MA_\tau N = \left[\begin{array}{cc} A_{\tau 11} & A_{\tau 12} \\ A_{\tau 21} & A_{\tau 22} \end{array} \right] \tag{10b}$$

$$\bar{P} := N^T P M^{-1} = \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right] \tag{10c}$$

$$\bar{Q} := N^T Q N = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array} \right] \tag{10d}$$

$$\bar{X} := N^T X N = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{array} \right] \tag{10e}$$

$$\bar{Y} := N^T Y N = \left[\begin{array}{cc} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{array} \right] \tag{10f}$$

$$\bar{Z} := M^{-T} Z M^{-1} = \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{array} \right]. \tag{10g}$$

From (8a) we have

$$\bar{P}\bar{E} = \bar{E}^T\bar{P}^T \geq 0. \tag{11}$$

Using the expression of $\bar{E}$ and $\bar{P}$ in (9) and (10c), it is clear that $P_{21} = 0, P_{11} \geq 0$. Furthermore, we can obtain form Schur complement and (8b) that P is nonsingular, and then $\bar{P}$ is nonsingular and $P_{11} > 0$. It follows from (8c) that

$$\begin{bmatrix} \bar{X} & \bar{Y} \\ * & \bar{E}^T\bar{Z}\bar{E} \end{bmatrix} \geq 0 \tag{12}$$

Substituting (10) into (12), one gets

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} & Y_{12} \\ * & X_{22} & Y_{21} & Y_{22} \\ * & * & Z_{11} & 0 \\ * & * & * & 0 \end{bmatrix} \geq 0, \tag{13}$$

By lemma 2, we have $Y_{12} = 0, Y_{22} = 0$, i. e.,

$$\bar{Y} = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix}. \tag{14}$$

It follows from (8b) that

$$\begin{bmatrix} \bar{\Gamma} & \bar{P}\bar{A}_\tau - \bar{Y} + \tau_m\bar{A}^T\bar{Z}\bar{A}_\tau \\ * & -\bar{Q} + \tau_m\bar{A}_\tau^T\bar{Z}\bar{A}_\tau \end{bmatrix} < 0, \tag{15}$$

with $\bar{\Gamma} = \bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \bar{Q} + \tau_m\bar{X} + \bar{Y} + \bar{Y}^T + \tau_m\bar{A}^T\bar{Z}\bar{A}$. Using the expression in (10) we can obtain

$$\begin{bmatrix} P_{22}A_{22} + A_{22}^TP_{22}^T + Q_{22} & P_{22}A_{\tau 22} \\ * & -Q_{22} \end{bmatrix} < 0, \tag{16}$$

which implies that A_{22} is nonsingular. Therefore, the pair (E, A) is regular and impulse free.

That the pair (E, A) is regular and impulse free means that there exist nonsingular matrices $\bar{M}, \bar{N}$ such that (E, A) is transformed into the Weierstrass canonical form $(\breve{E}, \breve{A})$:

$$\breve{E} = \bar{M}E\bar{N} = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \breve{A} = \bar{M}A\bar{N} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-p} \end{bmatrix}. \tag{17}$$

For simplicity, $\bar{M}, \bar{N}$ are still denoted by M, N, $\breve{E}, \breve{A}$ by $\bar{E}, \bar{A}$ and then other notations in (10) are still used with

$$\bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-p} \end{bmatrix}. \tag{18}$$

Under coordinate transformation

$$x(t) = Ny(t) = N\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{19}$$

with $y_1(t) \in R^p, y_2(t) \in R^{n-p}$, the system (3) is equivalently transformed into:

$$\begin{cases} \bar{E}\dot{y}(t) &= \bar{A}y(t) + \bar{A}_\tau y(t-\tau), \\ y(t) &= N^{-1}\phi(t) := \psi(t), \quad t \in [-\tau, 0] \end{cases} \tag{20}$$

that is,

$$\begin{cases} \dot{y}_1(t) &= A_1y_1(t) + A_{\tau 11}y_1(t-\tau) + A_{\tau 12}y_2(t-\tau), \\ 0 &= y_2(t) + A_{\tau 21}y_1(t-\tau) + A_{\tau 22}y_2(t-\tau), \\ y(t) &= \psi(t) := \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}, \quad t \in [-\tau, 0] \end{cases} \tag{21}$$

where $\psi_1(t) \in R^p, \psi_2(t) \in R^{n-p}$. Obviously, considering the asymptotical stability of system (21) is equivalent to considering that of system (3). Define the set S_1 as:

$S_1 := \{\psi(t) \in C_{n,\tau}, \psi(t)$ is the compatible initial function of system (21), and there exists a unique continuous solution of system (21) on $[0, +\infty)$ for $\psi(t)\}$.

2. To prove that the system (3) is asymptotically stable. As mentioned above, we only need to prove that the system (21) is asymptotically stable. For this purpose, the following auxiliary lemma is introduced.

Lemma 3: If there exist a continuous functional $V(y_t) : C_{n,2\tau} \rightarrow R$ and continuous nondecreasing functions $u, v, w : R^+ \rightarrow R^+$, with $u(0) = v(0) = 0$, $u(s) > 0, v(s) > 0, \forall s > 0$, and $V(y_t)$ satisfying:

i) $u(\| y_1(t) \|^2) \leq V(y_t) \leq v(\| y_t \|_c^2), \ \ t \geq \tau$;

ii) $D^+(V(y_t)) \leq -w(\| y_1(t) \|^2), \ \ t \geq \tau$

where $y_t := y(t+\theta), \theta \in [-2\tau, 0], t \geq \tau$, then the first p-dimensional component of the zero solution of system (21) is stable, i. e., for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\| y_1(t) \| \leq \epsilon, t \geq 0$ when the initial function $\psi(t) \in B(0, \delta(\epsilon)) \cap S_1$.

Furthermore, if $w(s) > 0$ for $s > 0$, and there exist constant scalars l_0, m_0, such that $\| \dot{y}_1(t) \| \leq m_0, t \geq 0$ when $\| y_1(t) \| \leq l_0, t \geq 0$, then the first p-dimensional component of the zero solution of system (21) is asymptotically stable, i. e., i) the first p-dimensional component of the zero solution of system (21) is stable; ii) there exists a positive scalar δ_0, which may be sufficiently small, such that $\lim_{t\to\infty} y_1(t) = 0$ when the initial function $\psi \in B(0, \delta_0) \cap S_1$.

Proof: It is similar to the proof of Lemma 3 in [2] and omitted.

Define Lyapunov-Krasovskii functional

$$\begin{aligned} V(y_t) &= y_1^T(t)P_{11}y_1(t) + \int_{t-\tau}^{t} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix}^T \bar{Q} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} ds \\ &\quad + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha d\beta \\ &= y^T(t)\bar{P}\bar{E}y(t) + \int_{t-\tau}^{t} y^T(s)\bar{Q}y(s)ds \\ &\quad + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(\alpha)\bar{E}^T\bar{Z}\bar{E}\dot{y}(\alpha)d\alpha d\beta, \quad t \geq \tau \end{aligned} \tag{22}$$

Noticing that

$$\begin{aligned} &\int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(\alpha)\bar{E}^T\bar{Z}\bar{E}\dot{y}(\alpha)d\alpha d\beta \\ &\leq \tau^2(\| \bar{A}^T\bar{Z}\bar{A} \| + 2 \| \bar{A}^T\bar{Z}\bar{A}_\tau \| + \| \bar{A}_\tau^T\bar{Z}\bar{A}_\tau \|) \| y_t \|_c^2, \\ &\qquad t \geq \tau. \end{aligned} \tag{23}$$

where $\| y_t \|_c = \sup_{\theta \in [-2\tau, 0]} \| y(t+\theta) \|$, we have

$$\begin{aligned} \lambda_{min}(P_{11}) \| y_1(t) \|^2 \leq V(y_t) &\leq [\tau_m^2(\| \bar{A}^T\bar{Z}\bar{A} \| + 2 \| \bar{A}^T\bar{Z}\bar{A}_\tau \| \\ &+ \| \bar{A}_\tau^T\bar{Z}\bar{A}_\tau \|) + \lambda_{max}(P_{11}) + \tau_m \| \bar{Q} \|] \| y_t \|_c^2, t \geq \tau. \end{aligned} \tag{24}$$

Using $y_1(t) - y_1(t-\tau) = \int_{t-\tau}^{t} \dot{y}_1(\alpha)d\alpha, t \geq \tau$, calculate the time-derivative of $V(y_t)$ along with the solution of

(21) is

$$
\begin{aligned}
&\dot{V}(y_t)\mid_{(21)}\\
&= y_1^T(t)P_{11}\dot{y}_1(t) + \dot{y}_1^T(t)P_{11}y_1(t) + y^T(t)\bar{Q}y(t)\\
&\quad - y^T(t-\tau)\bar{Q}y(t-\tau) + \tau\dot{y}^T(t)\bar{E}^T\bar{Z}\bar{E}\dot{y}(t)\\
&\quad - \int_{t-\tau}^t \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha\\
&= \begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12}\\ 0 & P_{22}\end{bmatrix}\Big(\begin{bmatrix} A_1+A_{\tau 11}\\ A_{\tau 21}\end{bmatrix} y_1(t)\\
&\quad + \begin{bmatrix} 0\\ I\end{bmatrix} y_2(t) - \begin{bmatrix} A_{\tau 11}\\ A_{\tau 21}\end{bmatrix}\int_{t-\tau}^t \dot{y}_1(\alpha)d\alpha\\
&\quad + \begin{bmatrix} A_{\tau 12}\\ A_{\tau 22}\end{bmatrix} y_2(t-\tau)\Big) + \Big(\begin{bmatrix} A_1+A_{\tau 11}\\ A_{\tau 21}\end{bmatrix} y_1(t)\\
&\quad + \begin{bmatrix} 0\\ I\end{bmatrix} y_2(t) - \begin{bmatrix} A_{\tau 11}\\ A_{\tau 21}\end{bmatrix}\int_{t-\tau}^t \dot{y}_1(\alpha)d\alpha\\
&\quad + \begin{bmatrix} A_{\tau 12}\\ A_{\tau 22}\end{bmatrix} y_2(t-\tau)\Big)^T \begin{bmatrix} P_{11} & P_{12}\\ 0 & P_{22}\end{bmatrix}^T \begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}\\
&\quad + y^T(t)\bar{Q}y(t) - y^T(t-\tau)\bar{Q}y(t-\tau)\\
&\quad + \tau\dot{y}^T(t)\bar{E}^T\bar{Z}\bar{E}\dot{y}(t) - \int_{t-\tau}^t \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha, t\geq\tau.
\end{aligned} \tag{25}
$$

From (13) we have

$$
\begin{aligned}
&-2\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12}\\ 0 & P_{22}\end{bmatrix}\begin{bmatrix} A_{\tau 11}\\ A_{\tau 21}\end{bmatrix}\int_{t-\tau}^t \dot{y}_1(\alpha)d\alpha\\
&= \int_{t-\tau}^t \begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{y}_1(\alpha)\end{bmatrix}^T \begin{bmatrix} 0 & 0 & \Theta_1\\ 0 & 0 & \Theta_2\\ \Theta_1^T & \Theta_2^T & 0\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{y}_1(\alpha)\end{bmatrix} d\alpha\\
&\leq \int_{t-\tau}^t \begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{y}_1(\alpha)\end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} & \Pi_1\\ X_{12}^T & X_{22} & \Pi_2\\ \Pi_1^T & \Pi_2^T & Z_{11}\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\\ \dot{y}_1(\alpha)\end{bmatrix} d\alpha\\
&= \tau\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12}\\ X_{12}^T & X_{22}\end{bmatrix}\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}\\
&\quad + 2\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}^T \begin{bmatrix} \Pi_1\\ \Pi_2\end{bmatrix}(y_1(t) - y_1(t-\tau))\\
&\quad + \int_{t-\tau}^t \dot{y}_1^T(\alpha)Z_{11}\dot{y}_1(\alpha)d\alpha,\ \ t\geq\tau
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
\Theta_1 &= -P_{11}A_{\tau 11} - P_{12}A_{\tau 21}, & \Theta_2 &= -P_{22}A_{\tau 21},\\
\Pi_1 &= Y_{11} - P_{11}A_{\tau 11} - P_{12}A_{\tau 21}, & \Pi_2 &= Y_{21} - P_{22}A_{\tau 21}.
\end{aligned}
$$

Substituting (26) into (25) and noticing (14) and (18), we get

$$
\begin{aligned}
&\dot{V}(y_t)\mid_{(21)}\\
&\leq \begin{bmatrix} y_1(t)\\ y_2(t)\\ y_1(t-\tau)\\ y_2(t-\tau)\end{bmatrix}^T \Xi \begin{bmatrix} y_1(t)\\ y_2(t)\\ y_1(t-\tau)\\ y_2(t-\tau)\end{bmatrix} + \tau_m\dot{y}^T(t)\bar{E}^T\bar{Z}\bar{E}\dot{y}(t)\\
&= \begin{bmatrix} y(t)\\ y(t-\tau)\end{bmatrix}^T \Big(\begin{bmatrix} \bar{P}\bar{A}+\bar{A}^T\bar{P}^T+\bar{Q} + \tau_m\bar{X}+\bar{Y}+\bar{Y}^T & \bar{P}\bar{A}_\tau - \bar{Y}\\ \bar{A}_\tau^T\bar{P}^T - \bar{Y}^T & -\bar{Q}\end{bmatrix}\\
&\quad + \tau_m\begin{bmatrix} \bar{A}^T\\ \bar{A}_\tau^T\end{bmatrix}\bar{Z}\begin{bmatrix} \bar{A} & \bar{A}_\tau\end{bmatrix}\Big)\begin{bmatrix} y(t)\\ y(t-\tau)\end{bmatrix}\\
&\leq -y^T(t)Wy(t),\ \ t\geq\tau
\end{aligned} \tag{27}
$$

where

$$
\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14}\\ \Xi_{12}^T & \Xi_{22} & -\Pi_2 & P_{22}A_{\tau 22}\\ \Xi_{13}^T & -\Pi_2^T & -Q_{11} & -Q_{12}\\ \Xi_{14}^T & (P_{22}A_{\tau 22})^T & -Q_{12}^T & -Q_{22}\end{bmatrix},
$$

$$
\begin{aligned}
\Xi_{11} &= P_{11}A_1 + A_1^TP_{11}^T + Q_{11} + \tau_mX_{11} + Y_{11} + Y_{11}^T\\
\Xi_{12} &= Y_{21}^T + P_{12} + \tau_mX_{12} + Q_{12}\\
\Xi_{13} &= P_{11}A_{\tau 11} + P_{12}A_{\tau 21} - Y_{11}\\
\Xi_{14} &= P_{11}A_{\tau 12} + P_{12}A_{\tau 22}\\
\Xi_{22} &= P_{22} + P_{22}^T + Q_{22} + \tau_mX_{22},
\end{aligned}
$$

and

$$
\begin{aligned}
W = &-[\bar{P}\bar{A}+\bar{A}^T\bar{P}^T+\bar{Q}+\tau_m\bar{X}+\bar{Y}+\bar{Y}^T+\tau_m\bar{A}^T\bar{Z}\bar{A}\\
&+ (\bar{P}\bar{A}_\tau - \bar{Y} + \tau_m\bar{A}^T\bar{Z}\bar{A}_\tau)(\bar{Q} - \tau_m\bar{A}_\tau^T\bar{Z}\bar{A}_\tau)^{-1}\\
&\times(\bar{P}\bar{A}_\tau - \bar{Y} + \tau_m\bar{A}^T\bar{Z}\bar{A}_\tau)^T].
\end{aligned} \tag{28}
$$

From (15) we know $W>0$, thus

$$
\begin{aligned}
\dot{V}(y_t)\mid_{(21)} &\leq -\lambda_{min}(W)y^T(t)y(t)\\
&\leq -\lambda_{min}(W)y_1^T(t)y_1(t)\ \ ,t\geq\tau.
\end{aligned} \tag{29}
$$

By Lemma 3, we have that the first p-dimensional component of the zero solution of system (21) is stable.

Pre-multiplying $\begin{bmatrix} -A_{\tau 22}^T & I\end{bmatrix}$ and post-multiplying $\begin{bmatrix} -A_{\tau 22}^T & I\end{bmatrix}^T$ on both sides of (16) and noticing (18), we have

$$
A_{\tau 22}^TQ_{22}A_{\tau 22} - Q_{22} < 0, \tag{30}
$$

which implies that $\rho(A_{\tau 22}) < 1$ since $Q_{22} > 0$. So there exist constants $\beta > 1$ and $\alpha\in(0,1)$ such that

$$
\| A_{\tau 22}^k \|\leq \beta\alpha^k,\ \ k = 0,1,\cdots. \tag{31}
$$

It is clear that for any $t\geq 0$, there exists a positive integer k such that $(k-1)\tau\leq t<k\tau$. Calculating $y_2(t)$ in (21), it gets that for $(k-1)\tau\leq t<k\tau, k=1,2,\cdots$

$$
y_2(t) = (-A_{\tau 22})^k y_2(t-k\tau) - \sum_{i=1}^k(-A_{\tau 22})^{i-1}A_{\tau 21}y_1(t-i\tau), \tag{32}
$$

Recall that the first p-dimensional component of the zero solution of system (21) is stable, so for any $\epsilon>0$, there exists $\delta(\epsilon), 0<\delta(\epsilon)\leq\epsilon$ such that when the initial function $\psi(t)\in B(0,\delta(\epsilon))\cap S_1$, we have

$$
\| y_1(t) \|\leq\epsilon,\ \ t\geq -\tau. \tag{33}
$$

Hence, from (31)-(33), and $\| y_2(\theta) \| = \| \psi_2(\theta) \|\leq\delta(\epsilon)\leq\epsilon, \theta\in[-\tau,0]$, it follows that for $t\geq 0$

$$
\| y_2(t) \|\leq \beta\alpha^k\epsilon + \frac{\beta}{1-\alpha}\| A_{\tau 21} \|\epsilon \leq \beta(1+\frac{1}{1-\alpha}\| A_{\tau 21} \|)\epsilon. \tag{34}
$$

Obviously, the equation (34) can be written as

$$\| y_2(t) \| \leq \beta(1+\frac{1}{1-\alpha}\| A_{\tau 21}\|)\epsilon, \ \ t\geq -\tau. \quad (35)$$

Thus from (33), (35) and (21) we have

$$\| \dot{y}_1(t) \| \leq \bar{\beta}\epsilon, \ \ t\geq 0 \quad (36a)$$

where

$$\bar{\beta} = \| A_1 \| + \| A_{\tau 11} \| + \| A_{\tau 12} \| \beta(1+\frac{1}{1-\alpha}\| A_{\tau 21} \|). \quad (36b)$$

And then using Lemma 3 gives that the first p-dimensional component of the zero solution of system (21) is asymptotically stable.

Next we will show that the zero solution of system (21) is asymptotically stable. We only need to prove: 1) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\| y_2(t) \| \leq \epsilon, t \geq 0$ when the initial function $\psi(t) \in B(0, \delta(\epsilon)) \cap S_1$. 2) There exists $\delta_0 > 0$, when the initial function $\psi(t) \in B(0, \delta_0) \cap S_1$, we have $y_2(t) \to 0, t \to \infty$. In fact, 1) can be concluded by (34). And 2) will follows from $y_1(t) \to 0, t \to \infty$ and (31), (32). Thus the asymptotical stability of the zero solution of system (21) is proved.

Equivalently, the asymptotical stability of system (3) is proved. It completes the proof. ■

Remark 2: The delay-independent asymptotical stability criterion of the singular time-delay system (3) given in [1] and [2] is: The system (3) is regular, impulse free and asymptotically stable if there exist matrices $Q > 0$ and P of appropriate dimensions satisfying

$$PE = E^T P^T \geq 0, \quad (37a)$$

$$\begin{bmatrix} A^T P^T + PA + Q & PA_\tau \\ * & -Q \end{bmatrix} < 0. \quad (37b)$$

Obviously, if there are solutions $Q > 0$ and P to (37), let $X = 0, Y = 0, Z = \epsilon I (\epsilon > 0$ is sufficiently small), then the above P, Q, X, Y, Z satisfy the condition (8) in Theorem 1. Therefore, the result in this paper is less conservative than that obtained in [1] and [2].

In view of this, we will present the delay-dependent robust stabilization result via Theorem 1. The following lemma is needed.

Lemma 4 [6]: Given matrices Ω, Γ and Ξ of appropriate dimensions with Ω symmetrical, then

$$\Omega + \Gamma F \Xi + (\Gamma F \Xi)^T < 0$$

for all F satisfying $FF^T \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$\Omega + \epsilon \Gamma \Gamma^T + \epsilon^{-1} \Xi^T \Xi < 0.$$

Theorem 2: Consider the uncertain singular time-delay system (1). If there exist matrices $Q > 0, X \geq 0, Z > 0, P, Y, W$ and a scalar $\epsilon > 0$ with P nonsingular, satisfying:

$$EP^T = PE^T \geq 0, \quad (38a)$$

$$\begin{bmatrix} \begin{matrix} PA^T + AP^T + W^T B^T + BW \\ +Q + \tau_m X + Y + Y^T \end{matrix} & A_\tau P^T - Y & \tau_m PA^T + \tau_m W^T B^T & PE_1^T + W^T E_2^T & \epsilon D \\ * & -Q & \tau_m PA_\tau^T & PE_\tau^T & 0 \\ * & * & -\tau_m Z & 0 & \epsilon \tau_m D \\ * & * & * & -\epsilon I & 0 \\ * & * & * & * & -\epsilon I \end{bmatrix} < 0, \quad (38b)$$

$$\begin{bmatrix} X & Y \\ * & EP^T Z^{-1} PE^T \end{bmatrix} \geq 0. \quad (38c)$$

then there exists state feedback control

$$u(t) = WP^{-T} x(t) \quad (39)$$

such that the the closed-loop system (7) constructed by (1) and (39) is regular, impulse free and robustly stable.

Proof: Noting that $K = WP^{-T}, A_c = A + BK, E_c = E_1 + E_2 K$ and using Schur complement, the proof can be easy obtained from Theorem 1 and Lemma 4. ■

Remark 3: It is clear that the nonlinear term $EP^T Z^{-1} PE^T$ in (38c) makes that (38c) is not conformable to a LMI. However, by using the cone complementarity linearization iterative algorithm proposed in [7] by minor modification, we can convert (38c) to solving a sequence of convex optimization problems subject to LMIs.

Without loss of generality, it is assumed that $E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, then the matrices P and Y satisfying (38) are of the forms:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, \ Y = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix}, \ P_{11} > 0 \quad (40)$$

where $P_{11} \in R^{p\times p}, Y_{11} \in R^{p \times p}$ and P is nonsingular. Introducing a new variable $U > 0$, (38c) can be replaced by

$$\begin{bmatrix} X & Y \\ * & EP^T UPE^T \end{bmatrix} \geq 0, \quad (41a)$$

and

$$UZ = I. \quad (41b)$$

Denote

$$X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \geq 0, \ U = \begin{bmatrix} U_{11} & U_{12} \\ * & U_{22} \end{bmatrix} > 0, \quad (42)$$

$X_{11} \in R^{p\times p}, U_{11} \in R^{p\times p}$. Thus (41a) can be rewritten as

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} \\ * & X_{22} & Y_{21} \\ * & * & P_{11} U_{11} P_{11} \end{bmatrix} \geq 0. \quad (43)$$

Introduce a matrix $T_{11} > 0$. It is easy to see that if

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} \\ * & X_{22} & Y_{21} \\ * & * & T_{11} \end{bmatrix} \geq 0 \quad (44a)$$

and

$$P_{11}U_{11}P_{11} \geq T_{11}, \tag{44b}$$

then (43) holds. By Schur complement argument, (44b) is equivalent to

$$\begin{bmatrix} U_{11} & P_{11}^{-1} \\ * & T_{11}^{-1} \end{bmatrix} \geq 0. \tag{45}$$

Introducing another two variables $S_{11} > 0$ and $V_{11} > 0$, then the condition (45) can be replaced by

$$\begin{bmatrix} U_{11} & V_{11} \\ * & S_{11} \end{bmatrix} \geq 0, \tag{46a}$$

and

$$P_{11}V_{11} = I, \;\; T_{11}S_{11} = I. \tag{46b}$$

To sum up, if there are solutions $X \geq 0, Z > 0, U > 0, P_{11} > 0, T_{11} > 0, S_{11} > 0, V_{11} > 0$ and P, Y with P nonsingular to (40), (41b), (42), (44a), (46a)-(46b), then (38c) has solutions $X \geq 0, Z > 0, P, Y$ with P nonsingular. Using the cone complementary linearization iterative algorithm proposed in [7], the robust stabilization problem can be considered as a cone complementary problem subject to LMIs:

$$\text{Minimize } \{\text{tr}(UZ) + \text{tr}(P_{11}V_{11} + T_{11}S_{11})\} \tag{47}$$

subject to LMIs: (38b), (40), (42), (44a), (46a) and

$$\begin{cases} Q > 0, X \geq 0, Z > 0, U > 0, \epsilon > 0, \\ P_{11} > 0, V_{11} > 0, T_{11} > 0, S_{11} > 0, \\ \begin{bmatrix} U & I \\ I & Z \end{bmatrix} \geq 0, \begin{bmatrix} P_{11} & I \\ I & V_{11} \end{bmatrix} \geq 0, \begin{bmatrix} T_{11} & I \\ I & S_{11} \end{bmatrix} \geq 0. \end{cases} \tag{48}$$

The algorithm is then described as follows.

Algorithm 1:

(1) Make singular value decomposition to matrix E: $\bar{U}E\bar{V} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$, where $\bar{U}, \bar{V}$ are orthogonal matrices and $\Sigma \in R^{p\times p}$ is a nonsingular diagonal matrix. Let $M = \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \bar{U}$ and $N = \bar{V} \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}$, then $\bar{E} = MEN = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$. Take M as the row full rank transformation matrix and N as the coordinate full rank transformation matrix, then system (1) is r. s. e. to:

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = (\bar{A} + \triangle\bar{A})\bar{x}(t) + (\bar{A}_\tau + \triangle\bar{A}_\tau)\bar{x}(t-\tau) \\ \qquad\qquad + (\bar{B} + \triangle\bar{B})u(t), \\ \bar{x}(t) = N^{-1}\phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{49}$$

where, $\bar{E} = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A} = MAN, \bar{A}_\tau = MA_\tau N, \bar{B} = MB$, $\triangle\bar{A} = \bar{D}F\bar{E}_1, \triangle\bar{A}_\tau = \bar{D}F\bar{E}_\tau$, $\triangle\bar{B} = \bar{D}F\bar{E}_2$, $\bar{D} = MD, \bar{E}_1 = E_1N, \bar{E}_\tau = E_\tau N, \bar{E}_2 = E_2N, \bar{x} = N^{-1}x$. For convenience, we still denote $\bar{E}, \bar{A}, \bar{A}_\tau, \bar{B}, \bar{D}, \bar{E}_1, \bar{E}_\tau, \bar{E}_2$ as $E, A, A_\tau, B, D, E_1, E_\tau, E_2$.

(2) For given $\tau_m > 0$, find a feasible set $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}, \epsilon$ satisfying (38b), (40), (42), (44a), (46a) and (48). If there are none, exit. Otherwise set $U^{(0)} = U, Z^{(0)} = Z, P_{11}^{(0)} = P_{11}, V_{11}^{(0)} = V_{11}, T_{11}^{(0)} = T_{11}, S_{11}^{(0)} = S_{11}$, and verify the condition (38c). If (38c) is satisfied, then the robustly stabilizing controller for system (1) is designed as $u(t) = WP^{-T}N^{-1}x(t)$. If (38c) is not satisfied, set the index of the objective function in the next step as $k = 0$ and go to step (3).

(3) Solve the following convex optimization problem for the variables $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}$ and ϵ:

$$\text{Minimize } \{ \text{tr}(U^{(k)}Z + Z^{(k)}U) + \text{tr}(P_{11}^{(k)}V_{11} + V_{11}^{(k)}P_{11} + T_{11}^{(k)}S_{11} + S_{11}^{(k)}T_{11}) \}$$

subject to LMIs: (38b), (40), (42), (44a), (46a) and (48).

Set $U^{(k+1)} = U, Z^{(k+1)} = Z, P_{11}^{(k+1)} = P_{11}, V_{11}^{(k+1)} = V_{11}, T_{11}^{(k+1)} = T_{11}, S_{11}^{(k+1)} = S_{11}$.

(4) Verify the condition (38c). If condition (38c) is satisfied, then the robustly stabilizing controller for system (1) is designed as $u(t) = WP^{-T}N^{-1}x(t)$. If condition (38c) is not satisfied within a specified number of steps of iterations, then exit. Otherwise, set the index k of the objective function in Step (3) as $k+1$ and go to Step (3).

IV. CONCLUSIONS

In this paper, the delay-dependent robust stability criterion and the problem of robust stabilization for singular time-delay systems with norm-bounded parameter uncertainty is investigated. The delay-dependent stability criterion guaranteeing the nominal singular time-delay systems to be regular, impulse free and asymptotically stable is established in terms of Lyapunov technique and LMIs. Then the robustly stabilizing controller is designed by using the cone complementarity linearization iterative algorithm via minor modification. The novelty of this paper is the delay-dependent stability criterion for singular time-delay systems, which improves the results in [1] and [2] to a certain extent.

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