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Delay-dependent Robust Resilient Guaranteed Cost Control for Uncertain Singular Time-delay Systems

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Abstract— **In this paper, the delay-dependent robust resilient guaranteed cost control for singular time-delay systems with norm-bounded parametric uncertainty is investigated. First, based on the delay-dependent stability criterion for the nominal singular time-delay system, a sufficient condition of the existence of the robust resilient guaranteed cost controller is established. Then the problem of robust resilient guaranteed cost control is solved via a resilient controller with respect to additive and multiplicative controller gain variations, which guarantees that the closed-loop system is regular, impulse free and asymptotically stable for all admissible uncertainties, and satisfies a cost bound. The explicit expression for the desired robust resilient guaranteed cost controller is designed by using the LMIs and the cone complementarity linearization iterative algorithm.**

Index Terms — **uncertain singular time-delay systems, guaranteed cost control, resilient controller, delay-dependent criteria, linear matrix inequality(LMI).**

I. INTRODUCTION

The guaranteed cost control for uncertain systems was first put forward in [1]. The main idea is to design a controller to robustly stabilize the uncertain system and guarantee an adequate level of performance. Recently, the study of guaranteed cost control for uncertain time-delay systems has attracted considerable attention, see [2]-[5]. In particular, the guaranteed cost control for a class of singular time-delay systems with norm-bounded parametric uncertainty is studied in [4]. However, the design method is delay-independent. Since the delay-dependent results are less conservative than the delay-independent ones, especially when the delay time is small, it is necessary to discuss the delay-dependent robust guaranteed cost control for uncertain singular time-delay systems.

On the other hand, it is worth noting that an implicit assumption inherent in these above design techniques is that the controller is precise, and exactly implemented. However, in practice, controllers do have a certain degree of errors due to finite word length in any digital systems, the imprecision inherent in analog systems and need for additional tuning of parameters in the final controller implementation. By means of several examples, see [6], it is demonstrated that the controllers designed by using weighted H_∞, μ and l_1 synthesis techniques may be very sensitive with respect to errors in the controller coefficients. Such controllers are often termed “fragile”. Hence, it is considered beneficial that the designed (nominal) controllers should be capable of tolerating some level of controller gain variations. Recently, some researchers have developed some efforts to tackle the non-fragile controller design problem [7]-[11]. In particular, The problem of designing resilient guaranteed cost controller for a class of generalized system is addressed in [10]. To the best of our knowledge, there are no previous results on delay-dependent robust resilient guaranteed cost control for singular time-delay systems in the literature, which has motivated our research.

In this paper, the problem of delay-dependent robust resilient guaranteed cost control for singular time-delay systems with norm-bounded parametric uncertainty is investigated. The considered time-delay is constant but unknown. Based on the delay-dependent stability criterion for the nominal singular time-delay system [12], a sufficient condition of the existence of the robust resilient guaranteed cost controller is established. Then by using the idea of generalized quadratic stabilization, the problem of delay-dependent robust resilient guaranteed cost control is solved via a resilient controller with respect to additive and multiplicative controller gain variations. And a cone complementarity linearization iterative algorithm is proposed to design the resilient guaranteed cost controller such that the resultant closed-loop system is regular, impulse free and asymptotically stable for all admissible uncertainties, and the close-loop value of the cost function satisfies a bound.

The paper is organized as follows: Section II briefly presents the problem statement and some preliminary lemmas. The problem of delay-dependent robust resilient guaranteed cost control is solved in Section III and final conclusions end the paper.

Notations: R denotes the set of real numbers, R^+ denotes 0 and axis of positive real numbers, R^n denotes the n-dimensional Euclidean space over the reals and $R^{n\times m}$ denotes the set of all $n \times m$ real matrices, $diag\{\cdots\}$ is a block-diagonal matrix. For real symmetric matrix X, the notation $X \geq 0$ $(X > 0)$ means the matrix X is positive-semidefinite (positive-definite), $\lambda_{min}(X)(\lambda_{max}(X))$ denotes the minimum (maximum) eigenvalue of matrix X . The superscript T represents the transpose. $C_{n,\tau} := C([-\tau, 0], R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into R^n, $x_t := x(t+\theta), \theta \in [-\tau, 0]$, obviously, $x_t \in C_{n,\tau}$. The following norms will be used, $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm,

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$\| \phi \|_c := \sup_{-\tau \le t \le 0} \| \phi(t) \|$ stands for the norm of a function $\phi \in C_{n,\tau}$.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a class of uncertain singular time-delay system represented by

$$\begin{cases} E\dot{x}(t) = (A + \triangle A)x(t) + (A_\tau + \triangle A_\tau)x(t-\tau) \\ \qquad\qquad + Bu(t), \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^q$ are the state and control input, respectively. E, A, A_τ and B are known real constant matrices with appropriate dimensions and $0 <$rank $E = p < n$. τ is an unknown constant delay and satisfies $0 < \tau \le \tau_m$, $\phi(t) \in C_{n,\tau}$ is a compatible vector valued initial function and it is assumed that $\phi(t)$ is differentiable. $\triangle A$ and $\triangle A_\tau$ are time-invariant matrices representing norm-bounded parametric uncertainties which are of the following forms:

$$\triangle A = D_1 F_1 E_1, \quad \triangle A_\tau = D_2 F_2 E_2 \tag{2a}$$

$$F_i^T F_i \le I, \quad i = 1, 2. \tag{2b}$$

Where, $D_i, E_i, i = 1, 2$, are known real constant matrices with appropriate dimensions, $F_i, i = 1, 2$ are uncertain real constant matrices with appropriate dimensions.

For this system, we will consider resilient controller of the form :

$$u(t) = (K + \triangle K)x(t). \tag{3}$$

About the controller gain variations, two forms will be considered:

a) Additive controller gain variations:

$$\triangle K = D_3 F_3 E_3, \quad F_3^T F_3 \le I; \tag{4a}$$

b) Multiplicative controller gain variations:

$$\triangle K = D_4 F_4 E_4 K, \quad F_4^T F_4 \le I. \tag{4b}$$

Where $D_i, E_i, i = 3, 4$, are known real constant matrices, $F_i, i = 3, 4$ are uncertain real constant matrices.

$\triangle A, \triangle A_\tau$ and $\triangle K$ are said to be admissible if (2) and (4) are satisfied.

Applying this controller to system (1) results in the closed-loop system:

$$\begin{cases} E\dot{x}(t) &= (A_k + \triangle A_k)x(t) + (A_\tau + \triangle A_\tau)x(t-\tau), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{5}$$

with $A_k = A + BK, \triangle A_k = \triangle A + B\triangle K$. The cost function associated with system (1) is

$$J = \int_0^\infty [x^T(t)Sx(t) + u^T(t)Ru(t)]dt \tag{6}$$

where S, R are given positive-definite symmetric matrices. Associated with the cost, the robust resilient guaranteed cost controller is defined as follows:

Definition 1: Consider the uncertain singular time-delay system (1). If there exist a resilient controller (3) and a positive scalar J^* such that for all admissible uncertainties (2) and (4), the closed-loop system (5) is regular, impulse free and asymptotically stable, and the closed-loop value of the cost function (6) satisfies $J \le J^*$, then J^* is said to be a guaranteed cost and the controller (3) is said to be a robust resilient guaranteed cost controller.

The objective of this paper is to design the robust resilient guaranteed cost controller (3) for system (1) and give an upper bound for the cost function (6).

To get the main result of this paper, two lemmas are introduced:

Lemma 1 [13]: For any matrices D, E, F with appropriate dimensions and a scalar $\epsilon > 0$, where F satisfies: $F^T F \le I$, the following inequality holds:

$$DFE + E^T F^T D^T \le \epsilon^{-1} DD^T + \epsilon E^T E. \tag{7}$$

Lemma 2 [12]: The singular time-delay system

$$\begin{cases} E\dot{x}(t) &= Ax(t) + A_\tau x(t-\tau), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{8}$$

is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0 < \tau \le \tau_m$, if there exist matrices $Q > 0, X \ge 0, Z > 0$ and P, Y satisfying

$$PE = E^T P^T \ge 0, \tag{9a}$$

$$\begin{bmatrix} \Gamma & PA_\tau - Y + \tau_m A^T Z A_\tau \\ * & -Q + \tau_m A_\tau^T Z A_\tau \end{bmatrix} < 0, \tag{9b}$$

$$\begin{bmatrix} X & Y \\ * & E^T Z E \end{bmatrix} \ge 0, \tag{9c}$$

where $\Gamma = A^T P^T + PA + Q + \tau_m X + Y + Y^T + \tau_m A^T Z A$.

III. ROBUST RESILIENT GUARANTEED COST CONTROL

In this section, first of all, we present a sufficient condition of the existence of robust resilient guaranteed cost controller (3) for the uncertain singular time-delay system (1), which will play a key role in solving the aforementioned problem.

Theorem 1: Consider the uncertain singular time-delay system (1) and the cost function (6). If there exist matrices $Q > 0, X \ge 0, Z > 0$ and P, Y such that

$$PE = E^T P^T \ge 0, \tag{10a}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix} < 0, \tag{10b}$$

$$\begin{bmatrix} X & Y \\ * & E^T Z E \end{bmatrix} \ge 0, \tag{10c}$$

hold for all admissible uncertainties (2) and (4), where

$$\begin{aligned}\Gamma_{11} &= (A_k + \triangle A_k)^T P^T + P(A_k + \triangle A_k) + Q + \tau_m X \\ &\quad + Y + Y^T + \tau_m (A_k + \triangle A_k)^T Z (A_k + \triangle A_k) \\ &\quad + S + (K + \triangle K)^T R (K + \triangle K) \\ \Gamma_{12} &= P(A_\tau + \triangle A_\tau) - Y \\ &\quad + \tau_m (A_k + \triangle A_k)^T Z (A_\tau + \triangle A_\tau) \\ \Gamma_{22} &= -Q + \tau_m (A_\tau + \triangle A_\tau)^T Z (A_\tau + \triangle A_\tau)\end{aligned}$$

then controller (3) is a robust resilient guaranteed cost controller for uncertain singular time-delay system (1).

Proof: From Lemma 2, we know that if (10) is satisfied, then the closed-loop system (5) constructed by (1) and (3) is regular, impulse free and asymptotically stable. Next, we will prove that there exists a positive scalar J^* such that the closed-loop value of the cost function (6) satisfies $J \leq J^*$. Noticing that the system (5) is regular and impulse free, and the initial function $\phi(t)$ is differentiable, so the solution $x(t)$ to system (5) is piecewise differentiable. Thus we have $x(t) - x(t-\tau) = \int_{t-\tau}^{t} \dot{x}(\alpha) d\alpha$ and the system (5) can be rewritten as

$$\begin{cases} E\dot{x}(t) &= (A_k + \triangle A_k + A_\tau + \triangle A_\tau) x(t) \\ &\quad -(A_\tau + \triangle A_\tau) \int_{t-\tau}^{t} \dot{x}(\alpha) d\alpha, \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{11}$$

Define Lyapunov-Krasovskii functional

$$\begin{aligned} V(x_t) &= x^T(t) P E x(t) + \int_{t-\tau}^{t} x^T(s) Q x(s) ds \\ &\quad + \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) E^T Z E \dot{x}(\alpha) d\alpha d\beta. \end{aligned} \tag{12}$$

Calculating the the time-derivative of $V(x_t)$ along with the solution of (11) one gets

$$\begin{aligned} \dot{V}(x_t) \mid_{(11)} &= 2x^T(t) P(A_k + \triangle A_k + A_\tau + \triangle A_\tau) x(t) \\ &\quad - 2x^T(t) P(A_\tau + \triangle A_\tau) \int_{t-\tau}^{t} \dot{x}(\alpha) d\alpha \\ &\quad + x^T(t) Q x(t) - x^T(t-\tau) Q x(t-\tau) \\ &\quad + \tau \dot{x}^T(t) E^T Z E \dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(\alpha) E^T Z E \dot{x}(\alpha) d\alpha. \end{aligned} \tag{13}$$

Combined with (10c) it is obtained that

$$\begin{aligned} & - 2x^T(t) P(A_\tau + \triangle A_\tau) \int_{t-\tau}^{t} \dot{x}(\alpha) d\alpha \\ &= \int_{t-\tau}^{t} \begin{bmatrix} x(t) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} 0 & \Theta_1 \\ \Theta_1^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(\alpha) \end{bmatrix} d\alpha \\ &\leq \int_{t-\tau}^{t} \begin{bmatrix} x(t) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} X & \Theta_2 \\ \Theta_2^T & E^T Z E \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(\alpha) \end{bmatrix} d\alpha \\ &\leq \tau_m x^T(t) X x(t) + 2x^T(t) \Theta_2 (x(t) - x(t-\tau)) \\ &\quad + \int_{t-\tau}^{t} \dot{x}^T(\alpha) E^T Z E \dot{x}(\alpha) d\alpha \end{aligned} \tag{14}$$

where

$$\Theta_1 = -P(A_\tau + \triangle A_\tau), \quad \Theta_2 = Y - P(A_\tau + \triangle A_\tau).$$

Then we have

$$\begin{aligned} \dot{V}(x_t) \mid_{(11)} &\leq \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \\ &\quad - x^T(t)[S + (K + \triangle K)^T R (K + \triangle K)] x(t). \end{aligned} \tag{15}$$

From (10b) we get

$$\dot{V}(x_t) \mid_{(11)} \leq -x^T(t)[S + (K + \triangle K)^T R (K + \triangle K)] x(t). \tag{16}$$

Integrating both sides of (16) from 0 to T results in

$$\begin{aligned} & -\int_{0}^{T} x^T(t)[S + (K + \triangle K)^T R (K + \triangle K)] x(t) dt \\ &\geq V(x_T) - V(x_0). \end{aligned} \tag{17}$$

Since the zero solution of the closed-loop system is asymptotically stable, let $T \to \infty$, we get

$$\begin{aligned} & \int_{0}^{\infty} [x^T(t) S x(t) + u^T(t) R u(t)] dt \\ &\leq V(x_0) = \phi^T(0) P E \phi(0) + \int_{-\tau}^{0} \phi^T(s) Q \phi(s) ds \\ &\quad + \int_{-\tau}^{0} \int_{\beta}^{0} \dot{\phi}^T(\alpha) E^T Z E \dot{\phi}(\alpha) d\alpha d\beta. \end{aligned} \tag{18}$$

This implies that

$$\begin{aligned} J^* &= \phi^T(0) P E \phi(0) + \int_{-\tau}^{0} \phi^T(s) Q \phi(s) ds \\ &\quad + \int_{-\tau}^{0} \int_{\beta}^{0} \dot{\phi}^T(\alpha) E^T Z E \dot{\phi}(\alpha) d\alpha d\beta \end{aligned} \tag{19}$$

is a guaranteed cost of system (1). Hence, controller (3) is a robust resilient guaranteed cost controller for system (1). It completes the proof. ■

Based on Theorem 1, we are now in the position to give the design method for the robust resilient guaranteed cost controller (3) in the light of two forms of controller gain variations.

For the additive controller gain variations, the uncertain matrix $\triangle A_k$ in the closed-loop system (5) can be written as

$$\begin{aligned} \triangle A_k &= D_k F_k E_k \\ &= \begin{bmatrix} D_1 & B D_3 \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & F_3 \end{bmatrix} \begin{bmatrix} E_1 \\ E_3 \end{bmatrix}. \end{aligned} \tag{20}$$

By using Theorem 1, we can get:

Theorem 2: Consider uncertain singular time-delay system (1) and the cost function (6). If there exist matrices $Q > 0, X \geq 0, Z > 0, P, Y, W$ and a scalar $\epsilon > 0$ with P nonsingular satisfying the following inequalities:

$$EP^T = PE^T \geq 0, \tag{21a}$$

$$\begin{bmatrix} \Lambda_{11} & A_\tau P^T - Y & \Lambda_{13} & P & W^T & PE_k^T & 0 & PE_3^T \\ * & -Q & \tau_m P A_\tau^T & 0 & 0 & 0 & PE_2^T & 0 \\ * & * & -\tau_m Z + \epsilon\tau_m^2 D_k D_k^T + \epsilon\tau_m^2 D_2 D_2^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -S^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & -R^{-1} + \epsilon D_3 D_3^T & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon I & 0 & 0 \\ * & * & * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{21b}$$

$$\begin{bmatrix} X & Y \\ * & EP^T Z^{-1} PE^T \end{bmatrix} \geq 0 \tag{21c}$$

with

$$\begin{aligned} \Lambda_{11} &= AP^T + PA^T + BW + W^T B^T + Q + \tau_m X \\ &\quad + Y + Y^T + \epsilon D_k D_k^T + \epsilon D_2 D_2^T \\ \Lambda_{13} &= \tau_m PA^T + \tau_m W^T B^T + \epsilon\tau_m D_k D_k^T + \epsilon\tau_m D_2 D_2^T, \end{aligned}$$

then the controller (3) with respect to the additive controller gain variations:

$$u(t) = (WP^{-T} + D_3 F_3 E_3) x(t) \tag{22}$$

is a robust resilient guaranteed cost controller for system (1), and the guaranteed cost is

$$\begin{aligned} J^* &= \phi^T(0) P^{-1} E\phi(0) + \int_{-\tau}^{0} \phi^T(s) P^{-1} Q P^{-T} \phi(s) ds \\ &\quad + \int_{-\tau}^{0}\int_{\beta}^{0} \dot\phi^T(\alpha) E^T Z^{-1} E \dot\phi(\alpha) d\alpha d\beta. \end{aligned} \tag{23}$$

Proof: By Lemma 1, it can be obtained that

$$\Xi := \begin{bmatrix} \Xi_1 & (A_\tau + \triangle A_\tau)P^T - Y & \tau_m P(A_k + \triangle A_k)^T & P & P(K + \triangle K)^T \\ P(A_\tau + \triangle A_\tau)^T - Y^T & -Q & \tau_m P(A_\tau + \triangle A_\tau)^T & 0 & 0 \\ \tau_m (A_k + \triangle A_k) P^T & \tau_m (A_\tau + \triangle A_\tau) P^T & -\tau_m Z & 0 & 0 \\ P^T & 0 & 0 & -S^{-1} & 0 \\ (K + \triangle K) P^T & 0 & 0 & 0 & -R^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \Xi_2 & A_\tau P^T - Y & \tau_m P A_k^T & P & PK^T \\ PA_\tau^T - Y^T & -Q & \tau_m P A_\tau^T & 0 & 0 \\ \tau_m A_k P^T & \tau_m A_\tau P^T & -\tau_m Z & 0 & 0 \\ P^T & 0 & 0 & -S^{-1} & 0 \\ KP^T & 0 & 0 & 0 & R^{-1} \end{bmatrix}$$

$$+ \begin{bmatrix} D_k & D_2 & 0 \\ 0 & 0 & 0 \\ \tau_m D_k & \tau_m D_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} F_k & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} \times \begin{bmatrix} E_k P^T & 0 & 0 & 0 & 0 \\ 0 & E_2 P^T & 0 & 0 & 0 \\ E_3 P^T & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ (\begin{bmatrix} D_k & D_2 & 0 \\ 0 & 0 & 0 \\ \tau_m D_k & \tau_m D_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} F_k & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} \times \begin{bmatrix} E_k P^T & 0 & 0 & 0 & 0 \\ 0 & E_2 P^T & 0 & 0 & 0 \\ E_3 P^T & 0 & 0 & 0 & 0 \end{bmatrix})^T$$

$$\leq \begin{bmatrix} \Xi_2 & A_\tau P^T - Y & \tau_m P A_k^T & P & PK^T \\ PA_\tau^T - Y^T & -Q & \tau_m P A_\tau^T & 0 & 0 \\ \tau_m A_k P^T & \tau_m A_\tau P^T & -\tau_m Z & 0 & 0 \\ P^T & 0 & 0 & -S^{-1} & 0 \\ KP^T & 0 & 0 & 0 & R^{-1} \end{bmatrix}$$

$$+\epsilon \begin{bmatrix} D_k & D_2 & 0 \\ 0 & 0 & 0 \\ \tau_m D_k & \tau_m D_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} D_k & D_2 & 0 \\ 0 & 0 & 0 \\ \tau_m D_k & \tau_m D_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_3 \end{bmatrix}^T$$

$$+\epsilon^{-1} \begin{bmatrix} PE_k^T & 0 & PE_3^T \\ 0 & PE_2^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} PE_k^T & 0 & PE_3^T \\ 0 & PE_2^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \tag{24}$$

with

$$\begin{aligned} \Xi_1 &= (A_k + \triangle A_k)P^T + P(A_k + \triangle A_k)^T + Q + \tau_m X \\ &\quad + Y + Y^T, \\ \Xi_2 &= A_k P^T + PA_k^T + Q + \tau_m X + Y + Y^T. \end{aligned}$$

Concerning that $A_k = A + BK$ and $K = WP^{-T}$, then by using a Schur complement argument, it follows that $\Xi < 0$ if (21b) is satisfied. Invoking again a Schur complement argument one obtains that $\Xi < 0$ is equivalent to

$$\Upsilon := \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ * & \Upsilon_{22} \end{bmatrix} < 0, \tag{25}$$

with

$$\begin{aligned} \Upsilon_{11} &= (A_k + \triangle A_k)P^T + P(A_k + \triangle A_k)^T + Q + \tau_m X \\ &\quad + Y + Y^T + \tau_m P(A_k + \triangle A_k)^T Z^{-1} (A_k + \triangle A_k) P^T \\ &\quad + PSP^T + P(K + \triangle K)^T R (K + \triangle K) P^T \\ \Upsilon_{12} &= (A_\tau + \triangle A_\tau)P^T - Y \\ &\quad + \tau_m P(A_k + \triangle A_k)^T Z^{-1} (A_\tau + \triangle A_\tau) P^T \\ \Upsilon_{22} &= -Q + \tau_m P(A_\tau + \triangle A_\tau)^T Z^{-1} (A_\tau + \triangle A_\tau) P^T. \end{aligned}$$

Pre-multiplying by $\mathrm{diag}\{P^{-1}, P^{-1}\}$ and post-multiplying by $\{P^{-T}, P^{-T}\}$ on both sides of (25), correspondingly, pre-multiplying by P^{-1} and post-multiplying by P^{-T} on both sides of (21a), pre-multiplying by $\text{diag}\{P^{-1}, P^{-1}\}$ and post-multiplying by $\text{diag}\{P^{-T}, P^{-T}\}$ on both sides of (21c), and denoting $\tilde{P} = P^{-1}, \tilde{Q} = P^{-1}QP^{-T}, \tilde{X} = P^{-1}XP^{-T}$, $\tilde{Y} = P^{-1}YP^{-T}, \tilde{Z} = Z^{-1}$, we can easy see that if there exist $Q > 0, X \geq 0, Z > 0, P, Y, W$ and a scalar $\epsilon > 0$ satisfying (21), then (10) has solutions $\tilde{Q} > 0, \tilde{X} \geq 0, \tilde{Z} > 0$ and $\tilde{P}, \tilde{Y}$. According to Theorem 1, it gets that controller (22) is a robust resilient guaranteed cost controller for system (1), and the guaranteed cost is

$$\begin{aligned} J^* &= \phi^T(0)\tilde{P}E\phi(0) + \int_{-\tau}^{0} \phi^T(s)\tilde{Q}\phi(s)ds \\ &\quad + \int_{-\tau}^{0}\int_{\beta}^{0} \dot{\phi}^T(\alpha)E^T\tilde{Z}E\dot{\phi}(\alpha)d\alpha d\beta \\ &= \phi^T(0)P^{-1}E\phi(0) + \int_{-\tau}^{0} \phi^T(s)P^{-1}QP^{-T}\phi(s)ds \\ &\quad + \int_{-\tau}^{0}\int_{\beta}^{0} \dot{\phi}^T(\alpha)E^TZ^{-1}E\dot{\phi}(\alpha)d\alpha d\beta. \end{aligned}$$

It completes the proof. ■

For the multiplicative controller gain variations, the uncertain matrix $\triangle A_k$ in the closed-loop system (5) can be written as

$$\begin{aligned} \triangle A_k &= D_cF_cE_c \\ &= \begin{bmatrix} D_1 & BD_4 \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} E_1 \\ E_4K \end{bmatrix}. \end{aligned} \quad (26)$$

Similar to Theorem 2, we have:

Theorem 3: Consider uncertain singular time-delay system (1) and the cost function (6). If there exist matrices $Q > 0, X \geq 0, Z > 0, P, Y, W$ and a scalar $\epsilon > 0$ with P nonsingular satisfying (21a), (21c) and

$$\begin{bmatrix} \Omega_{11} & A_\tau P^T - Y & \Omega_{13} & P & W^T & PE_1^T & W^TE_4^T & 0 & W^TE_4^T \\ * & -Q & \tau_m PA_\tau^T & 0 & 0 & 0 & 0 & PE_2^T & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -S^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -R^{-1} + \epsilon D_4D_4^T & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon I & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0 \quad (27)$$

with

$$\begin{aligned} \Omega_{11} &= AP^T + PA^T + BW + W^TB^T + Q + \tau_m X \\ &\quad + Y + Y^T + \epsilon D_cD_c^T + \epsilon D_2D_2^T \\ \Omega_{13} &= \tau_m PA^T + \tau_m W^TB^T + \epsilon\tau_m D_cD_c^T + \epsilon\tau_m D_2D_2^T \\ \Omega_{33} &= -\tau_m Z + \epsilon\tau_m^2 D_cD_c^T + \epsilon\tau_m^2 D_2D_2^T, \end{aligned}$$

then the controller (3) with respect to the multiplicative controller gain variations:

$$u(t) = (I + D_4F_4E_4)WP^{-T}x(t) \quad (28)$$

is a robust resilient guaranteed cost controller for system (1), and the guaranteed cost is (23).

It is clear, the nonlinear term $EP^TZ^{-1}PE^T$ in (21c) makes that (21c) is not conformable to a LMI . However, by using the cone complementarity linearization iterative algorithm proposed in [14] by minor modification, we can convert (21c) to solving a sequence of convex optimization problems subject to LMIs.

Without loss of generality, it is assumed that $E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, then

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, Y = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix}, \quad (29)$$

where $P_{11} \in R^{p\times p}, Y_{11} \in R^{p\times p}, P_{11} > 0$ and P is nonsingular. Introducing new variables $U > 0, T_{11} > 0, S_{11} > 0$ and $V_{11} > 0$, and denote

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} \quad (30)$$

with $X_{11} \in R^{p\times p}, U_{11} \in R^{p\times p}$. Then it is easy to see that if there are solutions P, X, Y, Z, U and V_{11}, S_{11}, T_{11} to (29), (30) and

$$UZ = I, \quad (31)$$

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} \\ * & X_{22} & Y_{21} \\ * & * & T_{11} \end{bmatrix} \geq 0, \quad (32)$$

$$\begin{bmatrix} U_{11} & V_{11} \\ * & S_{11} \end{bmatrix} \geq 0 \quad (33)$$

$$P_{11}V_{11} = I, \quad T_{11}S_{11} = I, \quad (34)$$

then (21c) has solutions X, Y, Z and P. Using the cone complementary linearization iterative algorithm proposed in [14], the problem to design the robust resilient guaranteed cost controller with respect to the additive controller gain variations can be considered as a cone complementary problem subject to LMIs:

Minimize $\{\text{tr}(UZ)\text{+tr}(P_{11}V_{11} + T_{11}S_{11})\}$ (35)
subject to

$$\left\{ \begin{array}{l} Q > 0, X \geq 0, Z > 0, U > 0, \epsilon > 0, \\ P_{11} > 0, V_{11} > 0, T_{11} > 0, S_{11} > 0, \\ \begin{bmatrix} U & I \\ I & Z \end{bmatrix} \geq 0, \begin{bmatrix} P_{11} & I \\ I & V_{11} \end{bmatrix} \geq 0, \begin{bmatrix} T_{11} & I \\ I & S_{11} \end{bmatrix} \geq 0 \end{array} \right. \quad (36)$$

and (21b), (29), (30), (32), (33).

Notice that the above problem is a non-convex optimization problem. To solve the problem, we follow the iterative linearization method via minor modification proposed in [14] to solve this problem, and get the algorithm described as follows.

Algorithm 1:

(1) Make singular value decomposition to matrix E: $\bar{U}E\bar{V} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$, where $\bar{U}, \bar{V}$ are orthogonal matrices and $\Sigma \in R^{p \times p}$ is a diagonal matrix. Let $M = \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \bar{U}$ and $N = \bar{V} \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}$, then $\bar{E} = MEN = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$. Take M, N as the transformation matrices, then system (1) is r. s. e. to:

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = (\bar{A} + \triangle\bar{A})\bar{x}(t) + (\bar{A}_\tau + \triangle\bar{A}_\tau)\bar{x}(t-\tau) \\ \qquad\qquad + \bar{B}u(t), \\ \bar{x}(t) = N^{-1}\phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{37}$$

where, $\bar{A} = MAN$, $\bar{A}_\tau = MA_\tau N$, $\bar{B} = MB$, $\triangle\bar{A} = \bar{D}_1 F_1 \bar{E}_1$, $\triangle\bar{A}_\tau = \bar{D}_2 F_2 \bar{E}_2$, $\bar{D}_1 = MD_1, \bar{D}_2 = MD_2$, $\bar{E}_1 = E_1 N, \bar{E}_2 = E_2 N$. Correspondingly, let $\bar{E}_3 = E_3 N, \bar{S} = N^T S N$. For convenience, we still denote $\bar{E}, \bar{A}, \bar{A}_\tau, \bar{B}, \bar{D}_1, \bar{D}_2, \bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{S}$ as $E, A, A_\tau, B, D_1, D_2, E_1, E_2, E_3, S$.

(2) For given $\tau_m > 0$, find a feasible set $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}, \epsilon$ satisfying (21b), (29), (30), (32), (33) and (36). If there are none, exit. Otherwise set $U^{(0)} = U, Z^{(0)} = Z, P_{11}^{(0)} = P_{11}, V_{11}^{(0)} = V_{11}, T_{11}^{(0)} = T_{11}, S_{11}^{(0)} = S_{11}$, and verify the condition (21c). If (21c) is satisfied, then the robust resilient guaranteed cost controller is designed as $u(t) = (WP^{-T} + D_3 F_3 E_3)N^{-1}x(t)$ and the guaranteed cost is

$$\begin{aligned} J^* \;=\; & \phi^T(0)N^{-T}P^{-1}EN^{-1}\phi(0) \\ & + \int_{-\tau}^{0} \phi^T(s)N^{-T}P^{-1}QP^{-T}N^{-1}\phi(s)ds \\ & + \int_{-\tau}^{0}\int_{\beta}^{0} \dot{\phi}^T(\alpha)N^{-T}E^T Z^{-1}EN^{-1}\dot{\phi}(\alpha)d\alpha d\beta. \end{aligned} \tag{38}$$

If (21c) is not satisfied, set the index of the objective function in the next step as $k = 0$ and go to step (3).

(3) Solve the following convex optimization problem for the variables $Q, X, Z, P, Y, W, U, P_{11}, V_{11}, S_{11}, T_{11}$ and ϵ:

$$\text{Minimize } \{ \text{ tr}(U^{(k)}Z + Z^{(k)}U) + \text{tr}(P_{11}^{(k)}V_{11} + V_{11}^{(k)}P_{11} + T_{11}^{(k)}S_{11} + S_{11}^{(k)}T_{11}) \}$$

subject to (21b), (29), (30), (32), (33) and (36).

Set $U^{(k+1)} = U, Z^{(k+1)} = Z, P_{11}^{(k+1)} = P_{11}, V_{11}^{(k+1)} = V_{11}, T_{11}^{(k+1)} = T_{11}, S_{11}^{(k+1)} = S_{11}$.

(4) Verify the condition (21c). If condition (21c) is satisfied, then the robust resilient guaranteed cost controller is designed as $u(t) = (WP^{-T} + D_3F_3E_3)N^{-1}x(t)$ and the guaranteed cost is (38). If condition (21c) is not satisfied within a specified number of steps of iterations, then exit. Otherwise, set the index k of the objective function in Step (3) as $k+1$ and go to Step (3).

Remark 1: Algorithm 1 gives a method to design the the robust resilient guaranteed cost controller with respect to the additive controller gain variations. Similarly, the algorithm to design the the robust resilient guaranteed cost controller with respect to the multiplicative controller gain variations can be obtained by minor modification to Algorithm 1.

IV. CONCLUSIONS

In this paper, the delay-dependent robust resilient guaranteed cost control for a class of uncertain singular time-delay systems is investigated. Based on the delay-dependent stability criterion for the nominal singular time-delay system, a sufficient condition of the existence of the robust resilient guaranteed cost controller is established. Then the resilient controller with respect to additive and multiplicative controller gain variations is designed by using the cone complementarity linearization iterative algorithm.

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