

Stability Criteria for Linear Discrete-Time Systems with Interval-Like Time-Varying Delay

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Abstract—This paper is concerned with the stability problem for a class of uncertain linear discrete-time systems with time-varying delay. The delay is of an interval-like type, which means that both lower and upper bounds for the time-varying delay are available. The uncertainty under consideration is norm-bounded uncertainty. Based on Lyapunov-Krasovskii functional approach, delay-dependent stability criteria are obtained using a sum inequality which is first introduced and plays an important role in deriving stability conditions. The criteria are formulated in the form of linear matrix inequalities (LMIs). A numerical example is given to show the effectiveness of the proposed criteria.

I. INTRODUCTION

During the last two decades the stability problem of linear continuous-time systems with time-delay has received considerable attention. The practical examples of time-delay systems include engineering, communications and biological systems [5]. The existence of delay in a practical system may induce instability, oscillation and poor performance [8]. For recent achievements, see [3] and reference therein. Compared with linear continuous-time systems with time-delay, less attention has been paid to linear *discrete-time* systems with time-delay. The reason is that for linear discrete-time systems with *constant* time-delay, one can transform them into the delay-free systems via state augmentation approach. However, the augmentation approach can not be applied to linear discrete-time systems with *time-varying* delay.

Similar to the case of linear continuous-time systems with time-delay [3], stability criteria for linear discrete-time systems with time-delay can be classified into two types: delay-independent stability criteria [9], [12] and delay-dependent stability criteria [6], [7]. In general, the delay-dependent stability criteria can provide some less conservative results than delay-independent stability criteria. Therefore, in the recent years, the delay-dependent stability problem of linear

discrete-time systems with time-delay, especially with time-varying delay, has attracted some researchers' interest.

For linear continuous-time systems, it is well known that there are some systems which are stable with some *nonzero* delay, but are unstable without delay [1], [2]. For such case, if there is a time-varying perturbation on the nonzero delay, it is of great significance to consider the stability of systems with *interval time-varying* delay. The stability of such kinds of systems was investigated in [4] using the Lyapunov-Krasovskii approach.

In this paper, we will consider the stability problem for a class of linear discrete-time delay systems with **interval-like** time-varying delay, the discrete analogues of linear continuous-time systems with *interval time-varying* delay. The linear discrete-time delay systems with **interval-like** time-varying delay appear in the field of networked control systems [10], [11]. Based on Lyapunov-Krasovskii functional approach, delay-dependent stability criteria will be derived by using a sum inequality which will be first established. A numerical example will be given to show the effectiveness of the criteria.

Notation: For symmetric matrices X and Y , the notation $X > Y$ ($X \geq Y$) means that matrix $X - Y$ is positive definite (positive semi-definite). I is an identity matrix of appropriate dimensions. Matrices, if not explicitly stated, are assumed to have compatible dimensions. For any real matrix A , A^T denotes the transpose of matrix A . For any nonsingular matrix A , A^{-1} denotes the inverse of matrix A . $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ matrices. $W^{\frac{1}{2}}$ denotes the square root of symmetric positive semi-definite matrix $W \geq 0$ ($W^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}V^T$ with V the eigenvector matrix of W satisfying $VV^T = I$ and Λ the diagonal eigenvalues matrix of W).

II. PROBLEM STATEMENT

Consider the following linear discrete-time system with time-varying delay

$$\begin{cases} x(k+1) = [A + \Delta A(k)]x(k) \\ \quad \quad \quad + [A_1 + \Delta A_1(k)]x(k-h(k)), \\ x(k) = \phi(k), \quad -h_M \leq k \leq 0, \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, A and A_1 are known real parameter matrices of appropriate dimensions, $\Delta A(k)$ and $\Delta A_1(k)$ are real-valued unknown matrices representing discrete-time varying parameter uncertainties of (1), and are assumed to be of the form

$$\begin{bmatrix} \Delta A(k) & \Delta A_1(k) \end{bmatrix} = DF(k) \begin{bmatrix} E & E_1 \end{bmatrix}, \quad (2)$$

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where D , E and E_1 are known real constant matrices of appropriate dimensions. $F(k) \in \mathbb{R}^{\alpha \times \beta}$ is a discrete-time varying uncertainty matrix satisfying

$$F^T(k)F(k) \leq I, \quad (3)$$

$\phi(k)$ is the initial condition of the system (1). $h(k)$ is a positive integer function representing the time-varying delay of the system (1) satisfying

$$0 < h_m \leq h(k) \leq h_M, \quad (4)$$

where h_m and h_M are two known positive integers, for this case, the $h(k)$ is called an **interval-like** time-varying delay.

The purpose of this paper is to develop delay-dependent robust stability criteria for the system (1) for any **interval-like** time-varying delay $h(k)$ satisfying (4).

To end this section, we introduce the following lemma that will play an important role in deriving the stability criteria.

Lemma 1: For any constant positive semi-definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $W^T = W \geq 0$, two positive integers r and r_0 satisfying $r \geq r_0 \geq 1$, the following inequality holds

$$\left(\sum_{i=r_0}^r x(i) \right)^T W \left(\sum_{i=r_0}^r x(i) \right) \leq \tilde{r} \sum_{i=r_0}^r x^T(i) W x(i).$$

where $\tilde{r} = r - r_0 + 1$.

Proof: It is easy to see that

$$\begin{aligned} & \left(\sum_{i=r_0}^r x(i) \right)^T W \left(\sum_{i=r_0}^r x(i) \right) \\ &= \frac{1}{2} \sum_{i=r_0}^r \sum_{j=r_0}^r 2x^T(i) W x(j) \\ &= \frac{1}{2} \sum_{i=r_0}^r \sum_{j=r_0}^r 2 \left(W^{\frac{1}{2}} x(i) \right)^T \left(W^{\frac{1}{2}} x(j) \right) \\ &\leq \frac{1}{2} \sum_{i=r_0}^r \sum_{j=r_0}^r (x^T(i) W x(i) + x^T(j) W x(j)) \\ &= (r - r_0 + 1) \sum_{i=r_0}^r x^T(i) W x(i). \end{aligned}$$

The proof is complete. ■

III. MAIN RESULT

Defining $h_{av} = \begin{cases} \frac{1}{2}(h_M + h_m), & \text{if } h_M + h_m \text{ is an even integer} \\ \frac{1}{2}(h_M + h_m - 1), & \text{if } h_M + h_m \text{ is an odd integer} \end{cases}$

and $\delta = \max\{h_{av} - h_m, h_M - h_{av}\}$, then $h(k)$ is a discrete-time time-varying sequence satisfying $h_{av} - \delta \leq h(k) \leq h_{av} + \delta$, where δ can be taken as the range of variation of the time delay $h(k)$.

In this section, employing the sum inequality in Lemma 1, a delay-dependent stability criterion in terms of an LMI

form is first presented for the following nominal system with interval-like time-varying delay $h(k)$ satisfying (4).

$$\begin{cases} x(k+1) = Ax(k) + A_1x(k-h(k)), \\ x(k) = \phi(k), \quad -h_M \leq k \leq 0. \end{cases} \quad (5)$$

Proposition 1: For some given positive integers h_m and h_M , the system (5) is asymptotically stable for any $h(k)$ satisfying (4), if there exist some matrices $P > 0$, $Q > 0$, $R > 0$ and $S > 0$ of appropriate dimensions such that the following LMI holds

$$\Xi \triangleq \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0, \quad (6)$$

where

$$\begin{aligned} \Xi_{11} &= \begin{bmatrix} -P + Q - R & R & 0 \\ R & -Q - R & 0 \\ 0 & 0 & -\delta S \end{bmatrix}, \\ \Xi_{12} &= \begin{bmatrix} A^T P & h_{av}(A - I)^T R & \tilde{\delta}(A - I)^T S \\ A_1^T P & h_{av} A_1^T R & \tilde{\delta} A_1^T S \\ \delta A_1^T P & \delta h_{av} A_1^T R & \tilde{\delta} \delta A_1^T S \end{bmatrix}, \\ \Xi_{22} &= \text{diag}\{ -P \quad -R \quad -\tilde{\delta} S \}, \\ \tilde{\delta} &= 2\delta + 1. \end{aligned}$$

Proof: Choose a Lyapunov-Krasovskii functional candidate as follows

$$V(k) \triangleq V_1(k) + V_2(k) + V_3(k) + V_4(k), \quad (7)$$

where

$$\begin{aligned} V_1(k) &= x^T(k) P x(k), \\ V_2(k) &= \sum_{i=k-h_{av}}^{k-1} x^T(i) Q x(i), \\ V_3(k) &= h_{av} \sum_{i=1}^{h_{av}} \sum_{j=k-i}^{k-1} e^T(j) R e(j), \\ V_4(k) &= \sum_{i=h_{av}-\delta}^{h_{av}+\delta} \sum_{j=k-i}^{k-1} e^T(j) S e(j), \\ e(j) &= x(j) - x(j+1), \end{aligned}$$

where $P > 0$, $Q > 0$, $R > 0$ and $S > 0$. It is easy to see that the system (5) can be rewritten as

$$\begin{aligned} & x(k+1) = Ax(k) + A_1x(k-h_{av}) \\ & + A_1[x(k-h(k)) - x(k-h_{av})] \\ & = \begin{cases} Ax(k) + A_1x(k-h_{av}) \\ + A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i), & \text{if } h(k) > h_{av} \\ Ax(k) + A_1x(k-h_{av}), & \text{if } h(k) = h_{av} \\ Ax(k) + A_1x(k-h_{av}) \\ - A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1). & \text{if } h(k) < h_{av} \end{cases} \quad (8) \end{aligned}$$

Case I: $h(k) > h_{av}$.

Taking the difference of $V_1(k)$, the increment of $V_1(k)$ is

$$\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k) \\
&= x^T(k)A^T P A x(k) \\
&\quad + 2x^T(k)A^T P A_1 x(k - h_{av}) \\
&\quad + 2x^T(k)A^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k-h_{av})A_1^T P A_1 x(k-h_{av}) \\
&\quad + 2x^T(k-h_{av})A_1^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + \left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right)^T A_1^T P A_1 \times \\
&\quad \left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right) - x^T(k)P x(k). \quad (9)
\end{aligned}$$

The increment of $V_2(k)$ is easily computed as

$$\begin{aligned}
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&= x^T(k)Q x(k) - x^T(k-h_{av})Q x(k-h_{av}). \quad (10)
\end{aligned}$$

The increment of $V_3(k)$ is

$$\begin{aligned}
\Delta V_3(k) &= V_3(k+1) - V_3(k) \\
&= h_{av}^2 e^T(k)R e(k) \\
&\quad - h_{av} \sum_{i=1}^{h_{av}} e^T(k-i)R e(k-i) \quad (11)
\end{aligned}$$

The increment of $V_4(k)$ is

$$\begin{aligned}
\Delta V_4(k) &= V_4(k+1) - V_4(k) \\
&= (2\delta+1)e^T(k)S e(k) \\
&\quad - \sum_{i=h_{av}-\delta}^{h_{av}+\delta} e^T(k-i)S e(k-i). \quad (12)
\end{aligned}$$

From (9)-(12) we have

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&= x^T(k)A^T P A x(k) \\
&\quad + 2x^T(k)A^T P A_1 x(k-h_{av}) \\
&\quad + 2x^T(k)A^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k-h_{av})A_1^T P A_1 x(k-h_{av}) \\
&\quad + 2x^T(k-h_{av})A_1^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + \left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right)^T A_1^T P A_1 \times \\
&\quad \left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right) - x^T(k)P x(k) \\
&\quad + x^T(k)Q x(k) - x^T(k-h_{av})Q x(k-h_{av}) \\
&\quad + h_{av}^2 e^T(k)R e(k) \\
&\quad - h_{av} \sum_{i=1}^{h_{av}} e^T(k-i)R e(k-i) \\
&\quad + (2\delta+1)e^T(k)S e(k) \\
&\quad - \sum_{i=h_{av}-\delta}^{h_{av}+\delta} e^T(k-i)S e(k-i). \quad (13)
\end{aligned}$$

Use Lemma 1 to obtain

$$\begin{aligned}
&\left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right)^T A_1^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\leq (h(k) - h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^T P A_1 e(k-i), \quad (14)
\end{aligned}$$

and

$$\begin{aligned}
&-h_{av} \sum_{i=1}^{h_{av}} e^T(k-i)R e(k-i) \\
&\leq -\left(\sum_{i=1}^{h_{av}} e(k-i) \right)^T R \left(\sum_{i=1}^{h_{av}} e(k-i) \right) \\
&= -[x(k-h_{av}) - x(k)]^T R [x(k-h_{av}) - x(k)]. \quad (15)
\end{aligned}$$

Noting that (8) we have

$$\begin{aligned}
&e^T(k)\Upsilon e(k) \\
&= x^T(k)(A-I)^T \Upsilon (A-I)x(k) \\
&\quad + 2x^T(k)(A-I)^T \Upsilon A_1 x(k-h_{av}) \\
&\quad + 2x^T(k)(A-I)^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k-h_{av})A_1^T \Upsilon A_1 x(k-h_{av}) \\
&\quad + 2x^T(k-h_{av})A_1^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + \left(\sum_{i=h_{av}+1}^{h(k)} e(k-i) \right)^T A_1^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i),
\end{aligned}$$

where $\Upsilon = h_{av}^2 R + (2\delta+1)S$. Using Lemma 1 again in the last term of the right hand of the above equality yields

$$\begin{aligned}
&e^T(k)\Upsilon e(k) \\
&\leq x^T(k)(A-I)^T \Upsilon (A-I)x(k) \\
&\quad + 2x^T(k)(A-I)^T \Upsilon A_1 x(k-h_{av}) \\
&\quad + 2x^T(k)(A-I)^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k-h_{av})A_1^T \Upsilon A_1 x(k-h_{av}) \\
&\quad + 2x^T(k-h_{av})A_1^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + (h(k) - h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^T \Upsilon A_1 e(k-i). \quad (16)
\end{aligned}$$

In addition, it is easy to see that

$$\begin{aligned}
& - \sum_{i=h_{av}-\delta}^{h_{av}+\delta} e^T(k-i)Se(k-i) \\
& \leq - \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)Se(k-i). \quad (17)
\end{aligned}$$

Then from (13)–(17) we have

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leq x^T(k)A^T P A x(k) + 2x^T(k)A^T P A_1 x(k - h_{av}) \\
&\quad + 2x^T(k)A^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k - h_{av})A_1^T P A_1 x(k - h_{av}) \\
&\quad + 2x^T(k - h_{av})A_1^T P A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + (h(k) - h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^T P A_1 e(k-i) \\
&\quad - x^T(k)P x(k) + x^T(k)Q x(k) \\
&\quad - x^T(k - h_{av})Q x(k - h_{av}) \\
&\quad - [x(k - h_{av}) - x(k)]^T R [x(k - h_{av}) - x(k)] \\
&\quad - \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)S e(k-i) \\
&\quad + x^T(k)(A - I)^T \Upsilon (A - I)x(k) \\
&\quad + 2x^T(k)(A - I)^T \Upsilon A_1 x(k - h_{av}) \\
&\quad + 2x^T(k)(A - I)^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + x^T(k - h_{av})A_1^T \Upsilon A_1 x(k - h_{av}) \\
&\quad + 2x^T(k - h_{av})A_1^T \Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&\quad + (h(k) - h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^T \Upsilon A_1 e(k-i) \\
&= \frac{1}{h(k) - h_{av}} \sum_{i=h_{av}+1}^{h(k)} \xi^T(k, i) \Phi \xi(k, i), \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
\xi^T(k, i) &= [x^T(k) \quad x^T(k - h_{av}) \quad e^T(k - i)], \\
\Phi &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} \end{bmatrix}, \\
\Phi_{11} &= A^T P A - P + Q - R \\
&\quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)(A - I), \\
\Phi_{12} &= A^T P A_1 + R \\
&\quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\Phi_{13} &= (h(k) - h_{av})[A^T P A_1 \\
&\quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\Phi_{22} &= A_1^T P A_1 - Q - R \\
&\quad + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\Phi_{23} &= (h(k) - h_{av})[A_1^T P A_1
\end{aligned}$$

$$\begin{aligned}
& + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\Phi_{33} &= (h(k) - h_{av})^2 [A_1^T P A_1 + A_1^T (h_{av}^2 R \\
& + (2\delta + 1)S)A_1] - (h(k) - h_{av})S.
\end{aligned}$$

Case II: $h(k) = h_{av}$.

For this case it is easy to get

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leq x^T(k)A^T P A x(k) \\
&\quad + 2x^T(k)A^T P A_1 x(k - h_{av}) \\
&\quad + x^T(k - h_{av})A_1^T P A_1 x(k - h_{av}) \\
&\quad - x^T(k)P x(k) + x^T(k)Q x(k) \\
&\quad - x^T(k - h_{av})Q x(k - h_{av}) \\
&\quad - [x(k - h_{av}) - x(k)]^T R [x(k - h_{av}) - x(k)] \\
&\quad + x^T(k)(A - I)^T \Upsilon (A - I)x(k) \\
&\quad + 2x^T(k)(A - I)^T \Upsilon A_1 x(k - h_{av}) \\
&\quad + x^T(k - h_{av})A_1^T \Upsilon A_1 x(k - h_{av}) \\
&= \eta^T(k) \hat{\Phi} \eta(k), \quad (19)
\end{aligned}$$

where

$$\begin{aligned}
\eta^T(k) &= [x^T(k) \quad x^T(k - h_{av})], \\
\hat{\Phi} &= \begin{bmatrix} \hat{\Phi}_{11} & \hat{\Phi}_{12} \\ \hat{\Phi}_{12}^T & \hat{\Phi}_{22} \end{bmatrix}, \\
\hat{\Phi}_{11} &= A^T P A - P + Q - R \\
&\quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)(A - I), \\
\hat{\Phi}_{12} &= A^T P A_1 + R \\
&\quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\hat{\Phi}_{22} &= A_1^T P A_1 - Q - R \\
&\quad + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1.
\end{aligned}$$

Case III: $h(k) < h_{av}$.

Similar to Case I, we have

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leq x^T(k)A^T P A x(k) + 2x^T(k)A^T P A_1 x(k - h_{av}) \\
&\quad - 2x^T(k)A^T P A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&\quad + x^T(k - h_{av})A_1^T P A_1 x(k - h_{av}) \\
&\quad - 2x^T(k - h_{av})A_1^T P A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&\quad + (h_{av} - h(k)) \sum_{i=h(k)}^{h_{av}-1} e^T(k-i-1)A_1^T P A_1 e(k-i-1) \\
&\quad - x^T(k)P x(k) + x^T(k)Q x(k) \\
&\quad - x^T(k - h_{av})Q x(k - h_{av}) \\
&\quad - [x(k - h_{av}) - x(k)]^T R [x(k - h_{av}) - x(k)] \\
&\quad - \sum_{i=h(k)}^{h_{av}-1} e^T(k-i-1)S(k-i-1) \\
&\quad + x^T(k)(A - I)^T \Upsilon (A - I)x(k) \\
&\quad + 2x^T(k)(A - I)^T \Upsilon A_1 x(k - h_{av}) \\
&\quad - 2x^T(k)(A - I)^T \Upsilon A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&\quad + x^T(k - h_{av})A_1^T \Upsilon A_1 x(k - h_{av})
\end{aligned}$$

$$\begin{aligned}
& -2x^T(k - h_{av})A_1^T \Upsilon A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
& + (h_{av} - h(k)) \sum_{i=h(k)}^{h_{av}-1} e^T(k-i-1)A_1^T \Upsilon A_1 e(k-i-1) \\
& = \frac{1}{h_{av} - h(k)} \sum_{i=h(k)}^{h_{av}-1} \zeta^T(k, i) \tilde{\Phi} \zeta(k, i), \quad (20)
\end{aligned}$$

where

$$\begin{aligned}
\zeta^T(k, i) &= \begin{bmatrix} x(k) \\ x(k - h_{av}) \\ [x(k - i) - x(k - i - 1)] \end{bmatrix}, \\
\tilde{\Phi} &= \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} \\ \tilde{\Phi}_{12}^T & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} \\ \tilde{\Phi}_{13}^T & \tilde{\Phi}_{23}^T & \tilde{\Phi}_{33} \end{bmatrix}, \\
\tilde{\Phi}_{11} &= A^T P A - P + Q - R \\
& \quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)(A - I), \\
\tilde{\Phi}_{12} &= A^T P A_1 + R \\
& \quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\tilde{\Phi}_{13} &= (h_{av} - h(k))[A^T P A_1 \\
& \quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\tilde{\Phi}_{22} &= A_1^T P A_1 - Q - R \\
& \quad + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\tilde{\Phi}_{23} &= (h_{av} - h(k))[A_1^T P A_1 \\
& \quad + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\tilde{\Phi}_{33} &= (h_{av} - h(k))^2 [A_1^T P A_1 + A_1^T (h_{av}^2 R \\
& \quad + (2\delta + 1)S)A_1] - (h_{av} - h(k))S.
\end{aligned}$$

Summarizing the above discussions, from (18), (19) and (20) we obtain

$$\Delta V(k) = \begin{cases} \frac{1}{h(k) - h_{av}} \sum_{i=h_{av}+1}^{h(k)} \xi^T(k, i) \Phi \xi(k, i), & \text{if } h(k) > h_{av} \\ \eta^T(k) \hat{\Phi} \eta(k), & \text{if } h(k) = h_{av} \\ \frac{1}{h_{av} - h(k)} \sum_{i=h(k)}^{h_{av}-1} \zeta^T(k, i) \tilde{\Phi} \zeta(k, i). & \text{if } h(k) < h_{av} \end{cases} \quad (21)$$

Noting that $|h(k) - h_{av}| \leq \delta$ for $k = 1, 2, 3, \dots$, $\Phi < 0$, $\hat{\Phi} < 0$ and $\tilde{\Phi} < 0$ are implied by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} \\ \Theta_{13}^T & \Theta_{23}^T & \Theta_{33} \end{bmatrix} < 0, \quad (22)$$

where

$$\begin{aligned}
\Theta_{11} &= A^T P A - P + Q - R \\
& \quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)(A - I), \\
\Theta_{12} &= A^T P A_1 + R \\
& \quad + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\Theta_{13} &= \delta [A^T P A_1
\end{aligned}$$

$$\begin{aligned}
& + (A - I)^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\Theta_{22} &= A_1^T P A_1 - Q - R \\
& \quad + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1, \\
\Theta_{23} &= \delta [A_1^T P A_1 + A_1^T (h_{av}^2 R + (2\delta + 1)S)A_1], \\
\Theta_{33} &= \delta^2 [A_1^T P A_1 + A_1^T (h_{av}^2 R \\
& \quad + (2\delta + 1)S)A_1] - \delta S.
\end{aligned}$$

So, if (22) holds, then $\Delta V(k) \leq -\lambda x^T(k)x(k)$ for some scalar $\lambda > 0$. Therefore, the system (5) is asymptotically stable. By Schur complement, (22) is equivalent to (6). The proof is complete. ■

Concerning the norm-bounded uncertainty, by Proposition 1, the following corollary is easily obtained for system (1).

Corollary 1: For some given positive integers h_m and h_M , the system (1) is robustly stable for any time-delay $h(k)$ satisfying (4) and all admissible parameter uncertainties satisfying (2) and (3), if there exist a scalar $\varepsilon > 0$, some matrices $P > 0$, $Q > 0$, $R > 0$ and $S > 0$ of appropriate dimensions such that the following LMI holds

$$\begin{bmatrix} \alpha_1 & \alpha_2^T \\ \alpha_2 & \alpha_3 \end{bmatrix} < 0. \quad (23)$$

where

$$\begin{aligned}
\alpha_1 &= \begin{bmatrix} -P + Q - R & R & 0 \\ R & -Q - R & 0 \\ 0 & 0 & -\delta S \end{bmatrix}, \\
\alpha_2 &= \begin{bmatrix} P A & P A_1 & \delta P A_1 \\ h_{av} R (A - I) & h_{av} R A_1 & \delta h_{av} R A_1 \\ \tilde{\delta} S (A - I) & \tilde{\delta} S A_1 & \tilde{\delta} \delta S A_1 \\ 0 & 0 & 0 \\ \varepsilon E & \varepsilon E_1 & \varepsilon \delta E_1 \end{bmatrix}, \\
\alpha_3 &= \begin{bmatrix} -P & 0 & 0 & P D & 0 \\ 0 & -R & 0 & h_{av} R D & 0 \\ 0 & 0 & -\tilde{\delta} S & \tilde{\delta} S D & 0 \\ D^T P & h_{av} D^T R & \tilde{\delta} D^T S & -\varepsilon I & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix}
\end{aligned}$$

$$\tilde{\delta} = 2\delta + 1.$$

Proof: Replace A and A_1 with $A + DF(k)E$ and $A_1 + DF(k)E_1$ in (6), respectively, to obtain

$$\Xi + \vartheta_1 F(k) \vartheta_2^T + \vartheta_2 F^T(k) \vartheta_1^T < 0, \quad (24)$$

where Ξ is defined in Proposition 1 and

$$\begin{aligned}
\vartheta_1^T &= [0 \quad 0 \quad 0 \quad D^T P \quad h_{av} D^T R^T \quad (2\delta + 1) D^T S^T] \\
\vartheta_2 &= [E \quad E_1 \quad \delta E_1 \quad 0 \quad 0 \quad 0]
\end{aligned}$$

It is clear to see that (24) is equivalent to

$$\Xi + \varepsilon^{-1} \vartheta_1 \vartheta_1^T + \varepsilon \vartheta_2 \vartheta_2^T < 0 \quad (25)$$

for a scalar $\varepsilon > 0$. By Schur complement and (6), (25) is equivalent to (23). This completes the proof. ■

Remark 1: The scalar $\varepsilon > 0$ in (23) can be absorbed by other variables by introducing $\tilde{P} = \varepsilon^{-1} P$, $\tilde{Q} = \varepsilon^{-1} Q$, $\tilde{R} = \varepsilon^{-1} R$ and $\tilde{S} = \varepsilon^{-1} S$.

Remark 2: Proposition 1 and Corollary 1 provide the delay-dependent stability conditions which are formulated in an LMI form. Hence, it is easy to compute the maximum bound of the allowable length δ of the interval-like of time-varying delay for given h_{av} or the maximum bound of h_{av} for given δ using efficient convex optimization algorithms.

Remark 3: Based on the obtained stability criteria, one can easily handle the synthesis problem for uncertain linear discrete-time systems with interval-like time-varying delay.

IV. A NUMERICAL EXAMPLE

To show the effectiveness of the proposed delay-dependent stability criteria, consider the system described by (1), (2) and (3) with

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.91 \end{bmatrix}, A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \\ E &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, E_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ D &= \alpha I, \alpha \geq 0. \end{aligned}$$

Case I $h_m = h_M = h(k) = h$ is a constant.

For $\alpha = 0$, it is pointed out in [7] that any delay-independent stability criteria fail to verify the stability of the system. Using Proposition 1, the maximum allowed bound h is obtained as $h_m = h_M = 42$ which is less conservative than the one in [7].

For $\alpha > 0$, we consider the effect of the uncertainty bound α on the maximum allowed bound h for robust stability. Numerical results are listed in Table I using Proposition 2. It is clear to see that as α increases, h decreases.

Table. Bound h calculated for different α

α	0	0.5	1	1.5	2	2.5	3
h	42	35	29	25	21	18	16

Case II $h(k)$ is a time-varying delay.

For $\alpha = 0$, by Proposition 1 the considered system is asymptotically stable for $h(k)$ satisfying $7 \leq h(k) \leq 13$, while for $\alpha = 1$, applying Proposition 2 one can guarantee that the system is robustly stable for $h(k)$ satisfying $8 \leq h(k) \leq 12$.

V. CONCLUSION

This paper has proposed stability criteria for a class of uncertain linear discrete-time systems with interval-like time-varying delay. A sum inequality has been established and employed to derive the criteria which are dependent on the lower and upper bounds of the time-varying delay. A numerical example has demonstrated the effectiveness of the criteria.

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