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Fault Diagnosis of Discrete Time Linear Systems Using Transmission Zeros and Zero Directions

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Abstract— **In this paper we present a new algorithm to generate a class of input signals corresponding to which the combination of outputs of a discrete time plant in its output zero direction is always zero irrespective of time. Using the above mentioned result and the output zeroing result of Tokarzewski (1999) we propose two tests to find the fault indicating row and fault indicating column of the plant transfer function matrix. Our fault detection scheme is based on the following triple: the output zero direction of the plant, the transmission zero of the plant, and the input zero direction of the plant. Based on the fault indicating rows and columns we identify those elements of the plant transfer function matrix which have undergone changes. The test is also applicable to those cases which have multiple fault indicating rows, columns and elements of the plant transfer function matrix. A system of four interconnected water tanks with two multivariable zeros was used to illustrate the results.**

I. Introduction

Single input single (SISO) discrete time systems have the unique property of blocking certain input signals based on their zeros. The transmission blocking property can be extended to the multiple inputs multiple outputs (MIMO) systems also. MacFarlane and Karcanias (1974) for the continuous time system and Tokarzewski (1999) for the discrete time system provided a class of inputs and initial conditions corresponding to which all the outputs of the plant are zero irrespective of time. We present here a novel algorithm to generate an input and the corresponding initial state vector so that the combinations of all outputs of a discrete time linear system in the output zero direction is always zero.

The literature is filled with various approaches towards fault diagnosis. Kramer and Palowitch (1987) used graph theory to predict the faults in a system. In the state space based methods, observers such as Kalman filters are used to predict faults. Lund (1992) used multiple Kalman filters to discriminate between two or more process models. In the probabilistic methods the occurrence of faults and disturbances is considered a stochastic process. Rojas-Guzman and Kramer (1993) used probability to find the most likely fault based on the available information about the state of the system. In the present work we use a special class of input signals to identify the fault indicating rows and columns and hence the changed elements of the transfer function matrix of a discrete time plant. The generated input signals are dependent on the triple of transmission zeros, input zero direction and output zero direction of the plant.

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II. Nomenclature and Definitions

A. Nomenclature

Since the two plants P and P', defined by (5) and (6) respectively, will be referred frequently in this paper therefore the nomenclature for input, output and state variable vectors and input, output and state zero directions for the plants P and P' are being provided.

$U(k)$ Input vector for both plants P and P'

$x(k)$ State variable vector for both plants P and P'

$y(k)$ Output vector for plant P

$y'(k)$ Output vector for plant P'

g Input zero direction of the plant P

g' Input zero direction of the plant P'

x_0 State zero direction of the plant P

x'_o State zero direction of the plant P'

v Output zero direction of the plant P

v' Output zero direction of the plant P'

q Transmission zero of the plant P

q' Transmission zero of the plant P'

Notice that the state variable vector and the input vector for both plants are the same. Before proceeding further it will be useful to provide some definitions of the terms which will be used in the rest of this paper. For a linear discrete time system defined as

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) \tag{1}$$

with n states, m inputs and r outputs the polynomial system matrix $P(z)$ is defined as

$$P(z)=\begin{bmatrix} zI-A & -B \\ C & 0 \end{bmatrix} \quad (2)$$

Here z is the z-transform variable. z has the same role in discrete time system as s has in the continuous time system. The transmission zeros are the values $z=q$ for which $P(z)$ loses rank. The state zero vector, x_0 and the input zero direction, g are defined as the solution to the following equation.

$$\begin{bmatrix} qI-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ g \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

The output zero direction v is defined as follows

$$\begin{bmatrix} x_v & v \end{bmatrix}\begin{bmatrix} qI-A & -B \\ C & 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \quad (4)$$

B. Transmission-blocking Theorem for Discrete-time system Tokarzewski (1999)

The transmission blocking problem as formulated by Isidori (1995) is as follows: find all pairs $(x_0, u(k))$, consisting of an initial state $x_0 \in R^n$ and a real-valued input vector sequence $u_0(k), k=0,1,2.....$, such that the corresponding output $y(k)$ is identically zero for all $k=0,1,2..$ In 1999 Tokarzewski came up with a solution to this problem. If $q \in C$ is a transmission zero of plant P then the input

$$u(k)=\begin{cases} g \text{ for k} = 0 \\ gq^k \text{ for k} = 1,2... \end{cases}$$

applied to P at the initial condition $x(0)=x_0$ yields the solution to the state equation of the form

$$x(k)=\begin{cases} x_0 \text{ for k} = 0 \\ \mathrm{x}_0 q^k \text{ for k} = 1,2,.. \end{cases}$$

and the system response $y(k)= 0$ for $k=0,1,2,..$

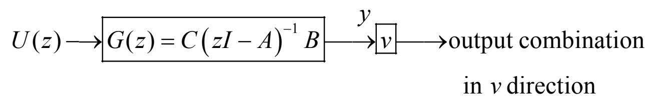
It is a well known fact that in the steady state each output of the plant goes to zero when the input is applied in the input zero direction. Also if the plant is in steady state then the combination of outputs in the output zero direction is always zero. MacFarlane and Karcanias showed for continuous time plants and Tokarzewski showed for discrete time plants that output zeroing property can be obtained even when the plant is not in the steady state. In the following sections it has been proved that the zeroing of the output combination in the output zero direction is also possible for the non-steady state of the discrete time plants.

C. Problem Formulation of the zeroing of outputs in output zero direction

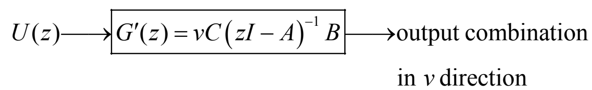
Consider a minimal plant P defined by the following equations

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k) \end{aligned} \quad (5)$$

with n states, m inputs and r outputs. Now if v is the output zero direction of the plant P then taking the combination of outputs in the output zero direction can be described by following block diagram



which can be further simplified to



Thus the problem of zeroing the output combination in output zero direction of plant P can be reduced to the problem of output zeroing of the plant P' which is defined as follows

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= vCx(k) \end{aligned} \quad (6)$$

where A,B and C the system matrices of original are plant P and v is the output zero direction of the original plant P. At first glance the solution to this problem seems very obvious because the transmission zero and input zero direction of P' can be calculated using the following equation

$$\begin{bmatrix} q'I-A & -B \\ vC & 0 \end{bmatrix}\begin{bmatrix} x_0' \\ g' \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

and then from the output zeroing result of Tokarzewski (1999) we can send the input signal of the form $g'(q')^k$ ($k=0,1,2...$) with initial state vector equal to x_0' in order to get the output of the plant P' always equal to zero or in other words get the combination of the outputs of the plant P in the output zero direction of P, always equal to zero. However the problem is not as trivial as it seems. It should be noted that the number of outputs for the plant P is one whereas the number of inputs to the plant P is m. Davison and Wang (1974) showed that if the number of inputs and outputs are not same for almost all (A,B,C) triples the system has no transmission zeros. Hence there is a need to approach this problem in an alternative way.

Let the j[th] column of the B matrix be denoted by b_j. Let q_j be the transmission zero corresponding to the j[th] input channel and is defined as the value $z = q_j$ for which the following matrices loses its rank

$$\begin{bmatrix} zI - A & -b_j \\ vC & 0 \end{bmatrix} \quad (8)$$

Let g_j and x_{0j} be the input zero direction and state zero vector respectively corresponding to the j[th] input channel and they are found by the following equation

$$\begin{bmatrix} q_j I - A & -b_j \\ vC & 0 \end{bmatrix} \begin{bmatrix} x_{0j} \\ g_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

Notice that existence of q_j is guaranteed for almost all cases since the number of output and input for the plant is equal (i.e. one).

III. Main result

If the input to the plants P and P' is given by

$$u(k) = \begin{bmatrix} g_1 q_1^k & .... & g_j q_j^k & .... & g_m q_m^k \end{bmatrix}^T \quad (10)$$

for all $k = 0,1,2.....$ then the following result holds.

Theorem 1: For previously defined plants P and P' and input $u(k)$ the state vector for both the plants is given by

$$x(k) = A^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) + \sum_{j=1}^{m} q_j^k x_{0j} \quad (11)$$

The output of the plant P' is given by

$$y'(k) = vCA^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) \quad (12)$$

and the output to the plant P is given by

$$y(k) = CA^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) + C \sum_{j=1}^{m} x_{0j} q_j^k \quad (13)$$

where $x(0)$ is the initial state vector for both the plants P and P' since the state vector for both P and P' is same for all k (change in the output matrix has no effect on the state variables).

Proof: The generalized solution for state vector for P and P' is given by

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^{k-l-1} B u(l) \quad (14)$$

Substituting for $u(l)$ we get

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^{k-l-1} \begin{pmatrix} b_1 g_1 q_1^l + b_2 g_2 q_2^l + . \\ .... + b_m g_m q_m^l \end{pmatrix}$$
$$= A^k x(0) + \sum_{l=0}^{k-1} A^{k-l-1} \left(\sum_{j=1}^{m} b_j g_j q_j^l \right) \quad (15)$$

For the j[th] input channel we have the following relations from (9)

$$\left(q_j I - A \right) x_{0j} = b_j g_j \quad (16)$$

$$vCx_{0j} = 0 \quad (17)$$

Substituting (16) in (15) we get

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^{k-l-1} \left(\sum_{j=1}^{m} b_j g_j q_j^l \right)$$
$$= A^k x(0) + \underbrace{\sum_{l=0}^{k-1} A^{k-l-1} \sum_{j=1}^{m} q_j^{l+1} x_{0j}}_{I} - \sum_{l=0}^{k-1} A^{k-l} \sum_{j=1}^{m} q_j^l x_{0j}$$

In I by doing change of variable $(l+1) \rightarrow l$ we get

$$x(k) = A^k x(0) + \sum_{l=1}^{k} A^{k-l} \sum_{j=1}^{m} q_j^l x_{0j} - \sum_{l=0}^{k-1} A^{k-l} \sum_{j=1}^{m} q_j^l x_{0j}$$
$$= A^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) + \sum_{j=1}^{m} q_j^k x_{0j} \quad (18)$$

Now, $y'(k) = vCx(k)$ (19)

Substituting (17) and (18) in (19) we get

$$y'(k) = vCA^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) + vC \sum_{j=1}^{m} q_j^k x_{0j}$$
$$= vCA^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) \quad (20)$$

Now output to the plant P is given by $y(k) = Cx(k)$. Substituting (18) in (20) we get

$$y(k) = CA^k \left(x(0) - \sum_{j=1}^{m} x_{0j} \right) + C \sum_{j=1}^{m} x_{0j} q_j^k$$

The above results can be generalized as follows.

Theorem 2: For previously defined plants P and P' and input $u(k)$ defined as

$$u(k) = \begin{bmatrix} \alpha_1 g_1 q_1^k & .... & \alpha_j g_j q_j^k & .... & \alpha_m g_m q_m^k \end{bmatrix}^T \quad (21)$$

where α_j is a scalar, the state vector for both the plants is given by

$$x(k) = A^k\left(x(0) - \sum_{j=1}^{m}\alpha_j x_{0j}\right) + \sum_{j=1}^{m}\alpha_j q_j^k x_{0j} \qquad (22)$$

The output of the plant P' is given by

$$y'(k) = vCA^k\left(x(0) - \sum_{j=1}^{m}\alpha_j x_{0j}\right) \qquad (23)$$

and the output to the plant P is given by

$$y(k) = CA^k\left(x(0) - \sum_{j=1}^{m}\alpha_j x_{0j}\right) + C\sum_{j=1}^{m}\alpha_j x_{0j} q_j^k \qquad (24)$$

where $x(0)$ is the initial state of plants P and P' since their state vectors are same for all k (change in the output matrix has no effect on the state variables).

Proof: The proof is similar to the proof of *Theorem 1*.

Lemma 1: In the results of theorem 1 and theorem 2 if we substitute $x(0) = \sum_{j=1}^{m} x_{0j}$ and $x(0) = \sum_{j=1}^{m}\alpha_j x_{0j}$ respectively, in both the cases we get $y'(k) = 0$ for all $k \geq 0$. It should be noted that even though the output of plant P is non-zero yet the output of the plant P' is zero for the above initial condition. In other words even though the components of the output of the plant P are non-zero yet their combination in the output zero direction of P is zero. This useful result will be used to obtain the combination of outputs of the original plant P in its output zero direction equal to zero.

Remark 1: Let $\mathbf{U} \in \mathbf{R}^m, \mathbf{X} \in \mathbf{R}^n, \mathbf{Y} \in \mathbf{R}^r$ be the input vector space, state vector space and the output vector space for the plant P respectively then

$$\mathbf{U} = span\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} \cdots \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$$

and $\mathbf{X} = span\begin{pmatrix}x_{01} & x_{02} & \cdots & x_{0m}\end{pmatrix}$ for $x(0) = \sum_{j=1}^{m}\alpha_j x_{0j}$.

Thus the relationship between the input space, state space and the output space for the zeroing of the output combination of plant P in the output zero direction of plant P can be shown by the geometrical relationships in fig. 1.

Using Lemma 1 an algorithm to obtain a set of input signals and the corresponding initial state vector such that the combinations of output components of the discrete time plant P in the output zero direction of plant P is always zero, is presented below. The steps are as follows

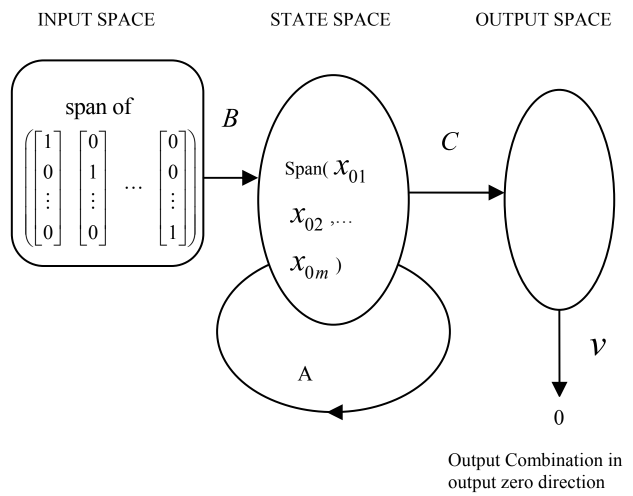


Figure 1: Geometric relationship between the input, output and state spaces

STEP 1: Find the transmission zero, input zero direction, output zero direction and state zero vector of the plant P using (2), (3) and (4).

STEP 2 : If b_j is the j[th] column of the B matrix then find the transmission zero q_j, input zero direction g_j and state zero vector using x_{0j} corresponding to j[th] input channel using (8) and (9).

STEP 3: Set the initial condition of the plant P as follows

$$x(0) = \sum_{j=1}^{m}\alpha_j x_{0j}$$

STEP 4: Use $u(k)$ defined by (21) as the input to P.

Remark 2: Theorem 2 helps us to upscale or downscale the input values for each input channel. Thus even though the $g_j q_j^k$ may not lie in normal range of u_j yet by careful selection of α_j we can bring it into the normal range of u_j.

IV. Novel Fault Detection Scheme for Discrete Time System

Based on Theorem 1, Theorem 2 and Lemma 1 below is a test to find the fault indicating column of the transfer function matrix $G(z)$ of plant P.

Column Test: If the input to the plant P and its initial conditions are given by $u(k) = \begin{bmatrix}0 & \ldots & 0 & g_j q_j^k & \ldots & 0\end{bmatrix}$ and $x(0) = x_{0j}$ then the combination of the outputs in the output zero direction should be zero. A non-zero value indicates that the elements of the plant transfer function matrix corresponding to the j[th] input (i.e. the j[th] column of $G(z)$) channel has changed.

Based on the output zeroing result of Tokarzewski (1999) stated before the following Lemma can be stated.

Lemma 2: Let q, x_0 and g be the transmission zero, state zero vector and the input zero direction of the plant respectively. Then for input $u(k) = gq^k$ for all $k \geq 0$ and initial condition $x(0) = x_0$ the non-zero value of the ith output indicates that the ith row of the transfer function matrix is faulty.
Proof: For the given input and initial condition all the outputs should be identically zero according to Tokarzewski. Since the ith output depends only on the ith row of $G(z)$ therefore the non-zero ith output indicates faulty ith row of $G(z)$.
Using Lemma 2 we get the following test for finding the faulty rows of the plant transfer function matrix of plant P.

Row Test: For input $u(k) = gq^k$ and initial condition $x(0) = x_0$ for the plant P the non-zero value of the ith output indicates that the ith row of the transfer function matrix is faulty.

Using the Row test and the column test in conjunction on the plant transfer function matrix $G(z)$ we can pin-point the faulty element of the plant transfer function matrix. Suppose using the Row Test we find that the ith row of $G(z)$ is faulty and using the Column Test we find that the jth column of $G(z)$ has faults.
Thus if we have only one faulty row and only one faulty column then we can easily deduce that g_{ij} element of the plant transfer function matrix is faulty.

V. AN ILLUSTRATIVE EXAMPLE

The above results were illustrated using a discrete time model of the quadruple tank system whose complete description and derivation of its continuous time model was provided by Johansson in 2000 [7]. A discrete time model with two inputs, two inputs and two transmission zeros was obtained using time steps of $Ts = 0.1$ seconds. The transfer function matrix s given by

$$G(z) = \begin{bmatrix} \dfrac{2.58}{620z-61} & \dfrac{0.74}{(620z-61)(115z-11)} \\ \dfrac{1.40}{(300z-29)(900z-89)} & \dfrac{2.82}{(900z-89)} \end{bmatrix} \quad (25)$$

Now the transmission zeros corresponding to the first and second input channels found using (8) are $q_1 = 0.0986$ and $q_2 = 0.0967$. Using (9) the corresponding input directions and state zero vectors are given by

$$g_1 = [-0.0220]; x_{01} = [-0.9178 \quad 0.3947 \quad -0.000 \quad -0.0358]^T$$

$$g_2 = [0.0675]; x_{02} = [-0.8042 \quad 0.3458 \quad 0.3182 \quad -0.3577]^T$$

For $u(k) = \left[-0.0220(0.0986)^k \quad 0.0675(0.0967)^k\right]^T$ and initial condition $x(0) = x_{01} + x_{02}$ we get the outputs as shown in Fig. 2. Fig. 2 verifies theorem 1. Now let us introduce some faults in the second column of $G(z)$ by changing the second column of the B matrix.

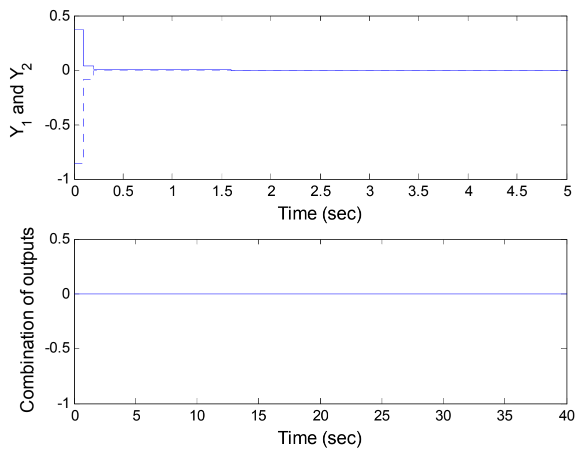


Figure 2: Plots of the outputs and their combination in output zero direction

Note that the changes to $G(z)$ can be made by changing either A, B or C matrix however changing second column of B only changes the second column of $G(z)$. Let the new $G(z)$ matrix be given as

$$G(z) = \begin{bmatrix} \dfrac{2.58}{620z-61} & \dfrac{17825z-1704.3}{(620z-61)(115z-11)} \\ \dfrac{1.40}{(300z-29)(900z-89)} & \dfrac{2.82}{(900z-89)} \end{bmatrix} \quad (26)$$

It can be noticed that $(1,2)$ element of $G(z)$ has changed.

A. Applying Column Test

For input $u(k) = \left[0 \quad 0.0675(0.0967)^k\right]^T$ and initial condition $x_{02} = [-0.8042 \quad 0.3458 \quad 0.3182 \quad -0.3577]^T$ the combination of outputs in output zero direction is shown in Fig. 3. Similarly by applying the column test to the first column we get the plot as shown in Fig. 4. From Fig. 3 and 4 we conclude that the fault lies in the second column.

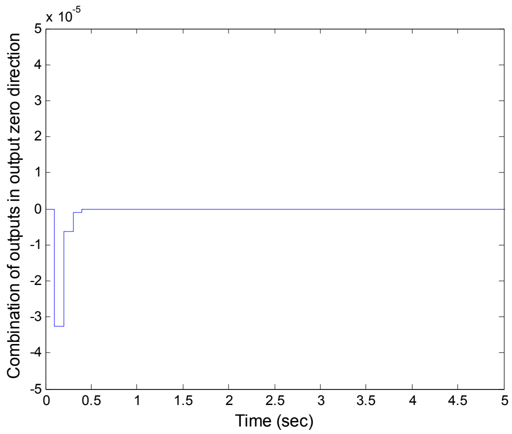


Figure 3: Combination of output in output zero direction

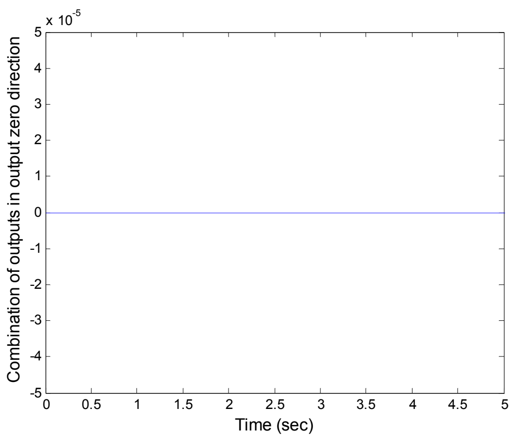


Figure 4: Combination of outputs in output zero direction

B. Applying Row Test

Now for our system the transmission zero, state zero vector and the input zero direction are as follows $z = 0.0941$; $x_0 = \begin{bmatrix} 0 & -0.000 & -0.7506 & -0.4699 \end{bmatrix}^T$ and $g = \begin{bmatrix} 0.3920 & 0.2494 \end{bmatrix}^T$ .For $u(k) = \begin{bmatrix} 0.3920 & 0.2494 \end{bmatrix}^T (0.0941)^k$ and $x(0) = \begin{bmatrix} 0 & -0.000 & -0.7506 & -0.4699 \end{bmatrix}^T$ we obtain the plots shown in Fig. 5. From Fig.5 it is clear that the first row (using the Row Test) of the plant transfer function matrix is faulty. Since in this case there is only one faulty row and one faulty column we can straightaway conclude that the fault lies in $(1,2)$ element of plant transfer function matrix. This matches with the result obtained by comparing the transfer function matrices given in (25) and (26).

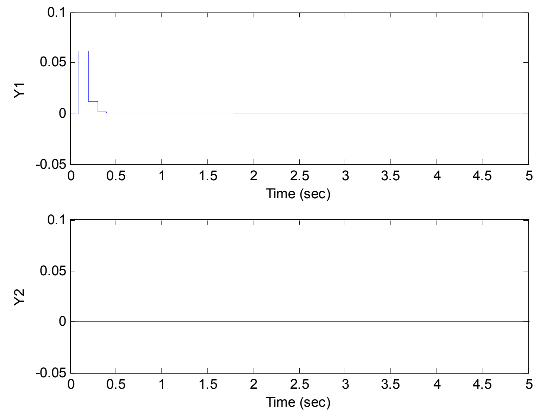


Figure 5: Outputs of the plant for an output zeroing input

VI. CONCLUSION

Plant's linearity was assumed before and after the occurrence of changes in the transfer function matrix. Though the result were illustrated on a plant with real transmission zeros, the results are equally applicable to systems having complex transmission zeros. Results in [6] help us to generate a pair of real initial state vector and input vector corresponding to complex conjugate transmission zero pair. Though the present work deals with only linear systems there is scope of extending this work to the non-linear systems.

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