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Observer with Multiple Sliding Modes for a Class of Nonlinear Uncertain Systems

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***Abstract*— Sliding mode observer design for a class of nonlinear uncertain systems is systematically developed in this paper. The observer is designed under less conservative conditions than existing sliding mode observers that are based on equivalent control concept. A methodology to select suitable transformation is formulated, and it divides the original system into three interconnected subsystems. Multiple sliding modes are then introduced to compensate for multiple disturbance terms in the subsystems by appending them to constant gain observer. The conditions for asymptotic stability of estimation error dynamics have been derived based on Lipschitz assumptions for nonlinear functions by standard Lyapunov analysis. Finally, simulation results are given to demonstrate the effectiveness of the proposed method.**

I. INTRODUCTION

Over the last decade, state estimation of nonlinear systems has been an active field of research [1], [2], [3], [4], [5], [6], [7]. The sliding mode approach is a well established method for handling disturbances and modelling uncertainties through sliding surface design and the equivalent control concept [4], [8], [9], [10]. Indeed, sliding mode observer (SMO) has been developed to robustly estimate the system states in the presence of uncertainties. In [5] the SMO design for linear systems is presented while [11] derived sliding mode observer by appending a switching term to a constant gain observer.

The equivalent control method for designing observer was adopted in [9] and was later extended to SISO nonlinear systems in [12]. Based on the idea in [13], an SMO was designed for general class of systems in [6] based on equivalent control method. However, one is required to filter the discontinuous estimation signal of the SMO to approximate the equivalent control [6]. In order to estimate the equivalent control, [6] imposed conservative assumptions on the structure of the system, i.e., the other parallel subsystem states appear only in the last equation of each subsystem. In this paper, we consider more general class of nonlinear systems than those considered in [6]. We remove the involutive conditions for disturbance vectors and design the observer based on a structural assumption in the transformed space. We introduce multiple sliding modes in order to handle multiple disturbances.

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II. PRELIMINARIES

Here we consider nonlinear systems that can be represented by the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}, u) + \sum_{i=1}^{m} \mathbf{p}_i(\mathbf{x}) d_i(\mathbf{x}, u, t) \\ y_1 = h_1(\mathbf{x}) \\ \cdots \qquad \cdots \\ y_c = h_c(\mathbf{x}) \end{cases} \tag{1}$$

in which $\mathbf{x} \in M$, a C^∞ connected manifold of dimension n, $\mathbf{f}(\mathbf{x}), \mathbf{G}(\mathbf{x}, u), \mathbf{P}(\mathbf{x}) = [\mathbf{p}_1(\mathbf{x}), \cdots, \mathbf{p}_m(\mathbf{x})]$ are smooth vector fields on M, and $h_i(\mathbf{x}), i = 1, \cdots, c$ are smooth functions from M to R. The disturbance vector is represented as $\mathbf{d}(\mathbf{x}, t) = [d_1, \cdots, d_m]$. The function $d_i(\mathbf{x}, u, t)$ denotes uncertainties and disturbances in the system, we assume $d_i(\mathbf{x}, u, t)$ to be bounded.

A. Nonlinear Transformation

By selecting $r_i - 1$ terms, (where r_i stands for relative degree of y_i output) instead of r_i terms in the traditional nonlinear transformation, we can avoid the disturbance inputs in the transformed space that normally appear in the r_i^{th} differentiation. A class of transformed subsystems that is not observable w.r.t. the unknown inputs is then obtained, and this is desired. The selection procedure is itemized as follows:

Step: 1 Assume there are s outputs that have relative degree > 2. The outputs are re-arranged such that the first s outputs have $r_j > 2$. We set

$$\boldsymbol{\phi}_j = \begin{bmatrix} h_j(\mathbf{x}) & L_{\mathbf{f}} h_j(\mathbf{x}) & \cdots & L_{\mathbf{f}}^{r_j - 2} h_j(\mathbf{x}) \end{bmatrix}^T \tag{2}$$

for all $j = 1, \cdots, s$.

Step: 2 From the above step, let $l = \sum_{j=1}^{s} (r_j - 1) + (c - s)$. If $l < n$, we choose all the outputs with relative degree $r_i = 2$ and some of the remaining outputs with the relative degree $r_i = 1$. Suppose that we have $a - s$ such outputs $y_{s+1}, \cdots, y_a$, we represent them as

$$\boldsymbol{\phi}_{w_0} = \begin{bmatrix} h_{s+1}(\mathbf{x}) & \cdots & h_a(\mathbf{x}) \end{bmatrix}^T \tag{3}$$

Step: 3 The remaining outputs are used to complete the transformation via

$$\boldsymbol{\phi}_{w_j} = \begin{bmatrix} h_i(\mathbf{x}) & L_{\mathbf{f}} h_i(\mathbf{x}) & \cdots & L_{\mathbf{f}}^{k_j - 1} h_i(\mathbf{x}) \end{bmatrix}^T \tag{4}$$

for $i = a+1, \cdots, c$ and $j = 1, \cdots, q$, where $q = c - a$. The selection of outputs and the value of k_j are done such that the transformation holds. That is, the outputs with relative degree one are shared between steps 2 and 3 so as to form n functions for the transformation.

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Our transformation matrix is thus given by

$$\begin{aligned}\tilde{\mathbf{x}} &= \left[\begin{array}{cccccccc}\boldsymbol{\phi}_1^T & \boldsymbol{\phi}_2^T & \cdots & \boldsymbol{\phi}_s^T & \boldsymbol{\phi}_{w_0}^T & \boldsymbol{\phi}_{w_1}^T & \cdots & \boldsymbol{\phi}_{w_q}^T\end{array}\right]^T \\ &= \boldsymbol{\Phi}(\mathbf{x}) \end{aligned} \tag{5}$$

The transformed coordinates are represented as $\tilde{\mathbf{x}}_i = \left[\begin{array}{cccc} x_1^i & x_2^i & \cdots & x_{r_i-1}^i \end{array}\right]^T$, $\tilde{\mathbf{x}}_{w_0} = \left[\begin{array}{cccc} x_1^{w_0} & x_2^{w_0} & \cdots & x_{a-s}^{w_0}\end{array}\right]^T$, $\tilde{\mathbf{x}}_{w_j} = \left[\begin{array}{cccc} x_1^{w_j} & x_2^{w_j} & \cdots & x_{k_j}^{w_j}\end{array}\right]^T$, where $\tilde{\mathbf{x}}_i = \boldsymbol{\phi}_i$, $\tilde{\mathbf{x}}_{w_0} = \boldsymbol{\phi}_{w_0}$ and $\tilde{\mathbf{x}}_{w_j} = \boldsymbol{\phi}_{w_j}$, for $i = 1, \cdots, s$, for $j = 1, \cdots, q$. Let $\tilde{\mathbf{x}}_z = \left[\tilde{\mathbf{x}}_1^T, \cdots, \tilde{\mathbf{x}}_s^T\right]$, $\tilde{\mathbf{x}}_w = \left[\tilde{\mathbf{x}}_{w_1}^T, \cdots, \tilde{\mathbf{x}}_{w_q}^T\right]$, then $\tilde{\mathbf{x}} = \left[\tilde{\mathbf{x}}_z^T, \tilde{\mathbf{x}}_{w_0}^T, \tilde{\mathbf{x}}_w^T\right]^T$.

System (1) can be transformed into the new coordinates with the transformation in (5) as

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \left[\frac{\partial \boldsymbol{\Phi}(\mathbf{x})}{\partial \mathbf{x}}\right]\dot{\mathbf{x}} \\ &= \left[\begin{array}{ccccccc}\dot{\tilde{\mathbf{x}}}_1^T & \cdots & \dot{\tilde{\mathbf{x}}}_s^T & \dot{\tilde{\mathbf{x}}}_{w_0}^T & \dot{\tilde{\mathbf{x}}}_{w_1}^T & \cdots & \dot{\tilde{\mathbf{x}}}_{w_q}^T\end{array}\right]^T \end{aligned} \tag{6}$$

The system (1) can be transformed into the new coordinates with the help of the above formulation so that we have three classes of subsystems.

Class A: The coordinates under the transformation $\boldsymbol{\phi}_1, \cdots, \boldsymbol{\phi}_s$ have the following structure. The disturbance inputs are completely decoupled in the subsystems described below.

$$\begin{aligned}\dot{x}_1^i &= x_2^i + \beta_1^i(\tilde{\mathbf{x}}, u) \\ \cdots & \quad \cdots \\ \dot{x}_{r_i-2}^i &= x_{r_i-1}^i + \beta_{r_i-2}^i(\tilde{\mathbf{x}}, u) \\ \dot{x}_{r_i-1}^i &= \alpha_i(\tilde{\mathbf{x}}) + \beta_{r_i-1}^i(\tilde{\mathbf{x}}, u) \\ y_i &= x_1^i \end{aligned} \tag{7}$$

where $\beta_j^i(\tilde{\mathbf{x}}, u) = \frac{\partial(L_{\mathbf{f}}^{j-1}h_i)}{\partial \mathbf{x}}\mathbf{G}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}), u)$ and $\alpha_i(\tilde{\mathbf{x}}) = L_{\mathbf{f}}^{r_i-1}h_i(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))$, for all $j = 1, \cdots, r_i - 1$ and for all $i = 1, \cdots, s$. From the above, it is clear that the unknown disturbance inputs do not appear and so the observer can be designed for these subsystems by existing techniques.

Class B: For the subsystem under the transformation $\boldsymbol{\phi}_{w_0}$, the disturbance terms exist, and has the following structure

$$\begin{aligned}\dot{x}_i^{w_0} &= \varphi_i(\tilde{\mathbf{x}}, u) + \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) \\ y_j &= x_i^{w_0} \end{aligned} \tag{8}$$

where $\varphi_i(\tilde{\mathbf{x}}, u) = L_{\mathbf{f}}h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}})) + \frac{\partial h_j}{\partial \mathbf{x}}\mathbf{G}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}), u)$, $\gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) = \sum_{v=1}^m L_{\mathbf{p}_v}h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v$, for $i = 1, \cdots, (a - s)$ and $j = s+1, \cdots, a$. The state of the above subsystems is measurable in the form of output and so the observer design problem can be handled with ease.

Class C: For the subsystems under the transformation $\boldsymbol{\phi}_{w_1}, \cdots, \boldsymbol{\phi}_{w_q}$, the following structure can be obtained

$$\begin{aligned}\dot{x}_1^{w_j} &= x_2^{w_j} + \mu_1^{w_j}(\tilde{\mathbf{x}}, u) + \xi_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ \cdots &= \cdots \\ \dot{x}_{k_j-1}^{w_j} &= x_{k_j}^{w_j} + \mu_{k_j-1}^{w_j}(\tilde{\mathbf{x}}, u) + \xi_{k_j-1}^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ \dot{x}_{k_j}^{w_j} &= \alpha_{w_j}(\tilde{\mathbf{x}}) + \mu_{k_j}^{w_j}(\tilde{\mathbf{x}}, u) + \xi_{k_j}^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ y_{w_j} &= x_1^{w_j} \end{aligned} \tag{9}$$

where $\alpha_{w_j}(\tilde{\mathbf{x}}) = L_{\mathbf{f}}^{k_j}h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))$, $\mu_i^{w_j}(\tilde{\mathbf{x}}, u) = \frac{\partial L_{\mathbf{f}}^i h_j}{\partial \mathbf{x}}\mathbf{G}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}), u)$, and $\xi_i^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) = \sum_{v=1}^m L_{\mathbf{p}_v}L_{\mathbf{f}}^i h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v$ for $j = 1, \cdots, q$, and $i = 1, \cdots, k_j$.

Assumption 1: There is atleast one output with relative degree equal to one w.r.t. unknown inputs.

Assumption 2: The mapping $\boldsymbol{\Phi}(x)$ is a diffeomorphism. Both the Jacobian matrix $\frac{\partial \boldsymbol{\Phi}(\mathbf{x})}{\partial \mathbf{x}}$ and the inverse Jacobian matrix $\left[\frac{\partial \boldsymbol{\Phi}(\mathbf{x})}{\partial \mathbf{x}}\right]^{-1}$ exist.

Assumption 3: The known functions $\mathbf{f}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$ and $\mathbf{p}_i(\mathbf{x})$ are bounded with respect to their arguments. The inputs of the nonlinear system (1) are bounded for some upper bounds. Further, the system (1) is Bounded-Input-Bounded-States (BIBS) stable.

III. Observer Design in Transformed Space

In this section, the observer designs for subsystems of **Class B** and **Class C** are discussed. The subsystems under **Class A** have the general form without disturbance inputs and the existing design techniques can be used for the design of observer based on some structural assumptions that ensure asymptotical convergence of the estimator similar to those employed in [6].

We define the estimation residual or error $\mathbf{e} \triangleq \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_n\end{array}\right]^T$ as $\mathbf{e} \triangleq \hat{\tilde{\mathbf{x}}} - \tilde{\mathbf{x}}$. In the analysis of each class of subsystems, the error dynamics will be represented with their corresponding superscripts (or) subscripts.

A. Sliding Mode Observer design for Class B subsystems

For the transferred **Class B** subsystem the observer can be designed as follows:

$$\begin{aligned}\dot{\hat{x}}_i^{w_0} &= \varphi_i(\hat{\tilde{\mathbf{x}}}, u) + l_{i,w_0}(y_j - \hat{x}_i^{w_0}) \\ &\quad + \rho_{i,w_0}\operatorname{sign}(y_j - \hat{x}_i^{w_0}) \end{aligned} \tag{10}$$

for $i = 1, \cdots, (a-s)$, $j = s+1, \cdots, a$. By proper selection of sliding mode gain ρ_{i,w_0}, we can ensure the asymptotic convergence of the above estimator. The following **Lemma 1** shows that the estimation error can converge asymptotically if we select the following sliding surface:

$$e_i^{w_0} = \hat{x}_i^{w_0} - x_i^{w_0} = 0 \tag{11}$$

Lemma 1: The subsystem described by (8) with sliding mode estimator of the form (10) ensures that the sliding surface $e_i^{w_0} = 0$ can be reached and maintained provided that the gain satisfies

$$\rho_{i,w_0} > \sup|\varphi_i(\hat{\tilde{\mathbf{x}}}, u) - \varphi_i(\tilde{\mathbf{x}}, u) - \gamma_i(\tilde{\mathbf{x}}, \mathbf{d})| \tag{12}$$

and $l_{i,w_0} > 0$ for all $i = 1, \cdots, (a-s)$.

Proof: With the help of error dynamics obtained from (8) and (10), and by choosing the Lyapunov function $V_i = \frac{1}{2}(e_i^{w_0})^2$, the result can be proven readily. ■

B. *Sliding Mode Observer design for* **Class C** *subsystems*

Subsystems described by (9) can be re-written in the following form:

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_{w_j} = \mathbf{A}_{w_j}\tilde{\mathbf{x}}_{w_j} + \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) + \boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ y_{w_j} = x_1^{w_j} = \mathbf{C}_{w_j}\tilde{\mathbf{x}}_{w_j} \end{cases} \quad (13)$$

where for all $j = 1, \cdots, q$. $\mathbf{A}_{w_j} = \begin{bmatrix} 0 & \mathbf{I}_{(k_j-1)\times(k_j-1)} \\ & \mathbf{0}_{1\times(k_j)} \end{bmatrix}$, $\mathbf{C}_{w_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ and

$$\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) = \begin{bmatrix} \mu_1^{w_j} & \mu_2^{w_j} & \cdots & \alpha_{w_j} + \mu_{k_j}^{w_j} \end{bmatrix}^T$$

$$\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, d) = \begin{bmatrix} \sum_{v=1}^{m} L_{\mathbf{p}_v} h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v \\ \sum_{v=1}^{m} L_{\mathbf{p}_v} L_{\mathbf{f}} h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v \\ \vdots \\ \sum_{v=1}^{m} L_{\mathbf{p}_v} L_{\mathbf{f}}^{k_j-1} h_j(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v \end{bmatrix}$$

For the analysis of subsystem $\tilde{\mathbf{x}}_{w_j}$, structural assumptions will be imposed in order to guarantee the convergence properties of each subsystem. All subsystems converge to the actual states one by one in a sequential fashion. The convergence of $\tilde{\mathbf{x}}_{w_j}$ subsystem depends on the convergence of $\tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}, \tilde{\mathbf{x}}_{w_1}, \cdots, \tilde{\mathbf{x}}_{w_{j-1}}$ subsystems. Once all the subsystems converged, the $\tilde{\mathbf{x}}_{w_j}$ subsystem will converge to the actual states asymptotically.

Assumption 4: The transferred system (13) satisfies:

$$\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) \triangleq \begin{bmatrix} \mu_1^{w_j}\left(x_1^{w_j}, x_2^{w_j}, \bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}, u\right) \\ \mu_2^{w_j}\left(x_1^{w_j}, x_2^{w_j}, x_3^{w_j}, \bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}, u\right) \\ \cdots \\ \mu_{k_j}^{w_j}\left(x_1^{w_j}, x_2^{w_j}, \cdots, x_{k_j}^{w_j}, \bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}, u\right) \end{bmatrix}$$

where $\bar{\tilde{\mathbf{x}}}_{w_j} = \left\{\tilde{\mathbf{x}}_{w_1}, \tilde{\mathbf{x}}_{w_2}, \cdots, \tilde{\mathbf{x}}_{w_{j-1}}\right\}$. $\bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}$ are regarded as inputs to the subsystem under consideration. Then according to [2], the $\tilde{\mathbf{x}}_{w_j}$ subsystem is uniformly observable for all inputs $\bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}$ and u.

The above assumption ensures uniform observability for any input. The following **Assumption** is the key requirement in the development of sliding mode observer.

Assumption 5: Each $\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, d)$ of (14) in **Class C** subsystems of (9) has the following structure:

$$\begin{aligned} & \boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, d) \\ = \; & \tilde{\mathbf{Z}}_{w_j}^1(\tilde{\mathbf{x}})\tilde{d}_{w_1}(\tilde{\mathbf{x}}, u, \mathbf{d}) + \cdots + \tilde{\mathbf{Z}}_{w_j}^{j-1}(\tilde{\mathbf{x}})\tilde{d}_{w_{j-1}}(\tilde{\mathbf{x}}, u, \mathbf{d}) \\ & + \tilde{\mathbf{Z}}_{w_j}^{j}(\tilde{\mathbf{x}})\tilde{d}_{w_j}(\tilde{\mathbf{x}}, u, \mathbf{d}) \end{aligned} \quad (14)$$

where, $\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})$ are the distribution vectors defined by $\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}}) = \begin{bmatrix} 0 & * & \cdots & * \end{bmatrix}^T$ if $i \neq j$ and $\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}}) = \begin{bmatrix} \bar{*} & * & \cdots & * \end{bmatrix}^T$ if $i = j$ with '$*$' represents an element that may be zero or finite, '$\bar{*}$' is a non-zero element.

From the above structure, it is clear that the output is influenced by a single disturbance in the transformed domain. This structure is instrumental for the development of SMO for the case of multiple disturbance terms.

The main idea is to use a robust term for each output that contains the disturbance measurement. Therefore, we have multiple robust terms to compensate for multiple disturbance inputs, i.e. we have multiple sliding modes in a single subsystem.

Assumption 6: All the distribution vectors $\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})$ of $\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, d)$ in (14) have the following structure:

$$\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}}) = \tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}}_{w_j}, \bar{\tilde{\mathbf{x}}}_{w_j}, \tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_{w_0}) \quad (15)$$

where $\bar{\tilde{\mathbf{x}}}_{w_j} = \left\{\tilde{\mathbf{x}}_{w_1}, \tilde{\mathbf{x}}_{w_2}, \cdots, \tilde{\mathbf{x}}_{w_{j-1}}\right\}$ is as defined earlier.

Assumption 7: The distribution vector $\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})$ and the function $\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u)$ are Lipschitzian w.r.t. the argument $\tilde{\mathbf{x}}$ for all $i = 1, \cdots, j$ and $j = 1, \cdots, q$.

1) Observer Design: Based on **Assumptions 4 - 7**, the **Class C** subsystems can be represented in the following form:

$$\dot{\tilde{\mathbf{x}}}_{w_j} = \mathbf{A}_{w_j}\tilde{\mathbf{x}}_{w_j} + \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) + \sum_{i=1}^{j}\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, \mathbf{d}) \quad (16)$$

For the subsystem (13) satisfying **Assumptions 4 - 7**, an observer of the following form can be designed to estimate the states:

$$\begin{aligned} \dot{\hat{\tilde{\mathbf{x}}}}_{w_j} = \; & \mathbf{A}_{w_j}\hat{\tilde{\mathbf{x}}}_{w_j} + \boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, u) + \mathbf{L}_{w_j}\left(y_{w_j} - \hat{x}_1^{w_j}\right) \\ & + \sum_{i=1}^{j}\tilde{\mathbf{Z}}_{w_j}^i(\hat{\tilde{\mathbf{x}}})u_r^{w_i}(t) \end{aligned} \quad (17)$$

where

$$\mathbf{L}_{w_j} = \begin{bmatrix} l_1^{w_j} & l_2^{w_j} & \cdots & l_{k_j-1}^{w_j} & l_{k_j}^{w_j} \end{bmatrix}^T \quad (18)$$

$l_1^{w_j}, l_2^{w_j}, \cdots, l_{k_j-1}^{w_j}, l_{k_j}^{w_j} > 0$, are properly chosen estimation gains such that the eigenvalues of $\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}$ are stable, and there exists a positive symmetric matrix $\mathbf{Q}_{w_j} = \mathbf{Q}_{w_j}^T > 0$ satisfying

$$\left(\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}\right)^T\mathbf{Q}_{w_j} + \mathbf{Q}_{w_j}\left(\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}\right) = -\mathbf{I} \quad (19)$$

Here $u_r^{w_i}(t)$ is a scalar-valued robust term given by the sliding mode estimation. A similar design for robust term was adopted in [4] and [7] for continuous and discrete-time sliding mode observer design for SISO nonlinear systems. It is given by

$$u_r^{w_i}(t) = -\frac{\rho_{w_i}}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})}\text{sign}\left(y_{w_i} - x_1^{w_i}\right) \quad (20)$$

for all $i = 1, \cdots, j$, where $z_1^{w_i}(\tilde{\mathbf{x}}) \neq 0 \; \forall \; \tilde{\mathbf{x}}$ is the first element in the column matrix corresponding to disturbance vector of $\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, d)$ in (14) of each subsystem in **Class C**. $u_r^{w_i}(t)$ corresponds to output y_{w_i} that only includes $\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, d)$ in its measurement. Now, we can obtain the error dynamics from (17) and (13) as

$$\begin{aligned} \dot{\mathbf{e}}_{w_j} = \; & \left(\mathbf{M}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}\right)\mathbf{e}_{w_j} + \boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) \\ & + \sum_{i=1}^{j}\left(\tilde{\mathbf{Z}}_{w_j}^i(\hat{\tilde{\mathbf{x}}})u_r^{w_i}(t) - \tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, \mathbf{d})\right) \end{aligned} \quad (21)$$

where $\mathbf{M}_{w_j} = \left(\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}\right)$, and here $u_r^{w_i}$ is introduced to increase the robustness of the system and to handle the uncertainties. Clearly, multiple robust terms are required to handle the multiple disturbances or unknown inputs present in the system.

2) Convergence Analysis:

Lemma 2: For the subsystem (9) satisfying **Assumptions 1-5** , the estimator (17), can ensure that $\tilde{\mathbf{x}}$ and $\mathbf{e}_{w_j}$ are all bounded provided that

$$\lambda_{max}(\mathbf{Q}_{w_j}) \leq \frac{1}{2}\left[l_{\boldsymbol{\mu}_j} + \sum_{i=1}^{j} l^i_{\mathbf{Z}_j} b_{\rho_i}\right]^{-1} \tag{22}$$

for some Lipschitz constants $l_{\boldsymbol{\mu}_j}, l^i_{\mathbf{Z}_j}$ and some upper bound b_{ρ_i}.

Proof: Based on **Assumption 7**, $\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u)$, $\tilde{\mathbf{Z}}^i_{w_j}(\tilde{\mathbf{x}})$ are Lipschitzian such that $\left\|\boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u)\right\| \leq l_{\boldsymbol{\mu}_j}\|\mathbf{e}\|$, and $\left\|\tilde{\mathbf{Z}}^i_{w_j}(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}^i_{w_j}(\tilde{\mathbf{x}})\right\| \leq l^i_{\mathbf{Z}_j}\|\mathbf{e}\|$. Since $z_1^{w_j}(\tilde{\mathbf{x}}) \neq 0$, $\forall \tilde{\mathbf{x}}$, then $\frac{\rho_{w_i}}{z_1^{w_j}(\tilde{\mathbf{x}})} \leq b_{\rho_i}$ for some upper bound b_{ρ_i}, and also we have $|\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, d)| \leq \bar{d}_i$. Therefore based on **Assumption 7** we have $\left\|\tilde{\mathbf{Z}}^i_{w_j}(\hat{\tilde{\mathbf{x}}})\left[\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, d) - u_r^{w_i}\right]\right\| \leq b^i_{\tilde{Z}_j}(\bar{d}_i + b_{\rho_i})$ for some upper bound $b^i_{\tilde{Z}_j}$. By choosing Lyapunov function candidate as $V_1^{w_j} = \mathbf{e}^T_{w_j}\mathbf{Q}_{w_j}\mathbf{e}_{w_j}$ and differentiating w.r.t. time, and using (21) and the above results, we can easily show $\dot{V}_1^{w_j} \leq -\|\mathbf{e}_{w_j}\|^2 + 2\lambda_{max}(\mathbf{Q}_{w_j})\sum_{i=1}^{j} b^i_{\tilde{Z}_j}(\bar{d}_i + b_{\rho_i})\|\mathbf{e}_{w_j}\|$ $+2\lambda_{max}\|\mathbf{e}_{w_j}\|(\mathbf{Q}_{w_j})\left[l_{\boldsymbol{\mu}_j} + \sum_{i=1}^{j} l^i_{\mathbf{Z}_j} b_{\rho_i}\right]\|\mathbf{e}_{w_j}\|$. As the states in **Class A** and **Class B** subsystems converge to actual states asymptotically, we have $\hat{\tilde{\mathbf{x}}}_z = \tilde{\mathbf{x}}_z$ and $\hat{\tilde{\mathbf{x}}}_{w_0} = \tilde{\mathbf{x}}_{w_0}$. Also $\tilde{\mathbf{x}}_{w_1}, \cdots, \tilde{\mathbf{x}}_{w_{j-1}}$ subsystem states converge to actual states in a sequential fashion. Based on **Assumptions 4 - 7**, we eventually have $\|\mathbf{e}\| \triangleq \|\mathbf{e}_{w_j}\|$ for subsystem w_j provided all the states up to w_{j-1} subsystems track the actual trajectories. Hence we finally have $\dot{V}_1^{w_j} \leq -\|\mathbf{e}_{w_j}\|^2 + 2\lambda_{max}(\mathbf{Q}_{w_j})\left[l_{\boldsymbol{\mu}_j} + \sum_{i=1}^{j} l^i_{\mathbf{Z}_j} b_{\rho_i}\right]\|\mathbf{e}_{w_j}\|^2$ $+ \ 2\lambda_{max}(\mathbf{Q}_{w_j})\sum_{i=1}^{j} b^i_{\tilde{Z}_j}(\bar{d}_i + b_{\rho_i})\|\mathbf{e}_{w_j}\|$. The estimation gain $\mathbf{L}_{w_j}$ should be designed such that the condition (22) is satisfied. Hence $\mathbf{e}_{w_j}$ is bounded. Since $\tilde{\mathbf{x}}_{w_j}$ is bounded, $\hat{\tilde{\mathbf{x}}}_{w_j}$ is also bounded. ■

Although the estimation error is bounded, the main purpose of $u_r^{w_i}$ in (17) is to improve estimation accuracy. We adopted two steps to improve accuracy by sliding mode estimation:

1) Define the following sliding surfaces,

$$e_1^{w_i} = 0 \tag{23}$$

for all $i = 1, \cdots j$, and design the sliding mode estimation as (20) to reach and maintain the sliding mode.

2) Ensure the estimation error $\mathbf{e}_{w_j}$ goes to zero in the sliding surface of $e_1^{w_i} = 0$ after all $\mathbf{e}_{w_1}, \cdots, \mathbf{e}_{w_j}$ convergence to zero sequentially.

The following **Lemma 3** and **Theorem 1** ensure the asymptotical convergence of the estimator.

Lemma 3: For the subsystem (9) satisfying **Assumptions 1-5** and the estimator (17), the sliding mode estimation (20) ensures that the sliding surfaces (23) can be reached and maintained provided the gain satisfies

$$\rho_{w_j} > \sup|\Upsilon(t)| \tag{24}$$

and $l_1^{w_j} > 0$, where $\Upsilon(t) = e_2^{w_j}(t) - z_1^{w_j}(\tilde{\mathbf{x}})\tilde{d}_{w_j}(\tilde{\mathbf{x}}, u, d) + \mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}, u) - \mu_1^{w_j}(\tilde{\mathbf{x}}, u)$.

Proof: It can be proven with ease by the dynamics of $e_1^{w_j}$ of the error dynamics $\mathbf{e}_{w_j}$ from (21) together with Lyapunov function $V_{j_1} = \frac{1}{2}(e_1^{w_j})^2$. ■

Now, we need to check the convergence of other state variables. Since the estimator design (17) using the robust term (20) ensures the sliding mode, we only need to examine the the convergence of the dynamics $\mathbf{e}_{w_j}$ during the sliding mode.

In the sliding mode when $e_1^{w_j} = 0$ and $\dot{e}_1^{w_j} = 0$, $\hat{\tilde{x}}_1^{w_j} = \tilde{x}_1^{w_j}$, the equivalent control $u_r^{w_j}$ can be obtained from $e_1^{w_j}$ of (21) as: $u_{eq}^{w_i} = -\frac{e^{w_i}_{2,d} + \mu_1^{w_i}(\hat{\tilde{\mathbf{x}}}, u) - \mu_1^{w_i}(\tilde{\mathbf{x}}, u) - z_1^{w_i}(\tilde{\mathbf{x}}, u)\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, \mathbf{d})}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}}, u)}$ for all $i = 1, \cdots, j$ [9]. In the following analysis, the superscript ^{d} denotes the estimated $\hat{\tilde{\mathbf{x}}}_{w_j}$-related variables in the sliding mode, i.e., $\mathbf{e}^d_{w_j} = \left[\begin{array}{cccc} e^{w_j}_{1,d} & e^{w_j}_{2,d} & \cdots & e^{w_j}_{k_j,d}\end{array}\right] \triangleq \hat{\tilde{\mathbf{x}}}^d_{w_j} - \tilde{\mathbf{x}}$.

Substituting the above equivalent control into (21) we have the estimation error dynamics in the sliding mode of $e_1^{w_j} = 0$,

$$\dot{\mathbf{e}}^d_{w_j} = \mathbf{M}_{w_j}\mathbf{e}^d_{w_j} + \Theta(t) + \Gamma(t) \tag{25}$$

where $\Theta(t) = \boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, u) + \sum_{i=1}^{j}\left(\tilde{\mathbf{Z}}^i_{w_j}(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}^i_{w_j}(\tilde{\mathbf{x}})\right)u_{eq}^{w_i}(t)$ and $\Gamma(t) = -\sum_{i=1}^{j}\tilde{\mathbf{Z}}^i_{w_j}(\tilde{\mathbf{x}})\left[\frac{e^{w_i}_{2,d}}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})} + \frac{\left(\mu_1^{w_i}(\hat{\tilde{\mathbf{x}}}, u) - \mu_1^{w_i}(\tilde{\mathbf{x}}, u)\right)}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})}\right] + \sum_{i=1}^{j}\tilde{\mathbf{Z}}^i_{w_j}(\tilde{\mathbf{x}})\left[\frac{z_1^{w_i}(\tilde{\mathbf{x}})}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})} - 1\right]\tilde{d}_{w_i}(\tilde{\mathbf{x}}, u, \mathbf{d})$. All the subsystems $\tilde{\mathbf{x}}_z, \tilde{\mathbf{x}}_d, \tilde{\mathbf{x}}_{w_1}, \cdots, \tilde{\mathbf{x}}_{w_{j-1}}$ track the actual trajectories. Based on the structural **Assumptions 4 - 7** we can see that the equilibrium point of error dynamics (25) is $\mathbf{e}^d_{w_j} = 0$, i.e., $\hat{\tilde{\mathbf{x}}}^d_{w_j} = \tilde{\mathbf{x}}_{w_j}$.

The feedback gain $\mathbf{L}_{w_j}$ is designed such that $\left(\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}\right)$ is a stable matrix. See **Remark 1**. The following theorem further examines the condition for the asymptotical stability of the estimator error.

Theorem 1: For the subsystem (9) satisfying **Assumptions 1-7** and the estimator (17), the sliding mode gain (20) ensures that the estimation error is asymptotically stable in the sliding mode $e_1^{w_j} = 0$ provided the gain $\mathbf{L}_{w_j}$ satisfies (19), (22) and

$$\lambda_{max}(\mathbf{Q}_{w_j}) \leq \frac{1}{2}\left[l_{\boldsymbol{\mu}_j} + \sum_{i=1}^{j}\left[l^i_{\mathbf{Z}_j} b_{u_i} + b^i_{\tilde{\mathbf{Z}}_j} b_{\frac{1}{z_i}}\left[1 + l^i_{\mu} + l^i_{z_1}\bar{d}_i\right]\right]\right]^{-1} \tag{26}$$

for some Lipschitz constants $l_{\boldsymbol{\mu}_j}$, $l^i_{\mathbf{Z}_j}$, l^i_{μ}, $l^i_{z_1}$ and some upper bounds b_{u_i}, $b^i_{\tilde{\mathbf{Z}}_j}$, $b_{\frac{1}{z_i}}$, $\bar{d}_j$.

Proof: Since $e_2^{w_i}, \tilde{\mathbf{x}}$ and $\tilde{d}_{w_i}$ are bounded, $u_{eq}^{w_i} \leq b_{u_i}$ for some upper bound b_{u_i}. Similar to the analysis in **Lemma 2** and from **Assumption 7**, $\mu_1^{w_j}$, $z_1^{w_j}$ are Lipschitz, hence we have $\left\|\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})\left[\frac{e_{2,d}^{w_i}}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})} + \frac{\left(\mu_1^{w_i}(\hat{\tilde{\mathbf{x}}},u)-\mu_1^{w_i}(\tilde{\mathbf{x}},u)\right)}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})}\right]\right\| \leq b_{\tilde{\mathbf{Z}}_j}^i b_{\frac{1}{z_i}}\left[1+l_\mu^i\right]\|\mathbf{e}\|$, $\left\|\tilde{\mathbf{Z}}_{w_j}^i(\tilde{\mathbf{x}})\left[\frac{z_1^{w_i}(\tilde{\mathbf{x}})}{z_1^{w_i}(\hat{\tilde{\mathbf{x}}})}-1\right]\tilde{d}_{w_i}(\tilde{\mathbf{x}},u,\mathbf{d})\right\| \leq b_{\tilde{\mathbf{Z}}_j}^i b_{\frac{1}{z_i}} \times l_{z_1}^i \times \bar{d}_i\|\mathbf{e}\|$, for some Lipschitz constants l_μ^j, $l_{z_1}^j$ and upper bound $b_{\frac{1}{z_j}}$. Now, we choose a Lyapunov function as $V_2^{w_j} = \left(\mathbf{e}_{w_j}^d\right)^T \mathbf{Q}_{w_j}\mathbf{e}_{w_j}^d$ and similar to the proof of **Lemma 2** we can obtain in the sliding surface of $e_1^{w_j} = 0$, $\dot{V}_2^{w_j} \leq -\|\mathbf{e}_{w_j}^d\|^2 + 2\lambda_{max}(\mathbf{Q}_{w_j})\left[l_{\boldsymbol{\mu}_j} + \sum_{i=1}^j\left[l_{\mathbf{Z}_j}^i b_{u_i} + b_{\tilde{\mathbf{Z}}_j}^i b_{\frac{1}{z_i}}^i\left[1+l_\mu^i+l_{z_1}^i\bar{d}_i\right]\right]\right] \times\|\mathbf{e}_{w_j}^d\|^2$. Hence condition (26) guarantees the asymptotical convergence of the robust estimator (17). ■

Remark 1: Design of Gain $\mathbf{L}_{w_j}$- The methods of [14] and [15] provide some insights about the distance to unobservability and the selection of the condition number that gives better stability. In the design, we need to calculate the Lipschitz constants and bounds for the function in the transformed space. Then, we need to design the observer such that the distance to unobservability of linear matrix $\mathbf{A}_{w_j} - \mathbf{L}_{w_j}\mathbf{C}_{w_j}$ is greater than the Lipschitz constants and bounded functions (right hand side of inequalities in (22) and (26)), together with the Lyapunov condition (19) in order to achieve stability for each subsystem.

IV. Observer in the Original Space

We need to produce the estimation of the original system. Under the **Assumption 2**, using (6), we have $\dot{\mathbf{x}} = \left[\frac{\partial\boldsymbol{\Phi}(\mathbf{x})}{\partial\mathbf{x}}\right]^{-1}\dot{\tilde{\mathbf{x}}}$. Similarly, for the combined observer in the transformed space described by (10) and (17), by means of inverse transformation, we can obtain the observer in the original space as $\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{G}(\hat{\mathbf{x}},u) + \left[\frac{\partial\boldsymbol{\Phi}(\mathbf{x})}{\partial\mathbf{x}}\right]^{-1}_{\mathbf{x}=\hat{\mathbf{x}}}\mathbf{L}_{trans}$, where $\mathbf{L}_{trans}$ matrix stands for the observer gains together with feedback terms designed in the transformed space. $\mathbf{L}_{trans}$ matrix constitutes the correction terms from all the subsystems and is given as follows: $\mathbf{L}_{trans} = [\mathbf{L}_1(y_1-h_1(x)),\cdots,\mathbf{L}_s(y_s-h_s(x)), l_{1,w_0}(y_{s+1}-x_1^{w_0}) + \rho_{1,w_0}\text{sign}(y_{s+1}-x_1^{w_0}),\cdots,l_{a,w_0}(y_a-x_{(a-s)}^{w_0})+\rho_{(a-s),w_0}\text{sign}(y_a-x_{(a-s)}^{w_0}),\mathbf{L}_{w_1}(y_{a+1} - x_1^{w_1}) + \tilde{\mathbf{Z}}_{w_1}^1 u_r^{w_1},\cdots,\mathbf{L}_{w_q}(y_c - x_1^{w_q}) + \sum_{i=1}^q \tilde{\mathbf{Z}}_{w_q}^i u_r^{w_i}]^T$ where terms $\mathbf{L}_1(y_1 - h_1(x)),\cdots,\mathbf{L}_s(y_s - h_s(x))$ correspond to feedback terms of **Class A** subsystems.

V. Illustrative Example

We consider the following nonlinear system.

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \sin x_1 - 2x_2x_3 \\ 2x_2 - 2x_3^2 \end{bmatrix} + \begin{bmatrix} -x_1u_1 \\ x_2u_2 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (27)$$

The output functions are $y_1 = h_1(\mathbf{x}) = x_1$ and $y_2 = h_2(\mathbf{x}) = x_3$. u_1, u_2 are known inputs whereas d_1 and d_2 are unknown inputs to the system, and they can be regarded as the disturbances to the system.

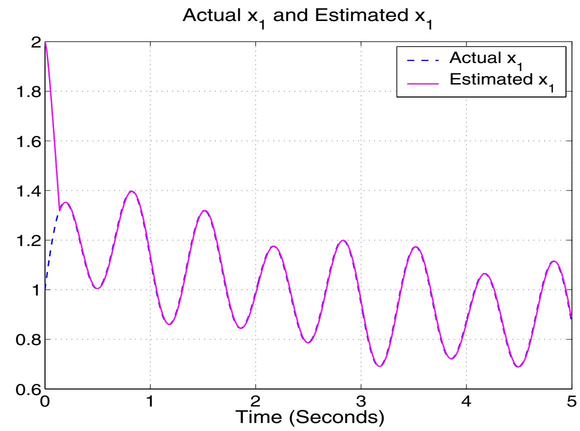

Fig. 1. x_1 and $\hat{x}_1$

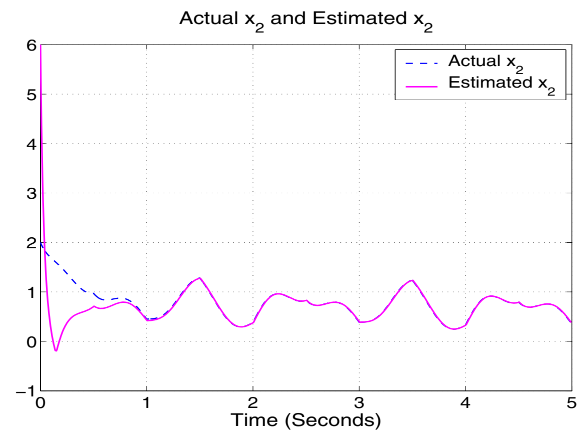

Fig. 2. x_2 and $\hat{x}_2$

A. Design of Proposed Observer

Both the outputs have relative degree equal to one and both the subsystems under transformation belong to **Class C**. For the subsystem under $h_1(\mathbf{x})$ we select $k_1 = 1$, and therefore $k_2 = 2$. We may also choose $k_1 = 2$ and $k_2 = 1$. Transformation exists for both cases.

The transformation for change of co-ordinates is selected as $x_1^{w_1} = x_1$; $x_1^{w_2} = x_3$; $x_2^{w_2}$=$L_{\mathbf{f}}h_2(\mathbf{x}) = 2x_2 - 2x_3^2$. The transformation matrix together with Jacobian and inverse Jacobian of transformation matrix can be formulated as

$$\tilde{\mathbf{x}} = \boldsymbol{\Phi}(x) = \begin{bmatrix} x_1 \\ x_3 \\ 2x_2 - 2x_3^2 \end{bmatrix}, \left[\frac{\partial\boldsymbol{\Phi}(x)}{\partial x}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -4x_3 \end{bmatrix}$$

$$\left[\frac{\partial\boldsymbol{\Phi}(x)}{\partial x}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2x_3 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

System (27) under the new coordinates can be expressed as follows:

$$\begin{aligned} \dot{x}_1^{w_1} &= \frac{x_2^{w_2} + 2(x_1^{w_2})^2}{2} - x_1^{w_1}u_1 + d_2 \\ \dot{x}_1^{w_2} &= x_2^{w_2} + \mu_1^{w_2}(\tilde{\mathbf{x}},u) + d_1 \\ \dot{x}_2^{w_2} &= \mu_2^{w_2}(\tilde{\mathbf{x}},u) + [2 - 4x_1^{w_2}]d_1 + 2d_2 \end{aligned} \quad (28)$$

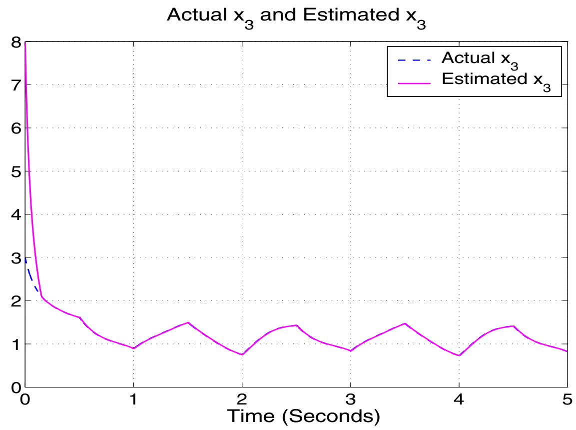

Fig. 3. x_3 and $\hat{x}_3$

where $\mu_1^{w_2}(\tilde{\mathbf{x}}, u) = u_2$ and $\mu_2^{w_2}(\tilde{\mathbf{x}}, u) = 2[\sin(x_1^{w_1}) - x_2^{w_2} + 2(x_1^{w_2})^2 x_1^{w_2}] - 4x_1^{w_2}x_2^{w_2} + [x_2^{w_2} + 2(x_1^{w_2})^2 - 4x_1^{w_1}]u_2$. In the transformed co-ordinates the observer design is done separately for two subsystems under **Class C**. Choose the proper estimation gain $l_1^{w_1}$ and sliding mode gain ρ_{w_1}. The sliding mode estimate for first equation of **Class C** subsystem can be designed according to (17) as follows:

$$\begin{aligned} \dot{\hat{x}}_1^{w_1} &= \frac{\hat{x}_2^{w_2} + 2(\hat{x}_1^{w_2})^2}{2} - \hat{x}_1^{w_1}u_1 + l_1^{w_1}(y_1 - \hat{x}_1^{w_1}) \\ &\quad + \rho_{w_1}\text{sign}(y_{w_1} - \hat{x}_1^{w_1}) \end{aligned} \quad (29)$$

For the last two equations of $\tilde{\mathbf{x}}_{w_2}$ subsystem, observer can be designed according to (17). The system satisfies **Assumption 5**. By choosing proper estimation gain $\mathbf{L}_{w_2} = [l_1^{w_2}, l_2^{w_2}]^T$ and the sliding mode gain ρ_{w_2}, the sliding mode observer for the above subsystem can be designed according to (17) as

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1^{w_2} \\ \dot{\hat{x}}_2^{w_2} \end{bmatrix} &= \begin{bmatrix} \hat{x}_2^{w_2} \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_1^{w_2}(\hat{\tilde{\mathbf{x}}}, u) \\ \mu_2^{w_2}(\hat{\tilde{\mathbf{x}}}, u) \end{bmatrix} + \begin{bmatrix} l_1^{w_2} \\ l_2^{w_2} \end{bmatrix} (y_{w_2} - \hat{x}_1^{w_2}) \\ &\quad + \begin{bmatrix} 1 \\ 2 - 4\hat{x}_1^{w_2} \end{bmatrix} \rho_{w_2}\text{sign}(y_{w_2} - \hat{x}_1^{w_2}) \\ &\quad + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rho_{w_1}\text{sign}(y_{w_1} - \hat{x}_1^{w_1}) \end{aligned}$$

By means of inverse transformation, the observer can be transformed back to original space as follows:

$$\begin{aligned} &\dot{\hat{\mathbf{x}}} \\ &= \begin{bmatrix} \hat{x}_2 \\ \sin(\hat{x}_1) - a_1\hat{x}_3\hat{x}_2 \\ 2\hat{x}_2 - 2\hat{x}_3^2 \end{bmatrix} + \begin{bmatrix} -\hat{x}_1 u_1 \\ \hat{x}_2 u_2 \\ u_2 \end{bmatrix} \\ &\quad + \left[\frac{\partial \mathbf{\Phi}(x)}{\partial x}\right]^{-1}_{\mathbf{x}=\hat{\mathbf{x}}} \mathbf{L}_{trans} \end{aligned} \quad (30)$$

where $\mathbf{L}_{trans} = [l_1^{w_1}(y_{w_1} - \hat{x}_1) + \rho_{w_1}\text{sign}(y_{w_1} - \hat{x}_1), l_1^{w_2}(y_{w_2} - \hat{x}_3) + \rho_{w_1}\text{sign}(y_{w_2} - \hat{x}_3), l_2^{w_2}(y_{w_2} - \hat{x}_3) + \left(2 - 4\hat{x}_1^{w_2}\right)\rho_{w_2}\text{sign}(y_{w_2} - \hat{x}_3) + 2\rho_{w_1}\text{sign}(y_{w_1} - \hat{x}_1)]^T$.

B. Simulation Results

The following values are chosen for simulation: $u_1 = 0.8$, $u_2 = 1.2$. The initial conditions for plant and estimator are $\mathbf{x}_o = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ and $\hat{\mathbf{x}}_o = \begin{bmatrix} 2 & 6 & 8 \end{bmatrix}^T$ respectively. The unknown input d_1 is selected to be a square wave of amplitude 1.5 and period 2π, whereas $d_2 = 3 \times \cos(3\pi t)$.

For the case where both disturbances are present, the following values are chosen for simulation. The estimation gains are $l_1^{w_1} = 1.2$, $l_1^{w_2} = 1.2$, and $l_2^{w_2} = 1.4$. The sliding mode gains are chosen to be $\rho_{w_1} = \rho_{w_2} = 4.5$. The maximum eigenvalue of $\mathbf{Q}_{w_2}$, the solution of (19), is 1.58, hence the right hand side of inequalities in (22) and (26) must be less than 0.63 in order to ensure convergence. Fig. 1 - Fig. 3 show the tracking performance when both disturbances are present. States which are measured quickly track the actual trajectories, whereas the robust estimate x_2 converges to the actual trajectory after both x_1 and x_3 converge in the sliding mode of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$. For the last equation of subsystem in **Class C**, we have multiple sliding modes where x_2 overcomes the uncertainties in the system and track the actual trajectory as shown in Fig. 2.

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