BMI Optimization Based on Unimodal Normal Distribution Crossover GA with Relaxed LMI Convex Estimation

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***Abstract*— This paper deals with the global optimization of the BMIEP(Bilinear Matrix Inequalities Eigenvalue Problem) based on the UNDX(Unimodal Normal Distribution Crossover) GA. First, analyzing the structure of the BMIEP, the existence of the typical difficult structures is confirmed. Then, based on the results of the problem structures analysis and the consideration of the BMIEP characteristic properties, an efficient UNDX GA algorithm with LMI convex estimation is proposed. The effectiveness of the proposed algorithm is evaluated by several numerical examples.**

I. INTRODUCTION

Since control system design problems can be naturally described by Bilinear Matrix Inequalities (BMIs)[1]−[3], there exist needs for algorithms which can solve such BMI problems directly. Since BMI optimization problem is in the class of NP hard, which means it is difficult to solve in case the number of variables is large, the efficient BMI algorithm solving the practical problem for control system design is not yet available.

In this research, Read-coded Genetic Algorithms (RCGA)[7]−[12], which is one of probabilistic optimization algorithm, is introduced for solving practical BMIs. Our approach aims the practical usefulness for solving BMIs. "Unimodal Normal Distribution Crossover (UNDX)"[13][14], which is currently considered as the most efficient crossover method for real-coded GA, is applied.

On BMIs, the problem includes the typical difficult structure, i.e. UV-valley, Big-valley and ridge line, which leads to the failure of the optimization, so-called deceptive phenomena. Considering the problem structure of BMIs, we apply "Extrapolation-directed Crossover (EDX)"[15], "Minimal Generation Gap (MGG) model"[16] and "Innately Split Model (ISM)"[17]. An algorithm for BMIs using GA techniques is proposed. In addition, in order to improve the performance, based on the consideration of BMI properties, a new primary search direction with relaxed LMI convex estimation is proposed. Then, BMI oriented real-coded GA algorithm is proposed. Finally, the effectiveness of the proposed method is confirmed by several numerical experiments.

II. PRELIMINARY

In this section, we summarize the notation and describe the problem formulation.

A. BMI Problem

Given real valued vectors $\boldsymbol{x} \in \mathcal{R}^{n_x}$ and $\boldsymbol{y} \in \mathcal{R}^{n_y}$, we define a biaffine matrix valued function $F : \mathcal{R}^{n_x} \times \mathcal{R}^{n_y} \rightarrow \mathcal{R}^{m\times m}$ by

$$F(\boldsymbol{x},\boldsymbol{y}) = F_{00} + \sum_{i=1}^{n_x} x_i F_{i0} + \sum_{j=1}^{n_y} y_j F_{0j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j F_{ij} \ , \tag{1}$$

where x_i is the ith element of $\boldsymbol{x}$, y_j is the jth element of $\boldsymbol{y}$ and $F_{ij} = F_{ij}^T \in \mathcal{R}^{m\times m}$. Define the sets of indices as $\mathcal{I} := \{1, 2, \cdots, n_x\}$ and $\mathcal{J} := \{1, 2, \cdots, n_y\}$.

Many control problems can be naturally described as BMI (Bilinear Matrix Inequalities) as $F(\boldsymbol{x},\boldsymbol{y}) \prec 0$. So, it is crucial to evaluate the feasibility of $F(\boldsymbol{x},\boldsymbol{y}) \prec 0$ (BMI Feasibility Problem) and also to minimize the linear objective function $\boldsymbol{c}_x^T\boldsymbol{x} + \boldsymbol{c}_y^T\boldsymbol{y}$ under this negative definite condition (BMI Linear Objectives Optimization Problem)[1][3].

In order to solve these problems, we must obtain the optimal solution $\lambda_{\rm opt}$ of the BMI eigenvalue problem as follows: Therefore, in this paper, we consider the global optimization of this non-convex BMI eigenvalue problem.

Given a closed hyper-rectangle $\mathcal{X} \times \mathcal{Y}$ as follows;

$$\mathcal{X} = [L_{x_1}, U_{x_1}] \times \cdots \times [L_{x_{n_x}}, \mathrm{U}_{x_{n_x}}], \tag{2}$$

$$\mathcal{Y} = [L_{y_1}, U_{y_1}] \times \cdots \times [L_{y_{n_y}}, U_{y_{n_y}}] \ , \tag{3}$$

where $-\infty < L_{x_i} \le U_{x_i} < \infty \ (i = 1, \cdots, n_x)$ and $-\infty < L_{y_j} \le U_{y_j} < \infty \ (j = 1, \cdots, n_y)$, BMIEP (BMI Eigenvalue Problem) is described as

$$\min_{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}\times\mathcal{Y}} \overline{\lambda}\{F(\boldsymbol{x},\boldsymbol{y})\} =: \min_{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}\times\mathcal{Y}} \Lambda(\boldsymbol{x},\boldsymbol{y}) =: \lambda_{\rm opt} \tag{4}$$

where $\overline{\lambda}(\cdot)$ denotes the maximal eigenvalue.

Defining a Linear Matrix valued function $F_L : \mathcal{R}^{n_x} \times \mathcal{R}^{n_y} \times \mathcal{R}^{n_x\times n_y} \rightarrow \mathcal{R}^{m\times m}$;

$$F_L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W}) = F_{00} + \sum_{i=1}^{n_x} x_i F_{i0} + \sum_{j=1}^{n_y} y_j F_{0j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{ij} F_{ij} \ , \tag{5}$$

where w_{ij} denotes the (i,j) element of $\boldsymbol{W}$, the lower bound of BMIEP $\lambda_{\rm Lopt}$ $(\le \lambda_{\rm opt})$ is obtained by

$$\lambda_{\rm Lopt} = \min_{(\boldsymbol{x},\boldsymbol{y},{\rm vec}(\boldsymbol{W}))\in\mathcal{X}\times\mathcal{Y}\times\mathcal{W}} \overline{\lambda}\{F_L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W})\} \ , \tag{6}$$

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and

$$\mathcal{W} \;:=\; \left\{ \;\mathrm{vec}(\boldsymbol{W}) \;\middle|\; (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}, \left.\begin{array}{l} w_{ij} \geq L_{y_j} x_i + L_{x_i} y_j - L_{y_j} L_{x_i} \\ w_{ij} \geq U_{y_j} x_i + U_{x_i} y_j - U_{y_j} U_{x_i} \\ w_{ij} \leq U_{y_j} x_i + L_{x_i} y_j - L_{y_j} U_{x_i} \\ w_{ij} \leq L_{y_j} x_i + U_{x_i} y_j - U_{y_j} L_{x_i} \\ i = 1, \cdots, n_x, \quad j = 1, \cdots, n_y \end{array} \right\} \right., \tag{7}$$

where $\mathrm{vec}(\boldsymbol{W})$ indicate $[w_{11} \quad w_{21} \cdots w_{n_x 1} \quad w_{12} \cdots w_{n_x n_x}]^{\mathrm{T}}$ [4][5].

This lower bound calculation is carried out with efficient LMI (Linear Matrix Inequalities) optimization technique.

B. Real-coded Genetic Algorithm

Real-coded genetic algorithms[7]−[12] is suitable for our BMIEP. In this subsection, we summarize several recent techniques of real-coded genetic algorithms.

First, for BMIEP, we define an individual, i.e. a real valued vector, as

$$\boldsymbol{g}_i := \mathrm{col}(\boldsymbol{x}_i, \boldsymbol{y}_i), \tag{8}$$

where $\mathrm{col}(\boldsymbol{x}, \boldsymbol{y})$ indicate $[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{y}^{\mathrm{T}}]^{\mathrm{T}}$. Suppose the population $\mathcal{G}$ composed of n_g individuals as

$$\mathcal{G} := \{\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_{n_g}\} \tag{9}$$

and define a fitness function $f(\boldsymbol{g}_i) > 0$ for the individual $\boldsymbol{g}_i$ as

$$f(\boldsymbol{g}_i) := \max_{\mathcal{G}} \Lambda(\boldsymbol{g}_i) - \Lambda(\boldsymbol{g}_i), \tag{10}$$

where $\Lambda(\boldsymbol{g}_i)$ indicate $\Lambda(\boldsymbol{x}_i, \boldsymbol{y}_i)$.

Unimodal normal distribution crossover GA is reported to generally demonstrate the higher performance than the other crossover method[13]. The UNDX procedure is summarized as follows;

[UNDX (Unimodal Normal Distribution Crossover)][13]

1) Select 3 individuals as parents $\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3} \in \mathcal{G}$, where $\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}$ are main parents, and $\boldsymbol{g}_{p3}$ is a sub parent.
2) Define the middle point of main parents $\boldsymbol{g}_m := \frac{(\boldsymbol{g}_{p1} + \boldsymbol{g}_{p2})}{2}$.
3) Let the direction from $\boldsymbol{g}_{p1}$ to $\boldsymbol{g}_{p2}$ be primary search direction $\boldsymbol{d}$, i.e. $\boldsymbol{d} := \boldsymbol{g}_{p2} - \boldsymbol{g}_{p1}$.
4) Define D as the distance from $\boldsymbol{g}_{p3}$ to the primary search direction $\boldsymbol{d}$.
5) Define $\boldsymbol{e}_i$ be the orthogonal basis vectors spanning the subspace perpendicular to primary search direction $\boldsymbol{d}$.
6) Two children $\boldsymbol{g}_{c1}$ and $\boldsymbol{g}_{c2}$ are now generated as follows;

$$\boldsymbol{g}_{c1} = \boldsymbol{g}_m + \xi \boldsymbol{d} + D \sum_{i=1}^{n-1} \eta_i \boldsymbol{e}_i, \tag{11}$$

$$\boldsymbol{g}_{c2} = \boldsymbol{g}_m - \xi \boldsymbol{d} - D \sum_{i=1}^{n-1} \eta_i \boldsymbol{e}_i, \tag{12}$$

$$\xi \sim N(0, \sigma_\xi^2), \quad \eta_i \sim N(0, \sigma_\eta^2), \tag{13}$$

where n is a dimension of search space, $N(a, v)$ represents a normal distribution whose average is a and variance is v, σ_ξ and σ_η are constant parameters which are recommended to be $\frac{1}{2}|\boldsymbol{d}|$ and $\frac{0.35D}{\sqrt{n}}$ respectively. We describe the UNDX procedure as $(\boldsymbol{g}_{c1}, \boldsymbol{g}_{c2}) = \mathrm{UNDX}(\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3})$. ∎

The values of σ_ξ and σ_η were heuristically decided based on numerical experiments in [13]. The theoretical optimality of these values is confirmed later, which preserve the stochastic properties, i.e. average, variance and covariance, of the parent population[14].In step 1), there exist several selection methods. One of the typical selection method is the following Roulett Wheel Selection.

[Roulette Wheel Selection] [6]

In roulette wheel selection, the probability of the selection of a individual $\boldsymbol{g}_i$ is set by

$$\mathrm{P}[\boldsymbol{g}_i] = \frac{f(\boldsymbol{g}_i)}{\sum_{i=1}^{n_g} f(\boldsymbol{g}_i)}. \tag{14}$$

where $\mathrm{P}[\cdot]$ denotes the probability density. ∎

Key for GA algorithms is the preservation of the diversity of the population. MGG (Minimal Generation Gap) model[16] is appropriate for keeping the diversity of the population;

[MGG(Minimal Generation Gap) model][16]

1) Generate an initial population.
2) Select a pair of individuals randomly from the population as parents.
3) Generate $2n_{\mathrm{cross}}$ offsprings by carrying out crossover at n_{cross} times,
4) Select two individuals from the family containing the two main parents and their $2n_{\mathrm{cross}}$ offsprings. One is the best individual and the other is selected by the rank-based roulette wheel selection[6]. Replace the two main parents in step 2) with the two individuals.
5) Repeat the procedure from step 2) to step 4) until a certain stop condition is satisfied.

There exists sampling bias on unimodal normal distribution crossover. The algorithm become to mainly search in the center area of the search space. This may lead to deceptive phenomena and fail to search the promising valley. In order to relax the sampling bias, extrapolation-directed crossover(EDX)[15] is applied. EDX is summarized as follows;

[EDX(Extrapolation-directed Crossover)][15]

1) [UNDX] step 1),3),4) and 5). In addition, suppose $\Lambda(\boldsymbol{g}_{p1}) < \Lambda(\boldsymbol{g}_{p2})$.

2) The child $\boldsymbol{g}_c$ is now generated as follows;

$$\boldsymbol{g}_c = \boldsymbol{g}_{p1} + D \sum_{i=1}^{n-1} v_i \boldsymbol{e}_i, \quad (15)$$

$$v_i \sim \; N(0, \sigma_\eta^2), \quad (16)$$

where n is a dimension of search space, σ_η is also recommended to be $\frac{0.35D}{\sqrt{n}}$. We describe the EDX procedure as $\boldsymbol{g}_c = \mathrm{EDX}(\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3})$. ∎

If there exists a promising solution in V-valley, i.e. steep valley, it might be missed. This kind of search failure is called as deceptive phenomena. In order to overcome the deceptive phenomena, concentrative search scheme is needed. To this end, Innately Split Model (ISM)[17] is introduced as follows.

[ISM (Innately Split Model)][17]

1) Carry out the optimization with small multi population.
2) Initialize each population in specified small search area.
3) Independently, carry out crossover and natural selection in each small population.
4) When there exist multi populations which search same valley, eliminate populations except one.
5) Eliminate the populations whose fitness cannot be improved for long term.

III. Genetic Algorithm for BMIEP with Conventional Techniques

In this section, we propose a real-coded genetic algorithms for BMIEP. The proposed algorithm is carefully designed based on the structure analyses of BMIEP. The algorithm itself is however general except the definition of individuals and fitness function. Specific algorithm for BMIEP is discussed in the next section.

First, analyzing the problem structures of BMIEP, the existence of Large Valley, UV valley and ridge line structure is confirmed. Then, due to overcome these difficulties, the carefully designed UNDX GA for BMIEP is proposed.

A. Problem Structure Analyses

In order to clarify the structure of BMIEP, we take the Helicopter stabilization problem[18][19], which is well-known as a difficult BMI problems. The local minima are plotted in Fig. 1, where green: normal local minima, blue: critical local minima, i.e. maximal eigenvalue is zero, and red: stabilizing local minima.

The landscape shows the ridge lines exist and stabilizing solutions exist far from the ridge line. In this case, since genetic algorithms mainly search around the ridge line, local optima far from the ridge line are missed. This structure make the problem difficult.

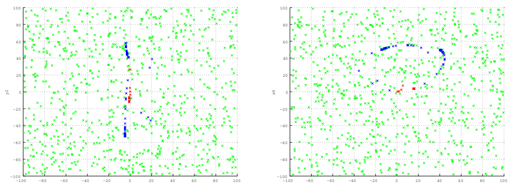

Fig. 1. Landscape of Helicopter stabilization problem

By additional analyses, the existence of UV-valley structure and Big-valley structure in Helicopter stabilizing problem is confirmed. UV-valley means the mixed existence of long curved valley (U-valley) and steep valley (V-valley). In this case, U-valley is mainly searched and the optimal points in V-valley are missed. In case Big-valley exists, genetic algorithms mainly search the mid area of the region. This structure makes the search of the boundary area difficult. These difficult structure leads to so-call deceptive phenomena, and cause the failure of the optimization. In order to overcome these hard structure of BMIEP, we design our real-coded genetic algorithms with the several techniques for GA-hard problems.

B. Genetic algorithm for BMIEP

The UNDX GA algorithms for BMIEP is described in **Algorithm 1**. In order to overcome the difficulties, i.e. big-valley, UV-valley and ridge line structure, **Algorithm 1** is designed with UNDX with MGG model, EDX and ISM techniques.

[Algorithm 1]

0 Set $k = 0$ and p_{EDX} $(0 < p_{\mathrm{EDX}} \leq 1)$.

1 Select p $(0 < p \leq 1)$, $c_{x_i} \in \mathcal{X}_i$, $i \in \mathcal{I}$ and $c_{y_j} \in \mathcal{Y}_j$, $j \in \mathcal{J}$ randomly.

2 Generate an initial population $\mathcal{P}(k)$ randomly.
$\mathcal{P}(k) := \{\boldsymbol{g}_1^{(k)}, \boldsymbol{g}_2^{(k)}, \cdots, \boldsymbol{g}_{n_{\mathrm{pop}}}^{(k)}\}$, $\boldsymbol{g}_\ell^{(k)} := (\boldsymbol{x}_\ell^{(k)}, \boldsymbol{y}_\ell^{(k)})$
$x_{\ell i}^{(k)} \in [pL_{x_i} - c_{x_i} \; pU_{x_i} - c_{x_i}] \cap \mathcal{X}_i, \; i \in \mathcal{I}$
$y_{\ell j}^{(k)} \in [pL_{y_j} - c_{y_j} \; pU_{y_j} - c_{y_j}] \cap \mathcal{Y}_j, \; j \in \mathcal{J}$

3 Select $\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3}$ from the population $\mathcal{P}(k)$ randomly. Suppose $\Lambda(\boldsymbol{g}_{p1}) < \Lambda(\boldsymbol{g}_{p2})$. Eliminate the main parents $\mathcal{P}(k) := \mathcal{P}(k) \backslash \{\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}\}$.

4 In probability p_{EDX}, carry out EDX for n_{EDX} times.
for $i = 1 : n_{\mathrm{EDX}}$
Calculate $\boldsymbol{g}_c = \mathrm{EDX}(\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3})$.
If $\Lambda(\boldsymbol{g}_c) < \Lambda(\boldsymbol{g}_{p1})$ holds, replace $\boldsymbol{g}_{p1}$ by $\boldsymbol{g}_c$.
End for. Go to Step 8.

5 Carry out UNDX for n_{cross} times.
Initialize $\mathcal{G}_{\mathrm{U}}(k) := \{\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}\}$
for $i = 1 : n_{\mathrm{cross}}$
$(\boldsymbol{g}_{c1}, \boldsymbol{g}_{c2}) = \mathrm{UNDX}(\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}, \boldsymbol{g}_{p3})$
$\mathcal{G}_{\mathrm{U}}(k) := \mathcal{G}_{\mathrm{U}}(k) \cup \{\boldsymbol{g}_{c1}, \boldsymbol{g}_{c2}\}$
End for.

6 Replace $\boldsymbol{g}_{p1}$ by the best individual in $\mathcal{G}_{\mathrm{U}}(k)$.

7 Replace $\boldsymbol{g}_{p2}$ by an individual selected from $\mathcal{G}_{\mathrm{U}}(k)$

by roulette wheel selection.

8 $\mathcal{P}(k) := \mathcal{P}(k) \cup \{\boldsymbol{g}_{p1}, \boldsymbol{g}_{p2}\}$ and Let $k := k+1$.

9 Repeat step 3 - 8 until convergence criterion is satisfied.

10 Repeat step 1 - 8 until stopping criterion is satisfied.

$\mathcal{P}(k)$ and $\mathcal{G}(k)$ are composed of $n_{\rm pop}$ and $2n_{\rm cross}$ individuals respectively.

IV. GA for BMIEP based on Relaxed LMI Convex Estimation

In this section, we introduce new mutation methodology using the properties of BMIEP. First, we clarify the properties of BMI local minima by means of the geometric representation. Then, a specified mutation method in BMIEP is derived.

A. Properties of BMIEP

In order to clarify the BMIEP properties geometrically, we define several characteristic sets. Associated with eq. (5), a convex level set is defined as follows:

$$\mathcal{L}(\lambda) := \{\ \mathrm{col}(\boldsymbol{x},\ \boldsymbol{y},\ \mathrm{vec}(\boldsymbol{W}))\ \mid\ F_L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W}) \preceq \lambda I\ \} \subset \mathcal{R}^{(n_x+n_y+n_xn_y)}, \tag{17}$$

From the definition, the convexity and the following relationship hold for $\mathcal{L}(\lambda)$;

$$\forall\ \lambda_1,\lambda_2,\quad \text{s.t.}\ \ \lambda_{\rm Lopt} \le \lambda_1 < \lambda_2,\quad \mathcal{L}(\lambda_2) \supset \mathcal{L}(\lambda_1) \ne \emptyset. \tag{18}$$

$F_L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W})$ is constructed by just replacing bilinear terms x_iy_j of new variables w_{ij} in eqn.(1). So, if we consider the constraints $x_iy_j = w_{ij}$ on eq.(5), both matrix valued functions become equivalent. To express the bilinear constraints, we define a subset $\mathcal{B}$ as

$$\mathcal{B} := \{\ \mathrm{col}(\boldsymbol{x},\ \boldsymbol{y},\ \mathrm{vec}(\boldsymbol{W}))\ \mid\ \mathrm{vec}(\boldsymbol{W}) = \boldsymbol{y}\otimes\boldsymbol{x}\ \} \subset \mathcal{R}^{(n_x+n_y+n_xn_y)}. \tag{19}$$

For the simplification of the notation, the following subset,

$$\mathcal{W}_{\mathcal{XY}} := \mathcal{X}\times\mathcal{Y}\times\mathcal{W}, \tag{20}$$

is also defined.

For BMIEP local minima, the following theorem holds.

Theorem 1: The following statements are equivalent.

1) $\boldsymbol{g}_{\rm loc} := \mathrm{col}(\boldsymbol{x}_{\rm loc}, \boldsymbol{y}_{\rm loc})$ is a local minimum in BMIEP whose function value is $\lambda_{\rm loc}$.
2) For $\tilde{\boldsymbol{g}}_{\rm loc} := \mathrm{col}(\boldsymbol{x}_{\rm loc},\ \boldsymbol{y}_{\rm locl},\ \boldsymbol{y}_{\rm loc}\otimes\boldsymbol{x}_{\rm loc})$ and in neighborhood of $\tilde{\boldsymbol{g}}_{\rm loc}$, i.e. $\tilde{\boldsymbol{g}} \in \mathcal{N}_r$ where

$$\mathcal{N}_r := \{\ \tilde{\boldsymbol{g}}\ \mid\ ||\tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_{\rm loc}|| \le r\ \}, \tag{21}$$

$r > 0$ is sufficiently small real scalar, there dose not exist $\tilde{\boldsymbol{g}}$ satisfying

$$\tilde{\boldsymbol{g}} \in \mathcal{L}(\lambda_{\rm loc} - \epsilon) \cap \mathcal{B} \cap \mathcal{N}_r, \tag{22}$$

where $^\exists\epsilon > 0$. ∎

In the practical point of view, Theorem 1 is not applicable due to the nonexistent condition of ϵ. In practice, we use the following necessary condition.

Lemma 1: If a $\tilde{\boldsymbol{g}}_{\rm loc} \in \mathcal{B}$ is a local minimum of BMIEP whose function value is $\lambda_{\rm loc}$, then the following condition holds;

$$\tilde{\boldsymbol{g}}_{\rm loc} \in \partial\{\mathcal{L}(\lambda_{\rm loc}) \cap \mathcal{W}_{\mathcal{XY}}\}, \tag{23}$$

where $\partial\{\cdot\}$ means the boundary of the subset.

[Proof]

If $\tilde{\boldsymbol{g}}_{\rm loc}$ is a local minimum whose value is $\lambda_{\rm loc}$, then

$$\tilde{\boldsymbol{g}}_{\rm loc} \in \mathcal{L}(\lambda_{\rm loc}), \tag{24}$$

$$\{\tilde{\boldsymbol{g}}_{\rm loc}\} \cap \mathcal{L}(\lambda_{\rm loc} - \epsilon) = \emptyset,\quad {}^\forall\epsilon > 0, \tag{25}$$

hold. Eq.(24) obviously holds from the definition. If eq. (25) dose not hold, it implies the function value of $\tilde{\boldsymbol{g}}_{\rm loc}$ is $\lambda_{\rm loc}-\epsilon$, i.e. not $\lambda_{\rm loc}$. Then, eq.(18), eq.(24) and eq.(25) imply $\tilde{\boldsymbol{g}}_{\rm loc} \in \partial\{\mathcal{L}(\lambda_{\rm loc})\}$. From the class inclusion relationship;

$$\begin{aligned}\partial\{\{\mathcal{L}(\lambda_{\rm loc})\}\cap\mathcal{B} \ &\subset\ \partial\{\{\mathcal{L}(\lambda_{\rm loc})\}\cap\mathcal{W}_{\mathcal{XY}} \\ &\subset\ \partial\{\{\mathcal{L}(\lambda_{\rm loc})\cap\mathcal{W}_{\mathcal{XY}}\},\end{aligned} \tag{26}$$

the statements is proved. ∎

Considering eq. (18), we can see that Lemma1 indicate the better local minima, i.e. $\Lambda(\cdot) \le \lambda_{\rm loc}$, exist in the convex region $\mathcal{L}(\lambda_{\rm loc})\cap\mathcal{W}_{\mathcal{XY}}$. In this context, the most promising search direction is obtained by

$$\begin{aligned}&\arg\min\quad \lambda,\\ &\text{s.t.}\quad \mathcal{L}(\lambda)\cap\mathcal{W}_{\mathcal{XY}} \ne \emptyset.\end{aligned} \tag{27}$$

This calculation eq. (27) is easily carried out by eq.(6) based on the LMI optimization technique.

B. GA for BMIEP based on new primal search direction

Based on the result in the former subsection, we propose a new primary search direction. Suppose $\mathcal{G} := \{\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_{n_g}\}$ and $\boldsymbol{g}_\ell := \mathrm{col}(\boldsymbol{x}_\ell, \boldsymbol{y}_\ell)$. First, define the region including all individuals of the population as follows.

$$\mathcal{Q}_{xy} := \mathcal{Q}_x \times \mathcal{Q}_y \tag{28}$$

$$\begin{aligned}&\mathcal{Q}_x := \mathcal{Q}_{x_1}\times\mathcal{Q}_{x_2}\times\cdots\times\mathcal{Q}_{x_{n_x}}\\ &\mathcal{Q}_{x_i} := [L_{x_i}\ \ U_{x_i}] \subset \mathcal{X}_i,\ i\in\mathcal{I}\\ &L_{x_i} := \min_{\ell=1,\cdots,n_g} \boldsymbol{x}_{\ell i},\ \ U_{x_i} := \max_{\ell=1,\cdots,n_g} \boldsymbol{x}_{\ell i}\end{aligned} \tag{29}$$

$$\begin{aligned}&\mathcal{Q}_y := \mathcal{Q}_{y_1}\times\mathcal{Q}_{y_2}\times\cdots\times\mathcal{Q}_{y_{n_y}}\\ &\mathcal{Q}_{y_j} := [L_{y_j}\ \ U_{y_j}] \subset \mathcal{Y}_j,\ j\in\mathcal{J}\\ &L_{y_j} := \min_{\ell=1,\cdots,n_g} \boldsymbol{y}_{\ell j},\ \ U_{y_j} := \max_{\ell=1,\cdots,n_g} \boldsymbol{y}_{\ell j}\end{aligned} \tag{30}$$

$\mathcal{Q}_x \times \mathcal{Q}_y$ is minimal orthogonal hyper rectangle including the population $\mathcal{G}$. Replacing $\mathcal{X}\times\mathcal{Y}$ by $\mathcal{Q}_x\times\mathcal{Q}_y$, define

$\mathcal{W}_Q$ by eq. (7). Carrying out eq. (6), we can get the special individual $\boldsymbol{g}_{\mathrm{lmi}}$ as follows;

$$\boldsymbol{g}_{\mathrm{lmi}} := \arg_{(\boldsymbol{x},\boldsymbol{y})} \min_{\mathcal{Q}_x \times \mathcal{Q}_y \times \mathcal{W}_Q} F_L(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W}) \quad (31)$$

$$=: \mathrm{LMI}(\mathcal{G}), \quad (32)$$

where $\arg_{(\boldsymbol{x},\boldsymbol{y})}$ denotes the partial arguments $(\boldsymbol{x},\boldsymbol{y})$ from the complete arguments $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{W})$.Now, based on the geometrical consideration for BMIEP, new effective primal search direction is defined by

$$\boldsymbol{d}_{\mathrm{lmi}} := \boldsymbol{g}_{\mathrm{lmi}} - \boldsymbol{g}_i. \quad (33)$$

In view point of the preservation of the population diversity, the new primary search direction is not applied directly. Considering the procedure of the MGG model which is superior for keeping the population diversity, the new GA with new primary search direction based on relaxed lmi estimation is proposed as follows.

[Algorithm 2]

0 [Algorithm 1] step 0 - 6
1 Replacing $\boldsymbol{g}_{p2}$ by $\boldsymbol{g}_{\mathrm{lmi}} := \mathrm{LMI}(\mathcal{G}_{\mathrm{U}}(k))$.
2 [Algorithm 1] step 8 - 10

In the proposed algorithm, primary search direction with relaxed LMI convex estimation is set based on the local population for reproduction. Due to the diversity of the orthogonal hyper rectangle region, the new primary search direction $\boldsymbol{d}_{\mathrm{lmi}}$ is indirectly applied in the proposed method. In case that the individual $\boldsymbol{g}_{\mathrm{lmi}}$ is randomly selected, the UNDX search to the primary search direction with relaxed LMI convex estimation is carried out. In the next section, the effectiveness of the proposed algorithms are evaluated by several numerical experiments.

V. Numerical Experiments

In this section, the effectiveness of the proposed methods is evaluated by numerical experiments.

A. Randomly generated problems

By using the following randomly generated problems, the numerical experiments are carried out.

[Randomly generated BMI problems]

Problem 1: $m = 8,\ n_x = n_y = 2,$
Problem 2: $m = 16,\ n_x = n_y = 4,$
Problem 3: $m = 24,\ n_x = n_y = 6,$
Problem 4: $m = 40,\ n_x = n_y = 10,$
where $L_{x_i} = L_{y_j} = U_{x_i} = U_{y_j} = 10$, $i \in \mathcal{I}$ and $j \in \mathcal{J}$. F_{ij}s are randomly generated.

The numerical parameters are set by $p = 0.1$, $n_{\mathrm{pop}} = 30$, $p_{\mathrm{EDX}} = 0.5$, $n_{\mathrm{EDX}} = 30$ and $n_{\mathrm{cross}} = 15$. The calculation environment is Pentium Ⅲ 1GHz with 256MB memory. The convergence criterion is the no progress of the fitness value for 100 generation. The stopping criterion is based on the CPU time; 300[s] for [Problem 1], 600[s] for [Problem 2] and 1500[s] for [Problem 3]. Since the proposed methods are stochastic, the calculation of the proposed methods are carried out for 10 times per each problems and

TABLE I
The results for Problem 1-3

		Conventional (Alg.1)	Proposed (Alg.2)
Problem 1	Max	0.0087	-0.2849
	Average	-0.0442	-0.2850
	Min	-0.2114	-0.2850
	CPU	300	300
Problem 2	Max	-0.0081	-0.3584
	Average	-0.1066	-0.3598
	Min	-0.1472	-0.3636
	CPU	600	600
Problem 3	Max	0.1337	0.1259
	Average	0.1271	0.1235
	Min	0.1227	0.1210
	CPU	1500	1500
Problem 4	Max	0.1739	0.1389
	Average	0.1561	0.1372
	Min	0.1403	0.1356
	CPU	1500	1500

These results of the numerical experiments shows that [Algorithm 2] with new primary search direction based on the relaxed LMI convex estimation tremendously improves the performance of the genetic algorithms with [Algorithm 1].

B. Static output feedback control problem

Consider an example of Helicopter stabilizing problem with static output feedback control firstly described in [18][19]. The problem is briefly summarized as follows. The dynamics of the Helicopter is represented by

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ y = C\boldsymbol{x} \end{cases}, \quad (34)$$

where

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc} -0.0366 & 0.0271 & 0.0188 & -0.4555 & 0.4422 & 0.1761 \\ 0.0482 & -1.0100 & 0.0024 & -0.4555 & 3.5446 & -7.5922 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 & -5.5200 & 4.4900 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]. \quad (35)$$

The stabilizing problem with static output feedback is described by BMI;

$$\Phi(K,X) := \begin{bmatrix} (A+BKC)^{\mathrm{T}}X + X(A+BKC) & 0 \\ 0 & -X \end{bmatrix} < 0. \quad (36)$$

BMIEP is now defined as $\min \overline{\lambda}\{\Phi(K,X)\}$.

The optimizations by [Algorithm 1] and [Algorithm 2] are carried out with the same numerical parameters and environment in subsection V-A. Our numerical results are summarized in Table II.

”Achievement / Trial” means the number of the achievement for obtaining the stabilizing controller, i.e. $\min \overline{\lambda}\{\Phi(K,X)\} < 0$, versus the number of the trial. From

TABLE II

THE RESULT FOR HELICOPTER PROBLEM

	Conventional (Alg.1)	Proposed (Alg.2)
Achievement / Trial	1 / 10	8 / 10
The best optimal value	-0.006	-1.098
CPU time [s] for the best	43620	48161

Table II, the effectiveness of [Algorithms 2] compared to [Algorithm 1] is confirmed.

In the view point of the comparison by the conventional method, the achieved optimal value is reported in [21] as -0.228 and 0.000 by [2][4][5].

The detail of the best solution by [Algorithm 2] is

$$K = \begin{bmatrix} -22.0138 \\ -7.9822 \end{bmatrix}, \tag{37}$$

$$X = \begin{bmatrix} 5.9055 & -6.4664 & -0.8250 & -1.4841 \\ -6.4664 & 99.8780 & 14.7428 & 8.2783 \\ -0.8250 & 14.7428 & 3.3653 & 1.4971 \\ -1.4841 & 8.2783 & 1.4971 & 3.3477 \end{bmatrix}.$$

The closed-loop poles are -18.539, -0.1587 and $-0.2418 \pm 0.7749i$. The degree of the stability is superior than the controller in [19].

VI. CONCLUSION

In this paper, first, considering BMIs problem structure, i.e. UV-valley, Big-valley and ridge line, a real-coded genetic algorithms with UNDX(Unimodal Normal Distribution Crossover), EDX(Extrapolation-directed Crossover), MGG(Minimal Generation Gap) model and ISM(Innately Split Model) was proposed.

Then, taking account of the BMIEP's characteristic properties, BMI oriented real-coded GA algorithms was proposed. New scheme is based on the property that the better local optima of BMI exists some specific region. To search the promising region, the new primary search direction is derived, which is calculated by LMIs(Linear Matrix Inequalities) optimization technique. The effectiveness of the proposed algorithms is confirmed by several numerical experiments including practical static output feedback controller design problem.

In this note, the possibility that the utilization of BMI characteristic can improve the performance of the general genetic algorithm is proved. The stochastic approach including GA is promissing for lage-scale practical BMIEP compared with deterministic approach, i.e. branch and bound method. Further research is needed.

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