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Global Stability in the Presence of Distributed Communication Delays

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***Abstract*— We analyze a class of dynamical systems in the presence of distributed propagation delays under an optimization framework for rate allocation problems in communication networks. We develop a model of the rate control system with distributed communication delays, and derive a sufficient condition for global stability under a set of reasonable assumptions.**

I. Introduction

With emerging *networked* control systems, control of a distributed system in the presence of varying delays is becoming an interesting and important issue. Examples of such systems include a sensor network consisting of arrays of sensors that collect and provide feedback information for a control system, and a communication network with many end users that individually adjust their transmission rates based on the feedback information provided by active queue management (AQM) schemes [3], [5], [11]. One can think of such a system as one that consists of one or more control agents with a *collection* of other agents that deliver feedback information with varying delays. In this type of system modeling of feedback information collected by many individual agents experiencing heterogeneous delays and a study of the system stability require a new framework different from a traditional one that allows only a single (possibly time-varying) delay. In this paper we consider a simple network where one control agent adjusts the control variable based on a feedback information provided by the collection of agents with varying or distributed delays. Although our framework can be applied to more general settings, we use a rate control problem in a communication network as an example to describe the framework and study the stability of the system.

Our basic setup consists of one or more senders and multiple receivers. There is one control agent that aggregates the traffic from the senders and acts on their behalf to assume the responsibility of controlling the aggregate transmission rate of the connections into the network. We assume that these connections traverse a single bottleneck or there exists one dominant bottleneck in the network that is shared by the connections. This bottleneck could be, for instance, the access link that connects a subnet or a small domain to the Internet. A commercial device, such as PacketShaper [1], can intercept packets from a group of connections and perform congestion control on the aggregate traffic. An example of a congestion control scheme that can be used is MulTCP that allows the control agent to mimic the behavior of a set of Transmission Control Protocol (TCP) connections using the aggregate traffic [9]. This type of model also emerges naturally when a large number of connections share a bottleneck link, where the system dynamics can be well approximated by a deterministic process that resembles the behavior of a single flow scaled properly [16], [17].

The stability of a rate control system with a set of users with fixed delays has been studied in the past [2], [4], [14]. However, they consider the case where a set of users individually adjust their own rates based on their utility functions and delayed feedback information, and the feedback delay for each user is fixed. Our model in this paper assumes that one single control agent acts as the sole congestion control agent on behalf of *all* users on their aggregate traffic. However, the feedback information experiences varying delays due to the fact that the destinations of the users may be different and hence the feedback delay for different users may experience heterogeneous delays.

We approximate the distribution of feedback delays of the connections using a family of gamma distributions. Our results in the presence of distributed delays with the assumed distribution are similar to those in [15]. One interpretation of our results is that the feedback information filtered through a necessary averaging function is sufficient for ensuring the stability of a rate control system with certain utility and resource price functions under some assumptions. It essentially tells us that the effect of distributed delay is dynamically equivalent to adding one or more stages of stable low-pass filters, and the stability of original system remains the same as that of a system with the distributed delays replaced with an equivalent (single average) delay. These results provide an interesting perspective for designing end user rate control algorithms and AQM mechanisms in the presence of distributed delays.

This paper is organized as follows. Section II describes the optimization problem for rate control. Our main results are presented in Section III. We conclude the paper in Section VI.

II. Background

In this section we briefly describe the rate control problem in the proposed optimization framework. Consider a network with a set $\mathcal{L}$ of resources or links and a set $\mathcal{I}$ of users. Let C_l denote the finite capacity of link $l \in \mathcal{L}$. Each user has a fixed route R_i, which is a non-empty subset of

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$\mathcal{L}$. We define a zero-one matrix A, where $A_{i,l} = 1$ if link l is in user i's route R_i and $A_{i,l} = 0$ otherwise. When the throughput of user i is x_i, user i receives utility $U_i(x_i)$. This utility function could represent either the user's true utility or some function assigned to the user through the selected end user algorithm. We take the latter view and assume that the utility functions of the users are used to select the desired rate allocation among the users. The utility $U_i(x_i)$ is an increasing, strictly concave and continuously differentiable function of x_i over the range $x_i \geq 0$. The rate control problem can be formulated as the following optimization problem [8]:

SYSTEM(U,A,C):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \mathcal{I}} U_i(x_i) \qquad (1) \\ \text{subject to} \quad & A^T x \leq C \ , \ x \geq 0 \end{aligned}$$

where $C = (C_l, l \in \mathcal{L})$.[1] The first constraint is the capacity constraint.

Assume that every user adopts rate-based flow control. Let $w_i(t)$ and $x_i(t)$ denote user i's willingness to pay per unit time and rate at time t, respectively.[2] Now suppose that at time t each resource $l \in \mathcal{L}$ charges a price per unit flow of $\mu_l(t) = p_l(\sum_{i:l \in R_i} x_i(t))$, where $p_l(\cdot)$ is an increasing function of the total rate going through it. Consider the system of differential equations

$$\frac{d}{dt} x_i(t) = \kappa_i \left(w_i(t) - x_i(t) \sum_{l \in R_i} \mu_l(t) \right) \ , \qquad (2)$$

where $w_i(t) = x_i(t) \cdot U_i^{'}(x_i(t))$. For an explanation of (2), refer to [10]. Since we assume that the utility functions of the users are selected to decide the rate allocation amongst the users, under (2) one can see that, in fact, both the users' utility functions and resource price functions can be utilized to decide the operating point of the system. Therefore, the design of rate control algorithms is equivalent to selecting the users' utility functions and the price functions of the resources in the network.

Kelly *et al.* [10] have shown that under some conditions on $p_l(\cdot), l \in \mathcal{L}$, the above system of differential equations converges to a point that maximizes the following expression

$$\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_l \int_0^{\sum_{i:l \in R_i} x_i} p_l(y) dy. \qquad (3)$$

Note that the first term in (3) is the objective function in our *SYSTEM(U, A, C)* problem. Thus, the algorithm proposed by Kelly *et al.* solves a relaxation of the $SYSTEM(U, A, C)$ problem.

[1]All vectors are assumed to be column vectors.

[2]Throughout the rest of the paper we refer to the willingness to pay per unit time as simply willingness to pay.

III. Rate Control with Feedback Delay

We study the rate control problem in Kelly's optimization framework described in Section II with a single bottleneck link.[3] Specifically, we consider a system where a set of flows that traverse a same bottleneck link is controlled by a single rate control agent as shown in Fig. 1. The agent adjusts the *aggregate* rate of the flows based on the congestion level at the bottleneck link. The feedback from the bottleneck link is delayed due to the link transmission and propagation delays. These flows may have different round-trip delays.

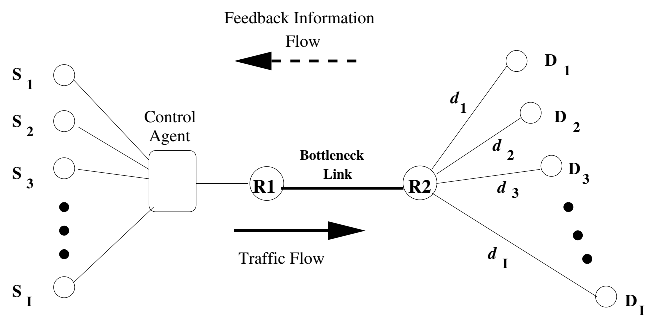

Fig. 1. A network with a single control agent and many connections.

This type of model can be motivated as follows. In [16], [17] it has been shown that when a large number of flows share a bottleneck link, the aggregate behavior of the flows can be well approximated by a deterministic process that mimics the behavior of a single flow that is scaled accordingly. Similarly, in [7] authors have modeled the interaction of a set of *homogeneous* TCP flows with the same round-trip delay with RED mechanism using a deterministic system given by a delay differential equation, where the interaction closely resembles the expected behavior of the aggregate rate of the flows as shown in [16], [17]. Hence, our model can be interpreted as a system where we approximate the aggregate behavior of a large number of flows with heterogeneous round-trip delays, using a single rate control agent that controls the aggregate rate of the flows based on the feedback signal provided by the bottleneck link. Another scenario where our model is useful is when the dominant bottleneck of flows originating within a domain is the access link and the aggregate rate of flows into, for example, an access link is rate controlled by the domain (*e.g.,* PacketShaper).

Since the rate of a user is limited in practice due to the link capacity and receiver buffer size, we assume that the (aggregate) rate is upper bounded by some constant X_{max}. Similarly user rates are bounded away from zero from the fact that there is a lower bound on the transmission rate. For instance, in the case of TCP the transmission rate of a connection cannot be smaller than one packet size divided by the round-trip time of the connection. We denote this

[3]Although there may be other links traversed by the flows sharing the bottleneck link, we assume that they are not bottlenecks.

lower bound on the rate by $X_{min} > 0$. This lower bound X_{min} can be arbitrarily close to 0.

Assumption 1: The rate belongs to a compact set $[X_{min}, X_{max}] \subset \mathbb{R}_+$.

A. Modeling Distributed Delays

Since we assume that the delays of the flows are heterogeneous and the control agent updates the rate based on the feedback signal provided by the flows, the feedback signal that the control agent sees at any given time consists of feedback signal provided by the bottleneck at different times over a period. We model this feedback signal delays spread over an interval using a distribution, and the value of the feedback signal used by the control agent is given by the *mean* of the feedback signal according to the given distribution of the delays.

Under this assumption the rate control action by the agent can be mathematically written as the following differential equation:[4]

$$\frac{d}{dt}x(t) = \kappa\left(g(x(t)) - \int_0^\infty f(x(t-s-T))K(s)ds\right) \quad (4)$$

where $T \geq 0$ is the minimum round-trip delay of the flows, $K(\cdot)$ is the kernel of the feedback delays with $\int_0^\infty K(s)ds = 1$,

$$g(x) = x \cdot U'(x) \quad \text{and} \quad f(x) = x \cdot p(x) \ . \quad (5)$$

Using a change of variable $\theta = t - s$, eq. (4) can be written as

$$\frac{d}{dt}x(t) = \kappa\left(g(x) - \int_{-\infty}^t f(x(\theta - T))K(t-\theta)d\theta\right) \quad (6)$$

In this paper we approximate the delay distribution $K(\cdot)$ using a family of generic delay kernels also known as gamma kernels, which is expressed as follows.

$$K(u) = \begin{cases} \frac{\alpha^{r+1}u^r}{r!}e^{-\alpha u} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases} , \quad (7)$$

where $\alpha > 0$ is a constant, and $r \geq 0$ is an integer.

The parameter r is called the order of delay kernel K [12], and the mean delay corresponding to the K with a given pair (α, r) is given by

$$\mathbf{E}[K] = \int_0^\infty u\frac{\alpha^{r+1}u^r}{r!}e^{-\alpha u}du = \frac{r+1}{\alpha} \ .$$

The kernel $K(\cdot)$ with $r = 0$ and $r = 1$ is called the weak and strong kernel, respectively, and is frequently used to model distributed delay in different disciplines [18]. The kernel K models a whole class of delay and the case of discrete delay can be realized by letting r and α go to ∞ simultaneously while keeping the mean delay $\frac{r+1}{\alpha}$ fixed.

We are interested in studying the global asymptotic behavior of the model in (4) under a set of reasonable assumptions on functions $f(x)$ and $g(x)$. In particular,

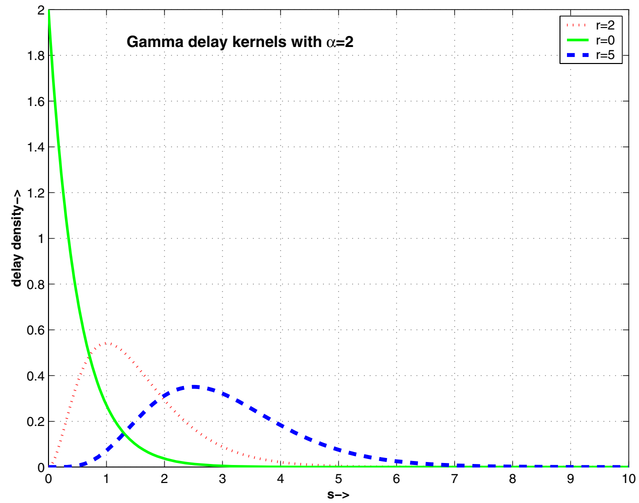

Fig. 2. Gamma delay kernels with increasing r and hence increasing average delay. It is possible to prioritize one feedback over other by choosing appropriate r.

we adopt the delay kernels in (7) and apply *linear chain technique* to establish the stability of the system [12]. The basic motivation for this method is to show that the effect of distributed delay is the same as addition of one or more low pass filters to the system. Mathematically, this can be described as follows: Let $x(t)$ be a positive solution of (6). We define

$$\omega_i(t) = \int_{-\infty}^t f(x(\theta - T))G^i(t-\theta)d\theta, \ i = 0, 1, \ldots, r \ , \quad (8)$$

where $G^i(u) = \frac{\alpha^{i+1}u^i}{i!}e^{-\alpha u}$, $u \geq 0$.

Note that for any $i \geq 1$,

$$\begin{aligned} \frac{d}{du}G^i(u) &= -\alpha G^i(u) + \alpha G^{i-1}(u) \ , \\ \frac{d}{du}G^0(u) &= -\alpha G^0(u) \ . \end{aligned} \quad (9)$$

Using (9) we find that $\mathbf{x}(t) := (x(t), \omega_r(t), \omega_{r-1}(t), \ldots, \omega_0(t))$ satisfies

$$\begin{aligned} \frac{d}{dt}x(t) &= \kappa\left(g(x(t)) - \omega_r(t)\right) \\ \frac{d}{dt}\omega_i(t) &= -\alpha\omega_i(t) + \alpha\omega_{i-1}(t) \ , \ i = 1, \ldots, r \\ \frac{d}{dt}\omega_0(t) &= -\alpha\omega_0(t) + \alpha f(x(t-T)) \end{aligned} \quad (10)$$

By substituting $t = T \cdot s$ and redefining $\beta = T \cdot \alpha$

$$\begin{aligned} \nu\frac{d}{dt}x(s) &= g(x(s)) - \omega_r(s) \\ \frac{d}{dt}\omega_i(s) &= -\beta\omega_i(s) + \beta\omega_{i-1}(s) \ , \ i = 1, \ldots, r \\ \frac{d}{dt}\omega_0(t) &= -\beta\omega_0(t) + \beta f(x(t-1)) \end{aligned} \quad (11)$$

where $\nu = 1/T \cdot \kappa$. Although these equations look similar to those studied in [6], they do not satisfy the assumptions

[4]From Assumption 1 we assume that when $x(t) = X_{min}$ (resp. $x(t) = X_{max}$), $\frac{d}{dt}x(t) = \max\{0, \text{eq. (4)}\}$ (resp. $\frac{d}{dt}x(t) = \min\{0, \text{eq. (4)}\}$).

required to apply the theorems in [6] due to the fact that gain is state-dependent in typical network rate allocation problems. Instead, we will extend the results in [6] for our networking scenario and derive new stability criteria.

We first make the following assumptions on the functions $g(x)$ and $f(x)$ defined in (5).

Assumption 2: (i) The function $g(x)$ as given in (12) is strictly decreasing with $g^{'}(x) < 0$ for all $x > 0$. (ii) The function $f(x)$ is increasing for all $x > 0$ (iii) Both $g(x)$ and $f(x)$ are Lipschitz continuous on strictly positive real axis.

We rewrite the system given in (11) in a form more amenable to analysis: Define

$$y(t) := g(x(t)) \ . \tag{12}$$

This allows us the following change of coordinate:

$$x(t) = g^{-1}(y(t)) \ \Rightarrow \ \frac{d}{dt}x(t) = \frac{\frac{d}{dt}y(t)}{g^{'}(g^{-1}(y(t)))} \tag{13}$$

where the inverse $g^{-1}(\cdot)$ exists from Assumption 2. Let $\overline{\kappa}(y(t)) := -g^{'}(g^{-1}(y(t)))$. Clearly, $\overline{\kappa}(y(t)) > 0$ under Assumption 2. Using this substitution in (13) we get the following form which resembles (6) closely, except for a state-dependent multiplicative gain $\overline{\kappa}(y(t))$.

$$\begin{aligned}&\frac{d}{dt}y(t) \\ &= \frac{\overline{\kappa}(y(t))}{\nu}\left(\int_{-\infty}^{t} f(g^{-1}(y(\theta-T)))K(t-\theta)d\theta - y(t)\right)\end{aligned} \tag{14}$$

It is (14) we are interested in studying and show that there is a close correspondence between invariance and global stability properties of map

$$y_{n+1} = f(g^{-1}(y_n)) := F(y_n) \tag{15}$$

and those of (14) for fixed and periodic orbits.

Using co-ordinate transformation given by (12), the dynamical system in (10) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}y(t) = \overline{\kappa}(y(t))\left[\omega_r(t) - y(t)\right] \\ &\frac{d}{dt}\omega_i(t) = -\beta\omega_i(t) + \beta\omega_{i-1}(t) \ , \ i = 1,\ldots,r \\ &\frac{d}{dt}\omega_0(t) = -\beta\omega_0(t) + \beta F(y(t-1))\end{aligned} \tag{16}$$

Suppose that $I = [a,\ b] \subset [X_{min},\ X_{max}]$ is a closed invariant interval under the map F in (15).

Let $Y := C([-1,0],\mathbb{R}_+)$ be the Banach space of continuous functions mapping the interval $[-1,0]$ to $\mathbb{R}_+$, and $Y_I := \{\phi \in Y | \phi(s) \in I \text{ for all } s \in [-1,0]\}$ and $\omega_r(0)$, $\ldots$, $\omega_0(0) \in I$.

B. Existence of a Unique Solution and Stability

Since the functions in (16) are Lipschitz continuous by assumption, solutions do exist for all $t \geq 0$ and are unique for any initial condition beginning in invariant interval I. Furthermore, the invariance property of the solutions stated below (Theorem 1), ensures that they stay positive and bounded by the initial set they start in, which is assumed to be invariant under the map F.

We first state the invariance result that is used to establish the stability of the system shortly.

Theorem 1: (Invariance) If initial function $\phi \in Y_I$ and $\omega_r(0),\ldots,\omega_0(0) \in I$, the corresponding solution $(y(t),\omega_r(t),\ldots,\omega_0(t);\phi,\omega_r(0),\ldots,\omega_0(0))$ satisfies $\mathbf{y}(t) := (y(t),\omega_r(t),\ldots,\omega_0(t)) \in I^{r+2}$ for all $t \geq 0$. This means that set I is invariant under (16).

Before proving Theorem 1 we first state a lemma that will be used in the proof of theorem.

Lemma 1: Suppose that I is a compact interval and $\eta : [0,\infty] \rightarrow R$ is a continuous function with values in I. If $\sigma : [0,\infty] \rightarrow R_+$ is a bounded, continuous, strictly positive function and $u(t)$ is a solution of following equation

$$\sigma(t)\dot{u}(t) + u(t) = \eta(t) \ , \tag{17}$$

then $u(t) \in I$ for all $t \geq 0$.

Proof: We will prove this lemma by contradiction. Suppose that $I = [a,\ b]$ and the lemma is not true. Define

$$t_0 = \inf\{t \geq 0 |\ \forall\ (t,t'), t' > t,\ \exists\ \tilde{t} \text{ such that } u(\tilde{t}) \notin I\}$$

First, suppose that $u(t_0) = b$. Then, every interval $(t_0, t_0 + \delta), \delta > 0$, contains a point τ such that $u(\tau) > b$ and $\dot{u}(\tau) > 0$. However, from (17) it follows that $\dot{u}(\tau) < 0$, which is a contradiction. The case for $u(t_0) = a$ can be shown to lead to a similar contradiction. This completes the proof. ∎

This lemma is essentially based on [6]. A similar lemma can be also found in [13] which proves the bounds in quantitative manner. We now proceed with the proof of Theorem 1.

Proof: (Theorem 1) Apply Lemma 1 to $\frac{d}{dt}\omega_0(t) = -\beta\omega_0(t) + \beta F(y(t-1))$. Clearly, if $\omega_0(0) \in I$ and initial function $\phi \in Y_I$, then $\omega_0(t) \in I$ for all $0 \leq t \leq 1$. By applying Lemma 1 to $\frac{d}{dt}\omega_1(t) = -\beta\omega_1(t) + \beta\omega_0(t)$, we can argue that $\omega_1(t) \in I$ for all $0 \leq t \leq 1$. Following this recursive argument, we can show that $\omega_i(t) \in I$ for $i = 1,\ldots,r$ and $y(t)$ belong to I for al $0 \leq t \leq 1$. Now by an induction argument on time the same can be argued for all $t \geq 0$. ∎

Next theorem considers the case where the map F has an attracting fixed point y^* with immediate basin of attraction $J_0 : F^n y_0 \rightarrow y^*$ with $y_0 \in J_0$. More specifically, suppose that there is a sequence of closed intervals$J_k, k = 0,1,\ldots$, such that $F(J_k) \subset int(J_{k+1}) \subset J_{k+1} \subset int(J_k)$ for all $k = 0,1,\ldots$, and $\cap_{k=1}^{\infty} J_k = \{y^*\}$. Let $Y_{J_0} = C([-1,0],J_0)$.

Theorem 2: (Stability) If $\nu > 0$, initial function $\phi \in Y_{J_0}$ and $\omega_r(0),\ldots,\omega_0(0) \in I$ then $\lim_{t\rightarrow\infty}(y(t),\omega_r(t),\ldots,\omega_0(t);\phi,\omega_r(0),\ldots,\omega_0(0)) = (y^*,\ldots,y^*)$.

Before proving the theorem we first introduce the following lemma.

Lemma 2: Fix $k = 0, 1, \ldots$. Suppose that L is an open interval that satisfies $J_{k+1} \subset L \subset J_k$. Consider the eq. (17) with $\eta(t) \in F(J_k)$ for all $t \geq 0$. Let $u(t)$ be the solution of (17) with $u(0) = u_0 \in J_k$. Then, there exists a finite time $t_0 = t_0(u_0, L)$ such that $u(t) \in L$ for all $t \geq t_0$.

Proof: If there exists a time t_0 such that $u(t_0) \in J_{k+1}$, then Lemma 1 implies that $u(t) \in J_{k+1}$ for all $t \geq t_0$ and Lemma 2 follows. Now suppose that $u(t) \notin J_{k+1}$ for all $t \geq 0$. First, suppose that $u(t) > \sup J_{k+1}$ for all $t \geq 0$. Then, (17) tells us that $\dot{u}(t) < 0$ for all $t \geq 0$, and since $u(t)$ is bounded from below by $\sup J_{k+1}$, $u(t)$ converges to some limit u_* such that $u(t) \to u_*$ as $t \to \infty$. If $u_* = \sup J_{k+1}$, then the lemma follow since there exists some finite t_0 such that $u(t_0) \in L$ since $J_{k+1} \subset L$. If $u_* > \sup J_{k+1}$, then from (17) there exists some positive constant ε such that $\dot{u}(t) = (-u(t) + \eta(t))/\sigma(t) \leq -\varepsilon$ for all sufficiently large t. This implies that $u(t) \downarrow -\infty$ as $t \to \infty$, which contradicts the assumption that $u(t) > \sup J_{k+1}$ for all $t \geq 0$. The other case, $u(t) < \inf J_{k+1}$ can be treated in a similar manner. ∎

We now proceed with the proof of Theorem 2.

Proof: (Theorem 2) Since Lemma 2 is true for all $k = 0, 1, \ldots$, from the last equation in (16) we can first find a finite t_1' such that for all $t \geq t_1'$, $\omega_0(t) \in J_1$. Then, successively applying the same argument to the equations

$$\frac{d}{dt}\omega_i(t) = -\beta\omega_i(t) + \beta\omega_{i-1}(t)$$

for $i = 1, 2, \ldots$, and then finally to

$$\frac{d}{dt}y(t) = \overline{\kappa}(t)[\omega_r(t) - y(t)],$$

we can establish that $\omega_i(t) \in J_1$ and $y(t) \in J_1$ for all $t \geq t_1$ for some finite t_1. Now, by an induction argument for each $k = 2, 3, \ldots$, one can find a sequence of $t_k, k = 1, 2, \ldots$, such that, for all $t \geq t_k$, $y(t) \in J_k$ and $\omega_i(t) \in J_k, i = 0, 1, \ldots, r$. Now the theorem follows from the assumption that $\cap_{k=1}^{\infty} J_k = \{y^*\}$. ∎

IV. APPLICATION

Consider the rate control problem with N homogeneous users. We assume that utility function of the users is of the form in ([15]) and the price function used at the resource is of ([15]). Then, the end user algorithm is given by

$$\begin{aligned}
&\dot{x}^{(N)}(t) \\
&= k\left(x^{(N)}(t)U_a'(t) - x^{(N)}(t-T) \cdot p(N \cdot x^{(N)}(t-T))\right) \\
&= k\left(\frac{1}{x^{(N)}(t)^a} - x^{(N)}(t-T)\left(\frac{N \cdot x^{(N)}(t-T)}{C}\right)^b\right) \quad (18)
\end{aligned}$$

where a superscript (N) is used to denote the dependence on N. The underlying discrete-time difference equation is given by

$$\begin{aligned}
y_{n+1}^{(N)} &= \left(\frac{N}{C}\right)^b y_n^{-\frac{b+1}{a}} := F^{(N)}(y_n^{(N)}) \\
\frac{1}{\left(x_{n+1}^{(N)}\right)^a} &= x_n^{(N)}\left(\frac{N \cdot x_n^{(N)}}{C}\right)^b, \quad x_n^{(N)} > 0 \\
x_{n+1}^{(N)} &= \left(\frac{(C/N)^b}{{x_n^{(N)}}^{b+1}}\right)^{\frac{1}{a}} \qquad (19)
\end{aligned}$$

From (19) the fixed point $x^{(N)*}$ is $\left(\frac{C}{N}\right)^{\frac{b}{a+b+1}}$, and the eigenvalue is given by $\lambda^{(N)}(x^{(N)*}) = -\frac{b+1}{a}$ and is independent of N. Therefore, the stability of the system does not depend on the number of users in the system. This can also be explained using the price elasticity of demand. Since, given a utility function of the form in ([15]) for some $a > 0$, the price elasticity of demand is constant for all $x > 0$, one would expect the stability of the system to be independent of the operating point, *i.e.,* the fixed point, and capacity, but only on the choices of the utility and price functions that determine the responsiveness of the users and resource, respectively.

In the case of instability when $a < b + 1$, however, depending on the feedback delay T, the rate can oscillate around the fixed point. Clearly, the upper bound on the solution is given by the self-imposed limit $\frac{C}{N}$ of the users to avoid any capacity mismatch. According to Theorem [15] the lower bound will be given by $F(C/N) = \left(\frac{N}{C}\right)^{\frac{1}{a}}$. Hence, link can see a wide fluctuation from full to very low utilization irrespective of the number of users. Note that this lower bound increases with a as one would expect.

V. NUMERICAL SIMULATIONS

We take a flow (same as N homogenous flows, where N is an i arbitrary positive integer) with their utility functions of the form given by ([15]) with $a = 3$ and price function as in ([15]) with $b = 5$, and the link capacity C is set to 10. It is clear that for these values of a and b rate control algorithm is unstable since $\frac{5+1}{3} = 2 > 1$. The optimal rates for both users in the absence of delay will be given by $x^* = \left(\left(\frac{2}{5}\right)^5\right)^{\frac{1}{9}} = 1.6637$. Their self imposed upper rate limit will be $C/2 = 2.5$. The lower limit on the solution according to the period two orbit of map F will be given by $F(2/5) = \left(\frac{2}{5}\right)^{\frac{1}{3}} = 0.7368$.

Fig. 3(a) shows the rate waveform for a delay of 50 time unit and exhibits stable because of the way we have chosen the parameters a and b. Basic effect of providing feedback based on distributed delay is addition of low pass filters in the control loop. The number of these low pass filters depends on the particular gamma delay kernel parameter r. In Figure 3(b) and Fig. 3(c) we are showing the wave of system with one and two stages of additional low pass filters due to different choices of gamma delay kernel parameter r. What we want to show that stability reimans unaffected by the introduction of these filters and basic stability is determined by the original choice of utility and price functions. Clearly, the output of filters (shown in blue and green) approach zero as rate algorithm settles to optimal value (shown in red).

VI. Conclusion

We have shown that the dynamical stability of a rate control mechanism between users and a resource is determined by the interaction of underlying utility and price functions even in the case when delay is distributed. In particular, we demonstrated that the net effect of distributed nature of delay is an addition of one or more stages of stable low pass filters in the dynamical loop when delay distribution is modeled using Gamma distribution. We also proved that the stability of a rate control system is not affected by the distributed nature of delay and only depends on the functional form of the utility and resource price functions.

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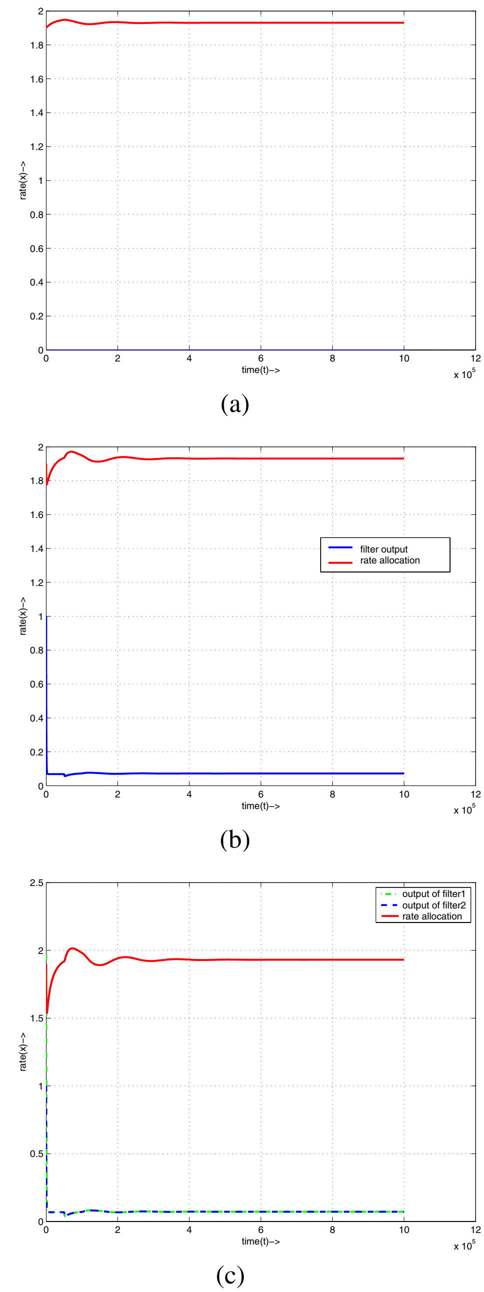

Fig. 3. (a) Rate waveform without any filters, (b) Rate waveform with one stage of low pass filter, and (c) Rate wavefor with two stage of low pass filters