2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

$\mathcal{H}_\infty$ Fuzzy Filter Design for Uncertain Nonlinear Systems with Markovian Jumps: An LMI Approach

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Abstract— **The paper addresses the problem of designing a robust filter for a class of uncertain Markovian jump nonlinear systems described by a class of uncertain Takagi-Sugeno fuzzy models with Markovian jumps. Based on an LMI approach, LMI-based sufficient conditions that guarantee the $\mathcal{L}_2$-gain from an exogenous input to an estimation error is less than a prescribed value are derived. A tunnel diode circuit is used to illustrate the effectiveness of the proposed design techniques.**

I. INTRODUCTION

In practice, structures of many technological systems are subject to random variations. These variations may result from component and interconnection failures, parameters shifting, tracking, sudden environmental disturbances, abrupt variations of the operating condition, etc. In the general sense, a variation is something that changes the behaviour of a technological system such that the system does no longer satisfy its purpose. In order to avoid production deteriorations or damage to machines and humans, variations have to be found as quickly as possible and decisions that stop the propagation of their effects have to be made. Stochastic differential equations have been widely used to model a system with structures subject to random changes. A differential equation with Markovian jumping parameters is one of the most well studied stochastic differential equations. The discrete and continuous dynamic behaviours of a Markovian jumping system are, respectively, described by a finite state Markov chain and differential equations.

A lot of achievements have been made in jumping linear quadratic control theory; see [1]-[16]. In [1], the problem of jumping linear quadratic control of a class of linear systems with discrete Markov processes which are not directly observable has been examined. Discrete-time linear quadratic optimal control problems for infinite Markov jump parameter systems have been studied in [2]. Using averaging and aggregation techniques, in [3] the control design for large scale jump linear systems where the process admits strong and weak interactions has been studied. In [4] and [5], results for linear dynamical systems with Markovian jumping parameters have been proposed. The issue of robust stability analysis and synthesis of Markovian jumping linear continuous-time systems has been investigated by [6] and [7]. Though linear Markovian jump systems have been extensively studied [1]-[16], to the best of our knowledge, the filter design for nonlinear Markovian jump dynamical systems remains as an open research area. Recently, there has been some attempt in this area. In [15], Hamilton-Jacobi-equation-based sufficient conditions for nonlinear Markovian jump systems to have an $\mathcal{H}_\infty$ performance have been derived . However, until now, it is still very difficult to find a global solution to the HJE either analytically or numerically.

Over the past two decades, there has been rapidly growing interest in application of fuzzy logic to control problem. Researches have been focused on its application to industrial processes and a number of successful results have been reported in the literature. In spite of these successes, there are many basic issues remain to be addressed. One of them is how to achieve a systematic design that guarantees closed-loop stability and performance. Recently, a great amount of effort has been devoted to describing a nonlinear system using a Takagi-Sugeno fuzzy model (see [17]-[19]). The Takagi-sugeno fuzzy model represents a nonlinear system by a family of local linear models which smoothly blended together through fuzzy membership functions. Unlike conventional modelling techniques which uses a single model to describe the global behavior of a nonlinear system, fuzzy modelling is essentially a multi-model approach in which simple sub-models (typically linear models) are fuzzily combined to described the global behavior of a nonlinear system. Based on this fuzzy model, a number of systematic model-based fuzzy control design methodologies have been developed.

The aim of this paper is to study the problem of designing an $\mathcal{H}_\infty$ filter for a class of uncertain Markovian jump nonlinear systems described by TS fuzzy model. The filtering problem can be stated as follows: given a dynamic system with exogenous input and measured output, design a filter to estimate an unmeasured output such that the mapping from the exogenous input to the estimation error is minimized or no larger than some prescribed level in terms of the $\mathcal{H}_\infty$ norm. Based on an LMI approach, LMI-based sufficient conditions for the filter to have an $\mathcal{H}_\infty$ performance are derived in terms of a family of linear matrix inequalities.

This paper is organized as follows. In Section II, system descriptions and definitions are presented. Based on an LMI approach, we develop a technique in Section III for

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designing a fuzzy $\mathcal{H}_\infty$ filter that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the estimation error is less than a prescribed value. The validity of this approach is demonstrated by using an illustrative system from the literature in Section IV. Finally, in Section V, the conclusion is drawn.

II. System Description

The class of uncertain nonlinear system with Markovian jumps under consideration is described by the following TS fuzzy system model with Markovian jumps:

$$\begin{aligned}\dot{x}(t) &= \textstyle\sum_{i=1}^{r}\mu_i(\nu(t))\Big[[A_i(\eta(t))+\Delta A_i(\eta(t))]x(t)\\ &\quad +[B_{1_i}(\eta(t))+\Delta B_{1_i}(\eta(t))]w(t)\Big]\\ z(t) &= \textstyle\sum_{i=1}^{r}\mu_i(\nu(t))[C_{1_i}(\eta(t))+\Delta C_{1_i}(\eta(t))]x(t)\\ y(t) &= \textstyle\sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_{2_i}(\eta(t))+\Delta C_{2_i}(\eta(t))]x(t)\\ &\quad +[D_{21_i}(\eta(t))+\Delta D_{21_i}(\eta(t))]w(t)\Big]\end{aligned} \tag{1}$$

where $\nu(t)=\begin{bmatrix}\nu_1(t) & \cdots & \nu_\vartheta(t)\end{bmatrix}$ is the premise variable that may depend on states in many cases, $\mu_i(\nu(t))$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\mu_i(\nu(t)) \geq 0$ and $\sum_{i=1}^{r}\mu_i(\nu(t)) = 1$), ϑ is the number of fuzzy sets, $x(t) \in \Re^n$ is the state vector; $x(0)=0$, $u(t)\in\Re^m$ is the input, $w(t)\in\Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, $y(t) \in \Re^\ell$ is the measurement, $z(t)\in\Re^s$ is the state to be estimated, and the matrix functions $A_i(\eta(t))$, $B_{1_i}(\eta(t))$, $C_{1_i}(\eta(t))$, $C_{2_i}(\eta(t))$, $D_{21_i}(\eta(t))$, $\Delta A_i(\eta(t))$, $\Delta B_{1_i}(\eta(t))$, $\Delta C_{1_i}(\eta(t))$, $\Delta C_{2_i}(\eta(t))$ and $\Delta D_{21_i}(\eta(t))$ are of appropriate dimensions. $\{\eta(t))\}$ is a continuous-time discrete-state Markov process taking values in a finite set $\mathcal{S} = \{1,2,\cdots,s\}$ with transition probability matrix $Pr \triangleq \{P_{\imath k}(t)\}$ given by

$$\begin{aligned}P_{\imath k}(t) &= Pr(\eta(t+\Delta)=k|\eta(t)=\imath)\\ &= \begin{cases}\lambda_{\imath k}\Delta+O(\Delta) & \text{if } \imath\neq k\\ 1+\lambda_{\imath\imath}\Delta+O(\Delta) & \text{if } \imath = k\end{cases}\end{aligned} \tag{2}$$

where $\Delta > 0$, and $\lim_{\Delta\longrightarrow 0}\frac{O(\Delta)}{\Delta}=0$. Here $\lambda_{\imath k}\geq 0$ is the transition rate from mode $\imath$ (system operating mode) to mode k ($\imath \neq k$), and

$$\lambda_{\imath\imath} = -\sum_{k=1,k\neq\imath}^{s}\lambda_{\imath k}. \tag{3}$$

For the convenience of notations, we let $\mu_i \triangleq \mu_i(\nu(t))$, $\eta=\eta(t)$, and any matrix $M(\mu,\imath)\triangleq M(\mu,\eta=\imath)$. The matrix functions $\Delta A_i(\eta)$, $\Delta B_{1_i}(\eta)$, $\Delta C_{1_i}(\eta)$, $\Delta C_{2_i}(\eta)$ and $\Delta D_{21_i}(\eta)$ represent the time-varying uncertainties in the system and satisfy the following Assumption.

Assumption 1:

$$\Delta A_i(\eta) = F(x(t),\eta,t)H_{1_i}(\eta),$$

$$\Delta B_{1_i}(\eta) = F(x(t),\eta,t)H_{2_i}(\eta),$$

$$\Delta C_{1_i}(\eta) = F(x(t),\eta,t)H_{3_i}(\eta),$$

$$\Delta C_{2_i}(\eta) = F(x(t),\eta,t)H_{4_i}(\eta),$$

$$\textit{and}\ \ \Delta D_{21_i}(\eta) = F(x(t),\eta,t)H_{5_i}(\eta)$$

where $H_{j_i}(\eta)$, $j=1,2,\cdots,5$ are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive function $\rho(\eta)$ such that the following inequality holds:

$$\|F(x(t),\eta,t)\| \leq \rho(\eta). \tag{4}$$

In this paper, we consider the following full order $\mathcal{H}_\infty$ fuzzy filter which is inferred as the weighted average of the local models of the form:

$$\begin{aligned}\dot{\hat{x}}(t) &= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big[\hat{A}_{ij}(\imath)\hat{x}(t)+\hat{B}_i(\imath)y(t)\Big]\\ \hat{z}(t) &= \textstyle\sum_{i=1}^{r}\mu_i\hat{C}_i(\imath)\hat{x}(t)\end{aligned} \tag{5}$$

where $\hat{A}_{ij}(\imath)$, $\hat{B}_i(\imath)$ and $\hat{C}_i(\imath)$ are filter gain matrices to be designed.

Before ending this section, we describe the problem under our study as follows.

Problem Formulation: Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$, design a robust $\mathcal{H}_\infty$ fuzzy filter of the form (5) such that the following inequality holds

$$\begin{aligned}\mathbf{E}\Big[\int_0^{T_f}&\Big\{\Big(z(t)-\hat{z}(t)\Big)^T\Big(z(t)-\hat{z}(t)\Big)\\ &-\gamma^2 w^T(t)w(t)\Big\}\,dt\Big] \leq 0,\ x(0)=0\end{aligned} \tag{6}$$

where $\mathbf{E}[\cdot]$ stands for the mathematical expectation, for all $T_f \geq 0$ and $w(t)\in[0,T_f]$.

Note that in the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

III. Robust Fuzzy $\mathcal{H}_\infty$ Filter Design

First, let us select our desired filter as follows:

$$\begin{aligned}\dot{\hat{x}}(t) &= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big[\hat{A}_{ij}(\imath,\varepsilon)\hat{x}(t)+\hat{B}_i(\imath)y(t)\Big]\\ \hat{z}(t) &= \textstyle\sum_{i=1}^{r}\mu_i\hat{C}_i(\imath)\hat{x}(t).\end{aligned} \tag{7}$$

Before presenting our first result, we establish the following lemma which will be used in the proof of our main result.

Lemma 1: Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$ and any positive constants $\delta(\imath)$, for $\imath=1,2,\cdots,s$, if there exist matrices $P(\imath)=P^T(\imath)$ such that the following linear inequalities hold:

$$P(\imath) > 0 \tag{8}$$

$$\begin{pmatrix}\begin{pmatrix}P(\imath)A_{cl}^{ij}(\imath)\\ +(A_{cl}^{ij}(\imath))^TP(\imath)\\ +\sum_{k=1}^{s}\lambda_{\imath k}P(k)\end{pmatrix} & (*)^T & (*)^T\\ (P(\imath)B_{cl}^{ij}(\imath))^T & -\gamma^2 I & (*)^T\\ C_{cl}^{ij}(\imath) & 0 & -I\end{pmatrix} < 0 \tag{9}$$

where $i,j=1,2,\cdots,r$,

$$A_{cl}^{ij}(\imath)=\begin{bmatrix}A_i(\imath) & 0\\ \hat{B}_i(\imath)C_{2_j}(\imath) & \hat{A}_{ij}(\imath)\end{bmatrix},$$

$$B_{cl}^{ij}(\imath) = \begin{bmatrix} \tilde{B}_{1_i}(\imath) \\ \hat{B}_i(\imath)\tilde{D}_{21_j}(\imath) \end{bmatrix},$$

$$C_{cl}^{ij}(\imath) = [\tilde{C}_{1_i}(\imath) \quad \tilde{D}_{12}(\imath)\hat{C}_j(\imath)]$$

with

$$\begin{aligned}
\tilde{B}_{1_i}(\imath) &= [\delta(\imath)I \ \ I \ \ 0 \ \ B_{1_i}(\imath) \ \ 0] \\
\tilde{C}_{1_i}(\imath) &= \Big[\frac{\gamma\rho(\imath)}{\delta(\imath)}H_{1_i}^T(\imath) \ \ \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{4_i}^T(\imath) \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{3_i}^T(\imath) \ \ \sqrt{2}\aleph(\imath)C_{1_i}^T(\imath)\Big]^T \\
\tilde{D}_{12}(\imath) &= \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2}\aleph(\imath)I \end{bmatrix}^T \\
\tilde{D}_{21_i}(\imath) &= [0 \ \ 0 \ \ \delta(\imath)I \ \ D_{21_i}(\imath) \ \ I] \\
\aleph(\imath) &= \Big(1 + \rho^2(\imath)\sum_{i=1}^{r}\sum_{j=1}^{r}\Big[\|H_{2_i}^T(\imath)H_{2_j}(\imath)\| \\
&\qquad + \|H_{5_i}^T(\imath)H_{5_j}(\imath)\|\Big]\Big)^{\frac{1}{2}},
\end{aligned}$$

then the inequality (6) is guaranteed.

Proof: Due to limited pages, the detail of the proof is omitted for brevity. ■

The left hand side of (9) can be re-expressed as follows:

$$\begin{aligned}
&P(\imath)A_{cl}^{ij}(\imath) + (A_{cl}^{ij}(\imath))^T P(\imath) + \gamma^{-2}P(\imath)B_{cl}^{ij}(\imath)\times \\
&(B_{cl}^{ij}(\imath))^T P(\imath) + \textstyle\sum_{k=1}^{s}\lambda_{\imath k}P(k) + (C_{cl}^{ij}(\imath))^T C_{cl}^{ij}(\imath).
\end{aligned} \tag{10}$$

Before providing LMI-based sufficient conditions for the system (1) to have an $\mathcal{H}_\infty$ performance, let us partition the matrix $P(\imath)$ given by Lemma 1 as follows:

$$P(\imath) = \begin{bmatrix} X(\imath) & Y^{-1}(\imath) - X(\imath) \\ Y^{-1}(\imath) - X(\imath) & X(\imath) - Y^{-1}(\imath) \end{bmatrix} \tag{11}$$

where $X(\imath) = X^T(\imath) \in \Re^{n\times n}$ and $Y(\imath) = Y^T(\imath) \in \Re^{n\times n}$. Utilizing the partition above, we define the new filter's input and output matrices as

$$\begin{aligned}
\mathcal{B}_i(\imath) &\triangleq \big[Y^{-1}(\imath) - X(\imath)\big]\hat{B}_i(\imath) \\
\mathcal{C}_i(\imath) &\triangleq \hat{C}_i(\imath)Y(\imath).
\end{aligned} \tag{12}$$

Using these changes of variable, we have the following theorem.

Theorem 1: Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and any positive constants $\delta(\imath)$, for $\imath = 1, 2, \cdots, s$, if there exist matrices $X(\imath) = X^T(\imath)$, $Y(\imath) = Y^T(\imath)$, $\mathcal{B}_i(\imath)$ and $\mathcal{C}_i(\imath)$, $i = 1, 2, \cdots, r$, satisfying the following linear matrix inequalities:

$$\begin{bmatrix} X(\imath) & I \\ I & Y(\imath) \end{bmatrix} > 0 \tag{13}$$

$$X(\imath) > 0 \tag{14}$$

$$Y(\imath) > 0 \tag{15}$$

$$\Psi_{11_{ii}}(\imath) < 0, \quad i = 1, 2, \cdots, r \tag{16}$$

$$\Psi_{22_{ii}}(\imath) < 0, \quad i = 1, 2, \cdots, r \tag{17}$$

$$\Psi_{11_{ij}}(\imath) + \Psi_{11_{ji}}(\imath) < 0, \quad i < j \le r \tag{18}$$

$$\Psi_{22_{ij}}(\imath) + \Psi_{22_{ji}}(\imath) < 0, \quad i < j \le r \tag{19}$$

where

$$\Psi_{11_{ij}}(\imath) = \begin{pmatrix} \begin{pmatrix} A_i(\imath)Y(\imath) \\ +Y(\imath)A_i^T(\imath) \\ +\lambda_{\imath\imath}Y(\imath) \\ +\frac{\tilde{B}_{1_i}(\imath)\tilde{B}_{1_j}^T(\imath)}{\gamma^{-2}} \end{pmatrix} & (*)^T & (*)^T \\ \begin{pmatrix} \tilde{C}_{1_i}(\imath)Y(\imath) \\ +\tilde{D}_{12}(\imath)\mathcal{C}_j(\imath) \end{pmatrix} & -I & (*)^T \\ \mathcal{J}^T(\imath) & 0 & -\mathcal{Y}(\imath) \end{pmatrix} \tag{20}$$

$$\Psi_{22_{ij}}(\imath) = \begin{pmatrix} \begin{pmatrix} A_i^T(\imath)X(\imath) \\ +X(\imath)A_i(\imath) \\ +\mathcal{B}_i(\imath)C_{2_j}(\imath) \\ +C_{2_i}^T(\imath)\mathcal{B}_j^T(\imath) \\ +\tilde{C}_{1_i}^T(\imath)\tilde{C}_{1_j}(\imath) \\ +\sum_{k=1}^{s}\lambda_{\imath k}X(k) \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T(\imath)X(\imath) + \tilde{D}_{21_i}^T(\imath)\mathcal{B}_j^T(\imath) & -\gamma^2 I \end{pmatrix} \tag{21}$$

with

$$\begin{aligned}
\mathcal{J}(\imath) &= \Big[\sqrt{\lambda_{1\imath}}Y(\imath) \ \cdots \ \sqrt{\lambda_{(i-1)\imath}}Y(\imath) \\
&\qquad \sqrt{\lambda_{(i+1)\imath}}Y(\imath) \ \cdots \ \sqrt{\lambda_{s\imath}}Y(\imath)\Big] \\
\mathcal{Y}(\imath) &= diag\,\Big\{Y(1), \ \cdots, \ Y(\imath-1), \\
&\qquad Y(\imath+1), \ \cdots \ , Y(s)\Big\} \\
\tilde{B}_{1_i}(\imath) &= [\delta(\imath)I \ \ I \ \ 0 \ \ B_{1_i}(\imath) \ \ 0] \\
\tilde{C}_{1_i}(\imath) &= \Big[\frac{\gamma\rho(\imath)}{\delta(\imath)}H_{1_i}^T(\imath) \ \ \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{4_i}^T(\imath) \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{3_i}^T(\imath) \ \ \sqrt{2}\aleph(\imath)C_{1_i}^T(\imath)\Big]^T \\
\tilde{D}_{12}(\imath) &= \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2}\aleph(\imath)I \end{bmatrix}^T \\
\tilde{D}_{21_i}(\imath) &= [0 \ \ 0 \ \ \delta(\imath)I \ \ D_{21_i}(\imath) \ \ I] \\
\aleph(\imath) &= \Big(1 + \rho^2(\imath)\sum_{i=1}^{r}\sum_{j=1}^{r}\Big[\|H_{2_i}^T(\imath)H_{2_j}(\imath)\| \\
&\qquad + \|H_{5_i}^T(\imath)H_{5_j}(\imath)\|\Big]\Big)^{\frac{1}{2}},
\end{aligned}$$

then the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable filter is of the form (7) with

$$\begin{aligned}
\hat{A}_{ij}(\imath) &= \big[Y^{-1}(\imath) - X(\imath)\big]^{-1}\mathcal{M}_{ij}(\imath)Y^{-1}(\imath) \\
\hat{B}_i(\imath) &= \big[Y^{-1}(\imath) - X(\imath)\big]^{-1}\mathcal{B}_i(\imath) \\
\hat{C}_i(\imath) &= \mathcal{C}_i(\imath)Y^{-1}(\imath)
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\mathcal{M}_{ij}(\imath) = \ & -A_i^T(\imath) - X(\imath)A_i(\imath)Y(\imath) \\
& -\big[Y^{-1}(\imath) - X(\imath)\big]\hat{B}_i(\imath)C_{2_j}(\imath)Y(\imath) \\
& -\textstyle\sum_{k=1}^{s}\lambda_{\imath k}Y^{-1}(k)Y(\imath) \\
& -\tilde{C}_{1_i}^T(\imath)\Big[\tilde{C}_{1_j}(\imath)Y(\imath) + \tilde{D}_{12}(\imath)\hat{C}_j(\imath)Y(\imath)\Big] \\
& -\gamma^{-2}\Big\{X(\imath)\tilde{B}_{1_i}(\imath) + \big[Y^{-1}(\imath) - X(\imath)\big]\times \\
& \hat{B}_i(\imath)\tilde{D}_{21_i}(\imath)\Big\}\tilde{B}_{1_j}^T(\imath).
\end{aligned} \tag{23}$$

Proof: Suppose there exist $X(\imath)$ and $Y(\imath)$ such that the inequalities (13) and (14)-(15) hold. The inequality (13) implies that the matrix P defined in (9) is a positive definite matrix. Using the partition (11), the controller (12) and multiplying (9) to the left by $\begin{bmatrix} Y(\imath) & I \\ Y(\imath) & 0 \end{bmatrix}$ and to the right by $\begin{bmatrix} Y(\imath) & Y(\imath) \\ I & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} \Phi_{11_{ij}}(\imath) & 0 \\ 0 & \Phi_{22_{ij}}(\imath) \end{bmatrix} \tag{24}$$

where

$$\begin{aligned} \Phi_{11_{ij}}(\imath) &= A_i(\imath)Y(\imath) + Y(\imath)A_i^T(\imath) + \lambda_{\imath\imath}Y(\imath) \\ &+ \Big\{ \big[Y(\imath)\tilde{C}_{1_i}^T(\imath) + \mathcal{C}_i^T(\imath)\tilde{D}_{12_j}^T(\imath)\big] \times \\ &\big[Y(\imath)\tilde{C}_{1_i}^T(\imath) + \mathcal{C}_i^T(\imath)\tilde{D}_{12_j}^T(\imath)\big]^T \Big\} \\ &+ \gamma^{-2}\tilde{B}_{1_i}(\imath)\tilde{B}_{1_j}^T(\imath) + \mathcal{J}^T(\imath)\mathcal{Y}^{-1}(\imath)\mathcal{J}(\imath) \end{aligned} \tag{25}$$

$$\begin{aligned} \Phi_{22_{ij}}(\imath) &= A_i^T(\imath)X(\imath) + X(\imath)A_i(\imath) \\ &+ \mathcal{B}_i(\imath)C_{2_j}(\imath) + C_{2_i}^T(\imath)\mathcal{B}_j^T(\imath) \\ &+ \gamma^{-2}\Big\{ \big[X(\imath)\tilde{B}_{1_i}(\imath) + \mathcal{B}_i(\imath)\tilde{D}_{21_j}(\imath)\big] \times \\ &\big[X(\imath)\tilde{B}_{1_i}(\imath) + \mathcal{B}_i(\imath)\tilde{D}_{21_j}(\imath)\big]^T \Big\} \\ &+ \tilde{C}_{1_i}^T(\imath)\tilde{C}_{1_j}(\imath) + \sum_{k=1}^{s} \lambda_{\imath k}X(k). \end{aligned} \tag{26}$$

Note that $\Phi_{11_{ij}}(\imath)$ and $\Phi_{22_{ij}}(\imath)$ are the Schur complements of $\Psi_{11_{ij}}(\imath)$ and $\Psi_{22_{ij}}(\imath)$. Using (16)-(19), we have (24) less than zero. Hence, by Theorem 1, we learn that the inequality (6) holds. ■

IV. An Illustrative Example

Consider a tunnel diode circuit shown in Fig. 1 where the tunnel diode is characterized by

$$i_D(t) = 0.002v_D(t) + \alpha v_D^3(t)$$

where α is the characteristic parameter. The circuit is governed by the following state equations:

$$\begin{aligned} C\dot{x}_1(t) &= -0.002x_1(t) - \alpha x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - Rx_2(t) + 0.1w_2(t) \\ y(t) &= Jx(t) + 0.1w_1(t) \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{27}$$

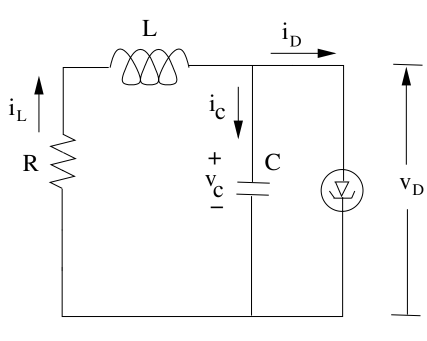

Fig. 1. Tunnel diode circuit.

where $w(t)$ is the disturbance noise input, $y(t)$ is the measurement output, $z(t)$ is the state to be estimated and J is the sensor matrix. Note that the variables $x_1(t)$ and $x_2(t)$ are the deviation variables (variables deviate from the desired trajectories). The parameters in the circuit are given as follows: $C = 20\ mF$, $L = 1000\ mH$ and $R = 10\ \Omega$. Suppose that this system is aggregated into 3 modes as shown in Table I:

TABLE I
System Terminology.

Mode $\imath$	$\alpha(\imath) \pm \Delta\alpha(\imath)$
1	0.01 $\pm 10\%$
2	0.02 $\pm 10\%$
3	0.03 $\pm 10\%$

with the nominal transition probability matrix that relates the three operation modes

$$P_{\imath k} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix}.$$

With these parameters, (27) can be rewritten as

$$\begin{aligned} \dot{x}_1(t) &= -0.1x_1(t) - \left(\tfrac{[\alpha(\imath)+\Delta\alpha(\imath)]}{C}x_1^2(t)\right) \cdot x_1(t) \\ &\quad +50x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - 10x_2(t) + 0.1w_2(t) \\ y(t) &= Jx(t) + 0.1w_1(t) \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{28}$$

For the sake of simplicity, we will use as few rules as possible. Assuming that $|x_1(t)| \leq 3$, the nonlinear network system (28) can be approximated by the following TS fuzzy model:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i \Big[(A_i(\imath) + \Delta A_i(\imath)]x(t) + B_1(\imath)w(t)\Big], \\ z(t) &= C_1(\imath)x(t), \\ y(t) &= C_2(\imath)x(t) + D_{21}(\imath)w(t) \end{aligned}$$

where μ_i is the normalized time-varying fuzzy weighting functions for each rule, $i = 1, 2$, where

$$A_1(1) = \begin{bmatrix} -0.1 & 50 \\ -1 & -10 \end{bmatrix}, \quad A_2(1) = \begin{bmatrix} -4.6 & 50 \\ -1 & -10 \end{bmatrix},$$

$$A_1(2) = \begin{bmatrix} -0.1 & 50 \\ -1 & -10 \end{bmatrix}, \quad A_2(2) = \begin{bmatrix} -9.1 & 50 \\ -1 & -10 \end{bmatrix},$$

$$A_1(3) = \begin{bmatrix} -0.1 & 50 \\ -1 & -10 \end{bmatrix}, \; A_2(3) = \begin{bmatrix} -13.6 & 50 \\ -1 & -10 \end{bmatrix},$$

$$B_{1_1}(\imath) = B_{1_2}(\imath) = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_{1_1}(\imath) = C_{1_2}(\imath) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_{2_1}(\imath) = C_{2_2}(\imath) = J, \;\; D_{21_1}(\imath) = D_{21_2}(\imath) = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$

$$\Delta A_1(\imath) = F(x(t), \imath, t) H_{1_1}(\imath)$$

$$\text{and} \;\; A_2(\imath) = F(x(t), \imath, t) H_{1_2}(\imath).$$

Now, by assuming that $\|F(x(t), \imath, t)\| \leq \rho(\imath) = 1$, we have

$$H_{1_1}(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; H_{1_2}(1) = \begin{bmatrix} -0.45 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H_{1_1}(2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; H_{1_2}(2) = \begin{bmatrix} -0.9 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H_{1_1}(3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } H_{1_2}(3) = \begin{bmatrix} -1.35 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Delta A_1(\imath) = \begin{bmatrix} -\left(\frac{\alpha(\imath)}{C} \cdot x_1^2(t)\right) & 0 \\ 0 & 0 \end{bmatrix} \triangleq F(x(t), \imath, t) H_{1_1}(\imath)$$

$$A_2(\imath) = \begin{bmatrix} -\left(\frac{\alpha(\imath)}{C} \cdot x_1^2(t)\right) & 0 \\ 0 & 0 \end{bmatrix} \triangleq F(x(t), \imath, t) H_{1_2}(\imath).$$

Note that the plot of the membership function Rules 1 and 2 is given in Fig. 2. Using the LMI optimization algorithm

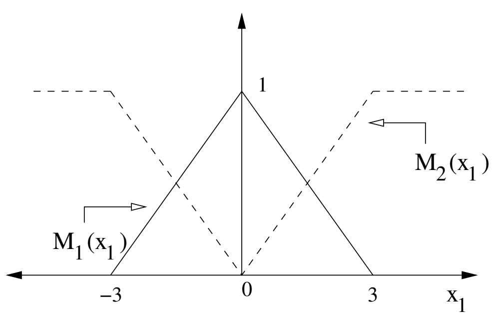

Fig. 2. Membership functions for the two fuzzy set.

and Theorem 1 with $\gamma = 1$, $J = [1 \;\; 0]$ and $\delta(1) = \delta(2) = \delta(3) = 1$, we obtain

$$\hat{A}_{11}(1) = \begin{bmatrix} -50.5324 & -1.7600 \\ -9.7924 & -0.5462 \end{bmatrix},$$

$$\hat{A}_{12}(1) = \begin{bmatrix} -50.5324 & -1.7600 \\ -9.7924 & -0.5462 \end{bmatrix},$$

$$\hat{A}_{21}(1) = \begin{bmatrix} -53.3639 & -1.8542 \\ -19.4469 & -0.3911 \end{bmatrix},$$

$$\hat{A}_{22}(1) = \begin{bmatrix} -53.3639 & -1.8542 \\ -19.4469 & -0.3911 \end{bmatrix},$$

$$\hat{B}_1(1) = \begin{bmatrix} 0.2743 \\ -0.9846 \end{bmatrix}, \quad \hat{B}_2(1) = \begin{bmatrix} 0.3067 \\ -1.2423 \end{bmatrix},$$

$$\hat{C}_1(1) = \begin{bmatrix} -35.3553 & -1.1213 \end{bmatrix},$$

$$\hat{C}_2(1) = \begin{bmatrix} -35.3553 & 0.1110 \end{bmatrix},$$

$$\hat{A}_{11}(2) = \begin{bmatrix} -52.3064 & -2.3475 \\ -3.8388 & -0.5670 \end{bmatrix},$$

$$\hat{A}_{12}(2) = \begin{bmatrix} -52.3064 & -2.3475 \\ -3.8388 & -0.5670 \end{bmatrix},$$

$$\hat{A}_{21}(2) = \begin{bmatrix} -58.4742 & -2.4526 \\ -25.9706 & -0.1006 \end{bmatrix},$$

$$\hat{A}_{22}(2) = \begin{bmatrix} -58.4742 & -2.4526 \\ -25.9706 & -0.1006 \end{bmatrix},$$

$$\hat{B}_1(2) = \begin{bmatrix} 0.4488 \\ -1.6417 \end{bmatrix}, \quad \hat{B}_2(2) = \begin{bmatrix} 0.0851 \\ -0.5918 \end{bmatrix},$$

$$\hat{C}_1(2) = \begin{bmatrix} -35.3553 & -0.1998 \end{bmatrix},$$

$$\hat{C}_2(2) = \begin{bmatrix} -35.3553 & -0.2554 \end{bmatrix},$$

$$\hat{A}_{11}(3) = \begin{bmatrix} -53.3336 & -2.8124 \\ -0.7319 & -0.7547 \end{bmatrix},$$

$$\hat{A}_{12}(3) = \begin{bmatrix} -53.3336 & -2.8124 \\ -0.7319 & -0.7547 \end{bmatrix},$$

$$\hat{A}_{21}(3) = \begin{bmatrix} -63.4126 & -3.1736 \\ -22.7881 & -0.0209 \end{bmatrix},$$

$$\hat{A}_{22}(3) = \begin{bmatrix} -63.4126 & -3.1736 \\ -22.7881 & -0.0209 \end{bmatrix},$$

$$\hat{B}_1(3) = \begin{bmatrix} 0.7630 \\ -2.9262 \end{bmatrix}, \quad \hat{B}_2(3) = \begin{bmatrix} 0.0795 \\ -0.7686 \end{bmatrix},$$

$$\hat{C}_1(3) = \begin{bmatrix} -35.3553 & -1.6653 \end{bmatrix},$$

$$\hat{C}_2(3) = \begin{bmatrix} -35.3553 & 0.2665 \end{bmatrix}.$$

The resulting fuzzy filter is

$$\begin{aligned} \dot{\hat{x}}(t) &= \textstyle\sum_{i=1}^{2} \sum_{j=1}^{2} \mu_i \mu_j \hat{A}_{ij}(\imath) \hat{x}(t) \\ &\quad + \textstyle\sum_{i=1}^{2} \mu_i \hat{B}_i(\imath) y(t) \\ \hat{z}(t) &= \textstyle\sum_{i=1}^{2} \mu_i \hat{C}_i(\imath) \hat{x}(t) \end{aligned} \tag{29}$$

where

$$\mu_1 = M_1(x_1(t)) \;\text{ and }\; \mu_2 = M_2(x_1(t)).$$

Remark 1: Figure 3 shows the result of the changing between modes during the simulation with the initial mode 2. The disturbance input signal, $w(t)$, which was used during the simulation is the rectangular signal (magnitude 0.9 and frequency 1 Hz). The simulation results for the ratio of the filter error energy to the disturbance input noise energy obtained by using the $\mathcal{H}_\infty$ fuzzy filter are depicted in Figure 4. After 15 seconds, the ratio of the filter error energy to the disturbance input noise energy tends to a constant value which is about 0.11. Thus, $\gamma = \sqrt{0.11} = 0.332$ which is less than the prescribed value 1. □

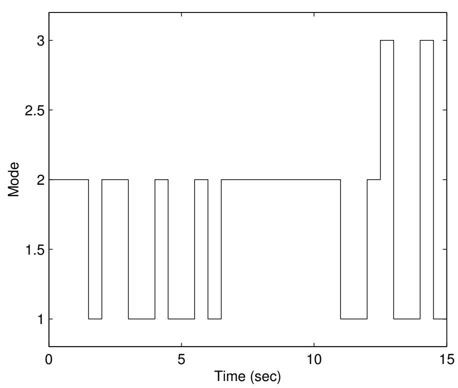

Fig. 3. The result of the changing between modes during the simulation with the initial mode 2.

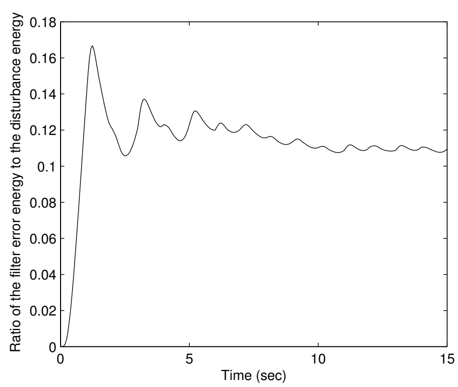

Fig. 4. The ratio of the filter error energy to the disturbance noise energy: $\frac{\int_0^{T_f}(z(t)-\hat{z}(t))^T(z(t)-\hat{z}(t))dt}{\int_0^{T_f} w^T(t)w(t)dt}$.

V. Conclusion

This paper has proposed a technique for designing an $\mathcal{H}_\infty$ filter for a class of fuzzy Markovian jump dynamic systems that guarantees the $\mathcal{L}_2$-gain from an exogenous input to an estimation error is less then a prescribed value. Based on an LMI approach, LMI-based sufficient conditions for the filter to have an $\mathcal{H}_\infty$ performance are established. The effectiveness of the proposed design methodology is demonstrated through a tunnel diode circuit.

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