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Robust fault detection of uncertain linear systems via quasi-LMIs

Alessandro Casavola, Domenico Famularo and Giuseppe Franzè

***Abstract*— Optimal H_∞ deconvolution filters robust fault detection of uncertain polytopic linear systems subject to unknown input disturbance are described.**

The filter must be capable to satisfy two sets of $\mathcal{H}_\infty$ constraints: the first is a disturbance attenuation and decoupling requirement whereas the second expresses the capability of the filter to enhance the fault signals. By means of the Projection Lemma and congruence transformations, a quasi-convex formulation of the problem is obtained via LMIs. The effectiveness of the design technique is illustrated via a numerical example.

I. INTRODUCTION

Fault Detection (FD) techniques are important topics in systems engineering from the viewpoint of improving system reliability. A fault represents any kind of malfunction in a plant that lead to unacceptable anomalies in the overall system behavior. Such a malfunction may occur due to component failures inside the main frame of the process, sensors and/or actuators.

The issue of fault detection has been addressed by many authors in several books and survey articles where many different design methodologies have been exploited (model based approach, parameter estimation, generalized likelihood ratio etc.). See [1], [2], [3], [4] and references therein for comprehensive and up-to-date tutorials.

Amongst many, model based techniques to fault detection (FD) have received considerable attention. This class of procedures makes use of model information to generate additional signals, referred to as residuals, that are conceived to be compared during the on-line operations with the corresponding measured quantities and generate a fault alarm when a large discrepancy arises. This approach is also known as analytical redundancy and hinges upon two components. The first, a Luenberger or a reduced-order observer whose role is to simultaneously decouple the residuals from the exogenous disturbances (robustness issue) and increase the sensitivity of the residuals w.r.t. faults. The second, a decision logic whose role is to possibly discriminate and isolate a single fault amongst many potential candidates [5], [6].

A drawback of the above approach in the case of uncertain LTI systems is that a nominal realization of the plant is necessary in order to design the observer [7], [8]. Such a simplistic design condition may be objectionable if the model uncertainty structure is defined in such a way that a nominal realization is not given or it is not clear which plant belonging to the uncertainty structure is the best candidate.

In order to overcome the arbitrariness related to the avalability of a nominal plant, a deconvolution filter approach for FDI purposes is used. As it is well known, a deconvolution filter aims to determine the unknown inputs from measured outputs. In [9], the nature of the deconvolution problems has been introduced and studied in depth. In particular, the definitions of exact, almost, optimal and suboptimal (in the $\mathcal{H}_2$-$\mathcal{H}_\infty$ sense) deconvolution problems have been stated. Solvability conditions have been derived by means of constructive geometric arguments. Then, using the same framework, in [10] Fault Estimation problems together with their solvability conditions have also been exploited. Finally, the uncertain plants case has been also studied by considering LFT (full-block) uncertainty structures.

In this paper a novel procedure for designing robust residual generators in the frequency domain is proposed. To this end, a robust $\mathcal{H}_\infty$ FD procedure for polytopic uncertain LTI systems is detailed, where the residual generator is a deconvolution filter whose dynamic does not depend on any nominal plant realization. Such a filter will be designed so as to robustly decouple the residuals from the disturbances and conversely to enhance the sensitivity of the residual vector to the fault signal. In particular, as proposed by [5], the first condition consists in minimizing the $\mathcal{H}_\infty$-norm of the disturbance-to-residual map whereas the second to solve an optimization on the lowest singular value of residual-to-faults map over a prescribed frequency range. Note that the latter leads to a nonconvex constraint, which via a linearization of the corresponding feasibility region can be recast into a convex one. As a consequence, the norm constraints are converted into quasi-LMIs via the Projection Lemma and congruence transformations.

The paper is organized as follows: in Section II the problem is formulated, in Section III the quasi-LMI procedure is outlined and the main results stated. Threshold computation is discussed in Section IV and in Section V a numerical example, supporting the effectiveness of the proposed approach, is reported. Some conclusions end the paper.

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Alessandro Casavola and Giuseppe Franzè are with the Dipartimento di Elettronica, Informatica e Sistemistica, Universitá degli Studi della Calabria, Via Pietro Bucci, Cubo 42-C, Rende (CS), 87036, ITALY {casavola,franze}@deis.unical.it

Domenico Famularo is with the DIMET - Università degli studi di Reggio Calabria, Reggio Calabria (RC) 89100, ITALY, domenico.famularo@unirc.it

Notation and Useful Results

Notation: Given a symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$, the inequality $X > 0$, $(X \geq 0)$ denotes matrix positive definiteness (semi-definiteness). Given two symmetric matrices X, Y, the inequality $X > Y$, $(X \geq Y)$ indicates that $X - Y > 0$, $(X - Y \geq 0)$. Given a vector signal $h(t)$, an average measure of its power content in the time window $[0,\, t_f]$ is the rms value

$$\|h(t)\|_{rms,t_f} := \sqrt{\frac{1}{t_f}\int_0^{t_f} h^T(\xi)h(\xi)\,d\xi}$$

Given a transfer function $G(s)$, which is analytic and bounded in the open right half plane s, the $\mathcal{H}_\infty$ norm of $G(s)$ is defined as $\|G(s)\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}(G(j\,\omega))$ and $\|G(s)\|_- := \underline{\sigma}(G(j\,0))$, (see [5]) here $\bar{\sigma}(\cdot)$, $\underline{\sigma}(\cdot)$ denote respectively the largest and the lowest singular value of a matrix.

II. Problem Formulation

Let us consider the following state space description

$$P: \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + [B_f\, B_d]\begin{bmatrix} f(t) \\ d(t) \end{bmatrix} \\ \\ y(t) & = & Cx(t) + [D_f\, D_d]\begin{bmatrix} f(t) \\ d(t) \end{bmatrix} \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $y(t) \in \mathbb{R}^p$ is the measured output, $f(t) \in \mathbb{R}^m$ is a set of detectable fault signals, $d(t) \in \mathbb{R}^d$ are sensors/actuator disturbances. If we suppose that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \left\{ \begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} = \sum_{i=1}^{s} \alpha_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \right. \\ \left. \alpha \in \Gamma \right\} \tag{2}$$

where $B := [B_f\, B_d]$, $D := [D_f\, D_d]$ and Γ is the unitary simplex

$$\Gamma := \left\{ (\alpha_1, \ldots, \alpha_s) : \sum_{i=1}^{s} \alpha_i = 1, \alpha_i \geq 0 \right\}.$$

W.l.o.g. we can assume that the polytopic system is quadratically stable (see [11]). This is the case e.g. when the system is pre-compensated. Notice that such a condition is necessary in order to apply the Bounded Real Lemma. In this paper, we shall consider a residual generator based on the deconvolution filter having the following structure

$$F: \left\{ \begin{array}{rcl} \dot{x}_F(t) & = & A_F x_F(t) + B_F y(t) \\ z(t) & = & L_F x_F(t) + H_F y(t) \end{array} \right. \tag{3}$$

where $x_F(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^q$, such that, considering the augmented system

$$G: \left\{ \begin{array}{rcl} \dot{x}_{cl}(t) & = & \mathcal{A}_{cl} x_{cl}(t) + \mathcal{B}_{cl}\begin{bmatrix} f(t) \\ d(t) \end{bmatrix} \\ \\ r_{cl}(t) & = & \mathcal{C}_{cl} x_{cl}(t) + \mathcal{D}_{cl}\begin{bmatrix} f(t) \\ d(t) \end{bmatrix} \end{array} \right. \tag{4}$$

$$x_{cl}(t) = \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix},\ r_{cl}(t) = z(t) - y(t),$$

$$\mathcal{A}_{cl} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix},\ \mathcal{B}_{cl} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$

$$\mathcal{C}_{cl} = \begin{bmatrix} H_F C - C & L_F \end{bmatrix},\ \mathcal{D}_{cl} = H_F D - D$$

and the residual vector, depending on disturbances, fault signals and command inputs, can be written as

$$r(s) = G_{rf}(s)\,f(s) + G_{rd}(s)\,d(s) + G_{ru}(s)\,u(s) \tag{5}$$

where the transfer functions $G_{rd}(s)$, $G_{rf}(s)$ and $G_{ru}(s)$ can be straightforwardly derived. Note that, due to the quadratic stability assumption and to the structure of the matrix $\mathcal{A}_{cl}$, the external input $u(t)$ can be omitted in the filter design phase. Its role will be instead taken into consideration in the threshold design stage. The problem we want to address is the synthesis of a stable residual generator such that:

1) the disturbance effects are minimized;
2) the fault effects are enhanced.

The objectives of robust residual generation are twofold and partially conflicting each other. In fact, a trade-off exists between the minimization of the disturbance effects on the residual and the maximization of the residual sensitivity to faults. The first leads to the minimization of the $\mathcal{H}_\infty$-norm of G_{rd} whereas the fault sensitivity enhancement would correspond to the maximization of the minimum singular values of G_{rf}, which is a nonconvex function of the convolution filter matrices. The design of the residual observer is accomplished by means of the following frequency windowed $\mathcal{H}_\infty$ optimization problem

Definition 1 *Given two scalars $\beta > \gamma > 0$. The filter (3) is called an $\mathcal{H}_\infty$ fault detection filter if the following conditions hold:*

$$\overline{\sigma}_{\omega\in[\omega_i,\,\omega_s]}\left(G_{r_{cl}d}(j\,\omega)\right) \leq \gamma,\ \gamma > 0. \tag{6}$$

$$\underline{\sigma}_{\omega\in[\omega_i,\,\omega_s]}\left(G_{r_{cl}f}(j\,\omega)\right) \geq \beta,\ \beta > 0, \tag{7}$$

$$\beta \geq \gamma. \tag{8}$$

where $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote respectively the maximum and minimum singular values of a matrix. We assume hereafter that both arguments of (6) and (7) are suitable weighted and will denote with $P_d(s)$ and $P_f(s)$ the corresponding system transfer functions. By considering two windowing filters $W_d(s)$, $W_f(s)$ such that,

$$\overline{\sigma}(W_d(j\,\omega)) = \left\{ \begin{array}{ll} 1 & \omega \in [\omega_i,\,\omega_s] \\ 0 & \omega \notin [\omega_i,\,\omega_s] \end{array} \right.$$

$$\underline{\sigma}(W_f(j\,\omega)) = \left\{ \begin{array}{ll} 0 & \omega \in [\omega_i,\,\omega_s] \\ 1 & \omega \notin [\omega_i,\,\omega_s] \end{array} \right.$$

conditions (6) and (7) translate into the following mixed robust filtering problem

Problem 1 - Given positive reals a and b, find a filter realization $F(s)$ such that

$$\min_{F(s)} a\gamma + b\beta$$
subject to
$$\|F(s)\,P_d(s)\,W_d(s)\|_\infty \le \gamma,\ \gamma > 0, \tag{9}$$
$$\|F(s)\,P_f(s)\,W_f(s)\|_- \ge \beta,\ \beta > 0, \tag{10}$$
$$\beta \ge \gamma. \tag{11}$$

Constants a and b are used to trade-off between the conflicting requirements (9) and (10).

The second important task for FD consists in the evaluation of the generated residuals. One widely adopted approach is to choose a threshold $J_{th} > 0$ and use the following logical relationships for fault detection

$$\begin{array}{lclcl} J_r(t) > J_{\text{th}} & \Rightarrow & \text{faults} & \Rightarrow & \text{alarm at instant } t, \\ J_r(t) \le J_{\text{th}} & \Rightarrow & \text{no faults} & & \end{array} \tag{12}$$

where

$$J_r(t) = \sqrt{\frac{1}{t}\int_0^t r^T(\tau)\,r(\tau)\,d\tau} \tag{13}$$

(see [14] for a detailed discussion about this index). Details and properties of the detection and isolation logic used in this paper will be given in next Section IV.

III. LMI Formulation

Let us consider a state space realization of $W_d(s)$ and $W_f(s)$ (identical state space dimension)

$$W_d(s) := \left[\begin{array}{c|c} A_{r,d} & B_{r,d} \\ \hline C_{r,d} & D_{r,d} \end{array}\right],\ W_f(s) := \left[\begin{array}{c|c} A_{r,f} & B_{r,f} \\ \hline C_{r,f} & D_{r,f} \end{array}\right]$$

The state space realization of $P_d(s)\,W_d(s)$ and $P_f(s)\,W_f(s)$ will be, as a consequence,

$$\begin{aligned} P_d(s)\,W_d(s) &:= \left[\begin{array}{cc|c} A_{r,d} & 0 & B_{r,d} \\ B\,C_{r,d} & A & B\,D_{r,d} \\ \hline D\,C_{r,d} & C & D\,D_{r,d} \end{array}\right] \\ &= \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right] \end{aligned}$$

$$\begin{aligned} P_f(s)\,W_f(s) &:= \left[\begin{array}{cc|c} A_{r,f} & 0 & B_{r,f} \\ B\,C_{r,f} & A & B\,D_{r,f} \\ \hline D\,C_{r,f} & C & D\,D_{r,f} \end{array}\right] \\ &= \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] \end{aligned}$$

The filter dimension, for convexity reasons, will be equal to $n_F = n_r$ (windowing) $+n$ (plant) . Let us examine first (10), by noting that

$$\|F(s)\,P_f(s)\,W_f(s)\|_- \ge \underline{\sigma}_{\omega\in[\omega_i,\,\omega_s]}\left(G_{r_{cl}f}(j\,\omega)\right) \ge \beta$$

and by applying the Bounded Real Lemma, such a condition is satisfied if there exists a matrix $X = X^T \in \mathbb{R}^{2\,n_F\times 2\,n_F}$, $X \ge 0$, such that the following inequality

$$\begin{array}{l} \dot{\xi}^T(t)\,X\,\xi(t) + \xi^T(t)\,X\,\dot{\xi}(t) \\ +\beta^2\,f^T(t)\,f(t) - r^T(t)\,r(t) < 0 \end{array} \tag{14}$$

holds true (here $\xi(t)$ represents the state of $F(s)\,P_f(s)\,W_f(s)$). Equation (14) is equivalent to

$$\begin{array}{l} (A_{cl,f}\,\xi(t) + B_{cl,f}\,f(t))^T\ X\,\xi(t) \\ +\xi^T(t)\,X\ (A_{cl,f}\,\xi(t) + B_{cl,f}\,f(t)) + \beta^2\,f^T(t)\,f(t) \\ -\left(C_{cl,f}\,\xi(t) + D_{cl,f}\,f(t)\right)^T\ (C_{cl,f}\,\xi(t) + D_{cl,f}\,f(t)) < 0 \end{array}$$

where

$$A_{cl,f} := \begin{pmatrix} \tilde{A}_d & 0 \\ B_F\,\tilde{C}_d & A_F \end{pmatrix},\ B_{cl,f} := \begin{pmatrix} \tilde{B}_d \\ B_F\,D_d \end{pmatrix}$$

$$C_{cl,f} := \begin{pmatrix} C_f - H_F\,C_f & -L_F \end{pmatrix},\ D_{cl,f} := D_f - H_F\,D_f$$

which yields the following matrix inequality

$$\begin{array}{l} \begin{bmatrix} A_{cl,f}^T\,X + X\,A_{cl,f} & X\,B_{cl,f} \\ B_{cl,f}^T\,X & \beta^2\,I \end{bmatrix} \\ -\begin{bmatrix} C_{cl,f}^T \\ D_{cl,f}^T \end{bmatrix} \begin{bmatrix} C_{cl,f} & D_{cl,f} \end{bmatrix} < 0. \end{array} \tag{15}$$

A necessary condition to satisfy (15) is

$$\beta^2\,I - D_{cl,f}^T\,D_{cl,f} < 0,$$

which corresponds to the outer region of a ball of radius β in the objective variable $D_{cl,f}$. This means that the feasible set is non convex. To overcome this difficulty, we restrict the search to a convex subset by imposing the following further constraint

$$(T\,H_F\,D_f)^T + (T\,H_F\,D_f) + 2\,\beta < 0.$$

which represents a convex region to the tangent hyperplane [12] of the ball

$$\beta^2\,I - D_{cl,f}^T\,D_{cl,f} = 0. \tag{16}$$

As a consequence, the matrix inequality (15) holds true if

$$\begin{bmatrix} C_{cl,f}^T \\ D_{cl,f}^T \end{bmatrix} \begin{bmatrix} C_{cl,f} & D_{cl,f} \end{bmatrix} \le \begin{bmatrix} \alpha_1\,X & 0 \\ 0 & \alpha_2\,I \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} A_{cl,f}^T\,X + X\,A_{cl,f} - \alpha_1\,X & X\,B_{cl,f} \\ B_{cl,f}^T\,X & (\beta^2 - \alpha_2)\,I \end{bmatrix} < 0 \tag{18}$$

$$(T\,H_F\,D_f)^T + (T\,H_F\,D_f) + 2\,\beta < 0. \tag{19}$$

By means of the projection lemma, constraint (18) can be written in the form

$$\mathcal{N}_{R_f}^T\,\Psi_f\,\mathcal{N}_{R_f} < 0 \tag{20}$$

where $\mathcal{N}_{R_f}$ is the null space of the block partitioned matrix

$$R_f = \begin{bmatrix} -I & A_{cl,f} & B_{cl,f} & I \end{bmatrix},$$

$$\mathcal{N}_{R_f} = \begin{bmatrix} A_{cl,f} & B_{cl,f} & I \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and Ψ_f is equal to

$$\Psi_f = \begin{bmatrix} 0 & X & 0 & 0 \\ X & -X - \alpha_1\,X & 0 & 0 \\ 0 & 0 & (\beta^2 - \alpha_2)\,I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By choosing the matrix

$$Q = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$$

the following inequality holds true

$$\mathcal{N}_Q^T \Psi_f \mathcal{N}_Q < 0 \tag{21}$$

$$\mathcal{N}_Q = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$\mathcal{N}_Q$ being the respective null space and

$$\mathcal{N}_Q^T \Psi_f \mathcal{N}_Q = \begin{bmatrix} -X - \alpha_1 X & 0 & 0 \\ 0 & (\beta^2 - \alpha_2) I & 0 \\ 0 & 0 & -X \end{bmatrix} < 0$$

which is satisfied if $\alpha_1 > 0$, $\alpha_2 > \beta^2$. Projection lemma states that the determination of $X = X^T$, A_F, B_F, L_F, H_F, such that

$$\mathcal{N}_{R_f}^T \Psi_f \mathcal{N}_{R_f} < 0 \text{ (eq. (18))}, \; \mathcal{N}_Q^T \Psi_f \mathcal{N}_Q < 0$$

is equivalent feasibility test on the inequalities

$$\Psi_f + R_f^T \, V \, Q + Q^T \, V^T \, R_f < 0. \tag{22}$$

Equation (22) translates into

$$\begin{bmatrix} -(V + V^T) & X + V^T A_{cl,f} & V^T B_{cl,f} & V^T \\ X + A_{cl,f}^T V & -X - \alpha_1 X & 0 & 0 \\ B_{cl,f}^T V & 0 & (\beta^2 - \alpha_2) I & 0 \\ V & 0 & 0 & -X \end{bmatrix} < 0$$

By considering block partitions of V and X

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \; X = \begin{bmatrix} X_1 & X_3^T \\ X_3 & X_2 \end{bmatrix}$$

and, having in mind the structure $\mathcal{A}_{cl,f}$, $\mathcal{B}_{cl,f}$, (22) becomes,

$$\begin{bmatrix} -(V_{11} + V_{11}^T) & -(V_{12} + V_{21}^T) & V_{11}^T \tilde{A}_f + V_{21}^T B_F \tilde{C}_f + X_1 & V_{21}^T A_F + X_3^T & V_{11}^T \tilde{B}_f + V_{21}^T B_F D_f & V_{11}^T & V_{21}^T \\ -(V_{21} + V_{12}^T) & -(V_{22} + V_{22}^T) & V_{12}^T \tilde{A}_f + V_{22}^T B_F \tilde{C}_f + X_3 & V_{12}^T A_F + X_2 & V_{12}^T \tilde{B}_f + V_{22}^T B_F D_f & V_{12}^T & V_{22}^T \\ (*) & (*) & -(1+\alpha_1) X_1 & -(1+\alpha_1) X_3^T & 0 & 0 & 0 \\ (*) & (*) & -(1+\alpha_1) X_3 & -(1+\alpha_1) X_2 & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & -(\beta^2 - \alpha_2) I & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & -X_1 & -X_3^T \\ (*) & (*) & (*) & (*) & (*) & -X_3 & -X_2 \end{bmatrix} < 0 \tag{23}$$

which, applying the following congruence transformation

$$\text{diag}\left[I, \, V_{22}^{-1} V_{21}, \, I \, V_{22}^{-1} V_{21}, \, I, \, I, \, V_{22}^{-1} V_{21}\right]$$

(23) becomes,

$$\begin{bmatrix} -(V_{11} + V_{11}^T) & -(S_2^T + S_2^T) & V_{11}^T \tilde{A}_f + \hat{B}_F \tilde{C}_f + \hat{X}_1 & \hat{A}_F + \hat{X}_3^T & V_{11}^T \tilde{B}_f + \hat{B}_F D_f & V_{11}^T & S_2^T \\ -(S_2^T + S_2^T) & -(S_1 + S_1^T) & S_2 \tilde{A}_f + \hat{B}_F \tilde{C}_f + \hat{X}_3 & \hat{A}_F + \hat{X}_2 & V_{12}^T \tilde{B}_f + V_{22}^T B_F D_f & S_2^T & S_1^T \\ (*) & (*) & -(1+\alpha_1) \hat{X}_1 & -(1+\alpha_1) \hat{X}_3^T & 0 & 0 & 0 \\ (*) & (*) & -(1+\alpha_1) \hat{X}_3 & -(1+\alpha_1) \hat{X}_2 & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & -(\beta^2 - \alpha_2) I & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & -\hat{X}_1 & -\hat{X}_3^T \\ (*) & (*) & (*) & (*) & (*) & -\hat{X}_3 & -\hat{X}_2 \end{bmatrix} < 0 \tag{24}$$

where

$$\hat{A}_F = V_{21}^T A_F V_{22}^{-1} V_{21}, \; \hat{B}_F = V_{21}^T B_F, \hat{L}_F = L_F V_{22}^{-1} V_{21}$$

$$S_1 = V_{21}^T V_{22}^{-T} V_{21}, \; S_2 = V_{21}^T V_{22}^{-T} V_{12}$$

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \hat{X}_3^T \\ \hat{X}_3 & \hat{X}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & V_{21}^T V_{22}^{-T} \end{bmatrix} X \begin{bmatrix} I & 0 \\ 0 & V_{22}^{-1} V_{21} \end{bmatrix}$$

so (24) is linear in $\hat{X}$, S_1, $S_2, \hat{A}_F$, $\hat{B}_F$, $V_{11}, \hat{L}_F$, H_F. Moreover, by Schur complements and using the previous congruence transformations, we have that inequality (17) becomes

$$\begin{bmatrix} \alpha_1 \hat{X}_1 & \alpha_1 \hat{X}_3^T & 0 & C^T H_F^T \\ \alpha_1 \hat{X}_3 & \alpha_1 \hat{X}_2 & 0 & \hat{L}^T \\ 0 & 0 & \alpha_2 I & D^T H_F \\ H_F C & \hat{L}_F & 0 & I \end{bmatrix} > 0 \tag{25}$$

The disturbance decoupling condition (9) holds instead true, for a given $\gamma \geq 0$ if the following quasi-convex Linear Matrix Inequality in the variables (see [13])

$$\begin{bmatrix} -\mu (V_{11} + V_{11}^T) & -\mu (S_2 + S_2^T) & V_{11}^T \tilde{A} + \hat{B}_F \tilde{C}_d + \hat{X}_1 & \hat{A}_F + \hat{X}_3^T & V_{11}^T \tilde{B}_d + \hat{B}_F D_d & 0 & V_{11}^T & S_2 \\ -\mu (S_2 + S_2^T) & -\mu (S_1 + S_1^T) & V_{12}^T \tilde{A} + \hat{B}_F \tilde{C}_d + X_3 & \hat{A}_F + X_2 & V_{12}^T B + \hat{B}_F D_d & 0 & S_2^T & S_1 \\ (*) & (*) & -\hat{X}_1 & -\hat{X}_3^T & 0 & \begin{pmatrix} 0 \\ C_d^T H_F^T - C_d^T \end{pmatrix} & 0 & 0 \\ (*) & (*) & -\hat{X}_3 & -\hat{X}_2 & 0 & \hat{L}_F^T & 0 & 0 \\ (*) & (*) & (*) & (*) & -\gamma I & (H_F D_d - D_d)^T & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & -\gamma I & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\mu \hat{X}_1 & -\mu \hat{X}_3^T \\ (*) & (*) & (*) & (*) & (*) & (*) & -\mu \hat{X}_3 & -\mu \hat{X}_2 \end{bmatrix} \tag{26}$$

is negative definite. The following result is so obtained:

Theorem 1 - *A feasible solution to the* **Problem 1** *is obtained by solving a sequence of μ-parameterized optimization problems*

$$\min_{\hat{X}, S_1, S_2, \hat{A}_F, \hat{B}_F, V_{11}, \hat{L}_F, H_F} \; a \, \gamma + b \, \beta$$

subject to

$$\hat{X} > 0$$

$$(25), (24), (26), (19), \; \textit{evaluated over the polytope vertices (2).} \tag{27}$$

For any choice of $\mu > 0$, if solvable, the above problem is convex and admits a unique solution.

Proof: By collecting all the previous discussion. □

Remark 1. The index $a \, \gamma_d + b \, \gamma_f$, $a > 0$, $b > 0$ is a scalar multi-objective cost functional and the weights a and b can be used to trade-off between decoupling and tracking.

Remark 2. The matrices A_F, B_F, L_F, H_F, defining the residual generator can be derived by means of the following procedure. Let us denote $\hat{X}$, S_1, $S_2, \hat{A}_F$, $\hat{B}_F$, $V_{11}, \hat{L}_F$ a solution of (27):

1) compute V_{22}, V_{21} ($n_F \times n_F$) by solving the following factorization problem

$$S_1 = V_{21}^T V_{22}^{-1} V_{21},$$

2) compute A_F, B_F, L_F

$$A_F = V_{21}^{-T}\hat{A}_F V_{21}^{-1}V_{22},\ B_F = V_{21}^{-T}\hat{B}_F,\\ L_F = \hat{L}_F V_{21}^{-1}V_{22}. \quad (28)$$

IV. Thresholds Computation

The detection and isolation decision logic is based on residual thresholds evaluation to be used during the on-line operations for improving the filter capability to avoid false alarm generation. To this end, a quite simple and effective adaptive methodology will be proposed to compute such quantities. The procedure here outlined is based on ideas proposed by [14]. Let the time-windowed rms-norm

$$J_r(t) \quad = \|r\|_{\text{rms},t} = \sqrt{\tfrac{1}{t}\int_0^t r^T(\tau)r(\tau)\,d\tau},$$

be a convenient residual measure. Under fault-free conditions, (5) becomes

$$r(s) = G_{rd}\,d(s) + G_{ru}\,u(s).$$

and via the Perseval's Theorem (see [15], pp. 98-99) one has that

$$\begin{aligned}\|r\|_{\text{rms},t,f=0} &= \|r_d + r_u\|_{\text{rms},t}\\ &\le \|G_{rd}\|_\infty\ \|d\|_{\text{rms}} + \|G_{ru}\|_\infty\ \|u\|_{\text{rms},t}\\ &= \gamma\alpha + \rho\ \|u\|_{\text{rms},t}\end{aligned}$$

where γ is the solution of the quasi-LMI optimization problem (27), ρ is the $\mathcal{H}_\infty$-norm of $G_{ru}(s)$ and α is a convenient upper-bound to the rms-norm of the worst disturbance acting on the plant. As a consequence, the following threshold results

$$J_{th}(t) := \gamma\alpha + \rho\ \|u\|_{\text{rms},t}\,. \quad (29)$$

V. Numerical example

The following example is taken from [7] and it shows the discrimination capabilities of the proposed method for the detection problem. Consider the following state description of the uncertain system with norm-bounded unstructured model uncertainty

$$\left\{\begin{array}{rcl}\dot{x}(t) &=& (A+\Delta A)\,x(t) + (B+\Delta B)\,u(t)\\ && +B_f\,f(t) + B_d\,d(t)\\ y(t) &=& C\,x(t) + D\,u(t) + D_f\,f(t) + D_d\,d(t)\end{array}\right.$$

where $[\Delta A\ \ \Delta B] = [E_1\Sigma_1 F_1\ \ E_2\Sigma_2 F_2],\ \ \Sigma_1^T\Sigma_1 \le I$, $\Sigma_2^T\Sigma_2 \le I$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ 6.52 & 24.77 & 33.82 & 35.56\\ 0 & 0 & 0 & 1\\ -12.56 & -49.71 & -56.25 & -68.50\end{bmatrix}$$

$$B = \begin{bmatrix} 0\\ 0.2471\\ 0\\ -0.4758\end{bmatrix},\ B_d = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0.5\end{bmatrix}$$

$$B_f = \begin{bmatrix} 0\\ 0.5\\ 0\\ -0.5\end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

$$D_d = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ D = 0,\ D_f = 0,$$

$$E = [0\ 0.1\ 0\ 0.1]^T,\ F_1 = [0.2\ 0.1\ 0.1\ 0.1],\ F_2 = 0.1.$$

In the sequel, the uncertain plant will be described by the multimodel realization whose vertices are:

$$A_1 = \begin{bmatrix} 0 & 1.0000 & 0 & 0\\ 6.5400 & 24.7800 & 33.8300 & 35.5700\\ 0 & 0 & 0 & 1.0000\\ -12.5400 & -49.7000 & -56.2400 & -68.4900\end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1.0000 & 0 & 0\\ 6.5000 & 24.7600 & 33.8100 & 35.5500\\ 0 & 0 & 0 & 1.0000\\ -12.5800 & -49.7200 & -56.2600 & -68.5100\end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0\\ 0.0347\\ 0\\ -0.4658\end{bmatrix},\ B_2 = \begin{bmatrix} 0\\ 0.0147\\ 0\\ -0.4858\end{bmatrix}$$

We are interested to design a robust filter which achieves robust disturbance decoupling and fault detection capabilities in the frequency range $\Omega = [\omega_i,\ \omega_s] = [0,\ 5]\frac{\text{rad}}{\text{s}}$. Then, the following windowing filters have been used

$$W_d(s) = \text{diag}\left(\frac{\omega_s^2}{s^2 + 15\,\omega_s\,s + \omega_s^2},\ \frac{\omega_s^2}{s^2 + 15\,\omega_s\,s + \omega_s^2}\right)$$

$$W_f(s) = \text{diag}\left(\frac{s+\omega_s}{\omega_s},\ \frac{s+\omega_s}{\omega_s}\right)$$

The disturbance $d(t)$ is assumed to be a unitary variance withe noise with the upper bound $\|d\|_2 = 0.15$, the fault signal is

$$f(t) = \left\{\begin{array}{ll} 0 & t < 5 \text{ sec.}\\ 1 & 5 \text{ sec.} \le t \le 10 \text{ sec.}\\ 0 & t > 10 \text{ sec.}\end{array}\right.$$

and the input $u(t)$ is taken as a unit step signal. The quasi-convex problem (27) has been solved choosing $a = 1$ and $b = 1$ (equal levels of importance in disturbance rejection and fault detection) in the objective function $a\,\gamma + b\,\beta$ and the corresponding optimal values of the objective function terms are $\gamma = 0.8252$ and $\beta = 2.8171$. Note that we have selected $T = [30\ 1]$ for constraint (19). Finally, for all simulations, we have imposed $\mu = 8$ for ensuring the feasibility of (27).

A 8-states, 2-inputs and 2-outputs filter $(A_F,\ B_F,\ L_F,\ H_F)$ has been computed by means of formulas (28). During all simulations the uncertain plant parameters have been kept constant at the vertex $(A_1,\ B_1)$.

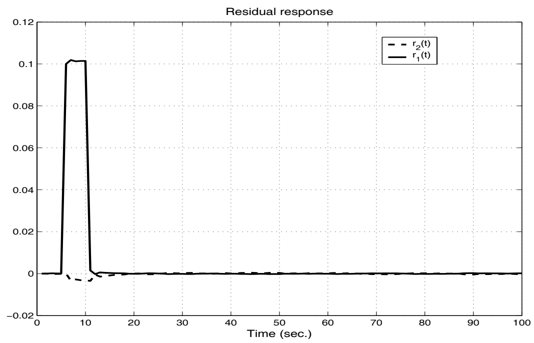

Fig. 1. Residual response, dashed line $r_2(t)$, continuous line $r_1(t)$ for $u(t) = 0$

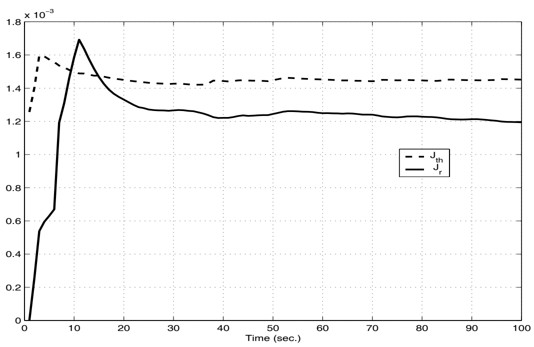

Fig. 2. Threshold $J_{th}(t)$ dashed line, continuous line time-windowed rms-norm $J_r(t)$

The two residuals are shown in Fig. 1, while in Fig. 2, the threshold responses $J_{th}(t)$ (dashed line) and $J_r(t)$ (continuous line) are reported for the same disturbance and fault signal of Fig. 1. It results that the alarm is detected $4\,sec.$ after its occurrence and the computed filter is capable to detect that the plant has recovered to no-alarm condition after $15\,sec.$

VI. Conclusions

A solution to the robust fault detection problem for linear polytopic uncertain plants has been proposed via deconvolution filters. By taking advantage of the projection lemma and using congruence transformations, the problem has been converted into a quasi-LMI optimization problem. In particular the sensitivity constraint has been recast in LMIs by a proper linearization of the feasibility region. A time-window based adaptive threshold logic has been used in order discriminate between real and false alarms. A numerical example showing the effectiveness of the approach has been described in details and the results testified good filter detection capabilities.

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