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Minimum Variance benchmark for Decentralized Controllers

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***Abstract*— This paper deals with performance assessment of decentralized controllers using a minimum variance (MV) benchmark. The available MV benchmarks do not take the structure of the controller into account and can give overly optimistic estimates of achievable performance, when applied to systems under decentralized control. We propose an approximate solution to this problem obtained by explicitly solving simple linear matrix equations. As a special case of this general result, we also present an upper bound on the achievable performance for systems under multi-loop proportional-integral-derivative control. These results are useful for assessing the feasibility of significant performance improvement by re-tuning of the decentralized controller and input-output pairing selection.**

I. Introduction

In the control literature, it is common to represent a non-linear, time-varying process by a linear time invariant model and design a controller based on this. In the presence of changing operating conditions and disturbance dynamics, the closed loop performance of the controller designed based on this approximation may deteriorate over time. Sustained benefits can be reaped by monitoring the performance and taking appropriate corrective actions, in the case of large deviations from the designed performance.

Poor controller tuning is one of the primary reasons for performance deterioration of industrial controllers. It is important to assess the feasibility of significant performance improvement, before the task of controller tuning is undertaken. This purpose is well served by the minimum variance (MV) benchmark, where the controller objective is defined in terms of output variance. The MV benchmark represents the theoretical lower bound on the achievable output variance. The output variance can be reduced by controller tuning, when the actual variance differs significantly from the MV benchmark; otherwise, different approaches should be considered *e.g.* the use of feedforward controller or additional manipulated variables.

The idea of MV control was introduced by Åström [1]. Harris [2] showed that with *a priori* knowledge of time delay, MV benchmark can be estimated using routine closed loop operating data and established it as a tool for performance monitoring of single-input, single-output (SISO) systems. This approach is further extended to multi-input, multi-output (MIMO) systems by Harris *et al.* [3] and Huang *et al.* [4]. Qin [5] and Harris *et al.* [6] provide comprehensive reviews of MV based and other performance assessment tools.

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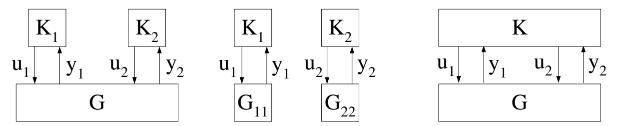

Fig. 1. Insufficiency of available Minimum Variance based benchmarks for performance assessment of systems under decentralized control

Though useful, the available MV benchmark shows limitations, when applied to systems under (block) decentralized or multi-loop control. The conventional approaches towards performance assessment of such controllers include:

- Loop by loop analysis
- Use of the MV benchmark for full multivariate controllers

The MV benchmark fails to take the process interactions into account, when applied in a loop-wise fashion; whereas, the full multivariable benchmark assumes more degrees of freedom for performance improvement than are available in the actual controller (see Figure 1). In either case, the bound on the achievable output variance is loose and can be overly optimistic. In many cases, it may lead the practicing engineer to search for the non-existent decentralized controller to match the performance of the MV benchmark. The gap between the benchmark and achievable performance further increases when the decentralized controller is restricted to be of reduced complexity, *e.g.* proportional integral derivative (PID) controller [7], [8]. Thus, a decentralized MV benchmark is required, which takes the controller structure into account. These arguments are further illustrated using the following motivating example taken from Huang and Shah [9]:

Example 1: Consider $\mathbf{y}(t) = \mathbf{G}(q^{-1})\mathbf{u}(t) + \mathbf{G}_w(q^{-1})\mathbf{a}(t)$, where q^{-1} is the backshift operator, $\mathbf{a}(t)$ is Gaussian noise with unit variance and

$$\mathbf{G} = \begin{bmatrix} \frac{q^{-2}}{1-0.4q^{-1}} & \frac{2q^{-2}}{1-0.5q^{-1}} \\ \frac{q^{-2}}{1-0.1q^{-1}} & \frac{q^{-2}}{1-0.2q^{-1}} \end{bmatrix}$$
$$\mathbf{G}_w = \begin{bmatrix} \frac{2}{1-0.9q^{-1}} & \frac{1}{1-0.3q^{-1}} \\ \frac{1}{1-0.4q^{-1}} & \frac{2}{1-0.5q^{-1}} \end{bmatrix} \quad (1)$$

The objective is to assess the performance of a multi-loop controller of the form $k\mathbf{I}$, $k = 0.17$. Under closed loop control, $E[\mathrm{tr}(\mathbf{y}(t)\mathbf{y}(t)^T] = 23.65$, where $E[.]$ is the expectation operator. The MV benchmark for full multivariate controller is 14.5, but no k or a dynamic compensator could be found that matches this benchmark closely. As shown later, the given controller structure inherently limits the achievable performance and the controller $0.17\mathbf{I}$ is nearly optimal for the given controller structure.

An explicit solution to the decentralized MV control problem has great theoretical and practical value, but is equally difficult to realize. The primary difficulty lies in enforcing the decentralized structure on the controller, as this yields a non-convex optimization problem [10]. Yuz and Goodwin [11] have suggested a two-step approach for determining an upper bound on the achievable output variance using a decentralized controller:

- A decentralized controller is designed based on only the diagonal elements of the system.
- The controller is redesigned to compensate for the ignored off-diagonal elements using an approximation of the sensitivity function.

Though the initial design based on the diagonal elements accommodates the controller structure, the controller redesign step requires some care and numerical search. Further, the utility of the method in its present form is limited to step disturbances only. A similar numerical search based method has been proposed by Ko and Edgar [8].

In this paper, we take a fundamentally different approach to derive an approximate solution for the decentralized MV control problem. The controller structure is posed as a constraint on the optimization problem and a suboptimal solution is obtained by explicitly solving the linear matrix equations defining the stationary point. As a special case, we present an upper bound on the achievable output variance for systems under multi-loop PID control. The results presented here do not require controller redesign [11] or numerical search [8]; however the simplicity of the result comes at the cost of sub-optimality. These results are useful for various purposes:

1) Performance assessment of existing decentralized or multi-loop controllers.
2) Selection of input-output pairings based on achievable decentralized performance.
3) Providing a good initial guess for non-convex parameter search methods.

II. Preliminaries

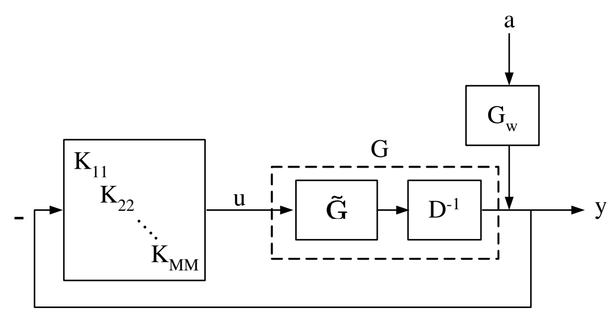

Fig. 2. Closed loop system with separation of interactor matrix

Consider the system shown in Figure 2, where $\mathbf{K}(q^{-1}) = \mathrm{diag}(\mathbf{K}_{ii}(q^{-1}))$, $i = 1, \cdots, M$. The objective is to find a controller such that the variance of $\mathbf{y}(t)$ or $E[\mathrm{tr}(\mathbf{y}(t)\mathbf{y}(t)^T]$ is minimized. We make the following simplifying assumptions:

1) $\mathbf{G}(q^{-1})$ and $\mathbf{G}_w(q^{-1})$ are stable, causal transfer matrices, contain no zeros outside the unit circle and are square having dimensions $n \times n$.
2) $\mathbf{a}(t)$ is a random noise sequence with unit variance and $\mathbf{y}(t)$ is stationary up to its second moment.

The assumption that $\mathbf{G}(q^{-1})$ and $\mathbf{G}_w(q^{-1})$ are square is made for notational simplicity and can easily be relaxed for generalization purposes. When $\mathbf{G}_w$ contains zeros outside the unit circle, these zeros can be factored through an all pass factor factorization without affecting the noise spectrum [9]. Further, there is no loss of generality in assuming that the system is affected by noise having unit variance. When $E[\mathbf{a}(t)\mathbf{a}^T(t)] \neq \mathbf{I}$, the noise model can always be scaled to satisfy this assumption.

A. Interactor Matrices

Before proceeding with the main development, we introduce the useful concept of interactor matrices.

***Definition** 1:* For every $n_1 \times n_2$ proper, rational polynomial transfer matrix $\mathbf{G}(q^{-1})$, there is a unique, non-singular, $n_1 \times n_1$ lower triangular polynomial matrix $\mathbf{D}(q)$, such that $|\mathbf{D}(q)| = q^r$ and [12]

$$\lim_{q^{-1}\to 0} \mathbf{D}(q)\mathbf{G}(q^{-1}) = \lim_{q^{-1}\to 0} \tilde{\mathbf{G}}(q^{-1}) = \tilde{\mathbf{G}}(0) \tag{2}$$

where $\tilde{\mathbf{G}}(0)$ is a full rank constant matrix [9]. The matrix $\mathbf{D}(q)$ is called the *interactor matrix*.

For univariate systems, the MV benchmark primarily depends on the time delay associated with $\mathbf{G}(q^{-1})$ [1]. This time delay can also be interpreted as the non-invertible part of the transfer matrix, as its inverse is non-causal. Similarly, the multivariate system $\mathbf{G}(q^{-1})$ can be factored as $\mathbf{G}(q^{-1}) = \mathbf{D}^{-1}(q^{-1})\tilde{\mathbf{G}}(q^{-1})$ such that $\tilde{\mathbf{G}}(q^{-1})$ and $\mathbf{D}^{-1}(q^{-1})$ contain the invertible and non-invertible parts of $\mathbf{G}(q^{-1})$ respectively. The interactor matrix generalizes the time delay for univariate systems to the multivariate case [9] and can be written as,

$$\mathbf{D}(q) = \mathbf{D}_0(q)q^d + \mathbf{D}_1(q)q^{d-1} + \cdots \mathbf{D}_{d-1}(q)q$$

where d denotes the order of the interactor matrix.

When $\mathbf{D}(q)$ assumes the form $\mathbf{D}(q) = q^d\mathbf{I}$, $\mathbf{D}(q)$ is called a simple interactor matrix. Similarly, an interactor matrix with the form $\mathbf{D}(q) = \mathrm{diag}(q^{d_1}, \cdots, q^{d_n})$ is called a diagonal interactor matrix. $\mathbf{D}(q)$ with no special structure is called a general interactor matrix.

The lower triangular form is only one of the possible realizations of the interactor matrices. In general, the interactor matrix can also be upper triangular or a full matrix. One realization of the interactor matrix that is of immediate interest to us, is when $\mathbf{D}(q)$ is a unitary transfer matrix.

***Definition** 2:* For a rational proper, transfer matrix $\mathbf{G}(q^{-1})$ having full rank, let the $\mathbf{D}(q)$ satisfying (2) also satisfies $\mathbf{D}^T(q^{-1})\mathbf{D}(q) = \mathbf{I}$. Then, $\mathbf{D}(q)$ is called a *unitary interactor matrix* [13].

The unitary interactor matrix is non-unique, but two unitary interactor matrices are related by transformation through a unitary matrix [9]. The unitary interactor matrix is useful for deriving the MV control law, which is independent of the order in which outputs are stacked in the output vector or the importance of individual outputs [9].

B. Problem Formulation

We formulate the optimization problem next that can be solved to obtain the solution to the decentralized MV control problem. In the remaining discussion, the arguments q^{-1} and t are dropped for ease of representation. Let the system shown in Figure 2 be expressed as

$$\begin{aligned} \mathbf{y} &= \mathbf{D}^{-1}\tilde{\mathbf{G}}\mathbf{u} + \mathbf{G}_w\mathbf{a} \\ \text{or} \quad \mathbf{D}_1\mathbf{y} &= q^{-d}\tilde{\mathbf{G}}\mathbf{u} + \bar{\mathbf{G}}_w\mathbf{a} \end{aligned} \tag{3}$$

where $\mathbf{D}_1 = q^{-d}\mathbf{D}$, $\bar{\mathbf{G}}_w = \mathbf{D}_1\mathbf{G}_w$ and d is the order or number of non-zero impulse response matrices of $\mathbf{D}$. Using Diophantine's identity,

$$\bar{\mathbf{G}}_w = \bar{\mathbf{F}} + q^{-d}\bar{\mathbf{R}}$$

For regulatory control, $\mathbf{u} = -\mathbf{K}\mathbf{y}$ and

$$\mathbf{D}_1\mathbf{y} = -q^{-d}\tilde{\mathbf{G}}\mathbf{K}\mathbf{y} + (\bar{\mathbf{F}} + q^{-d}\bar{\mathbf{R}})\mathbf{a} \tag{4}$$

By rearranging (3),

$$\mathbf{a} = \bar{\mathbf{G}}_w^{-1}(\mathbf{D}_1\mathbf{y} - q^{-d}\tilde{\mathbf{G}}\mathbf{u}) \tag{5}$$

Using (5) and simple algebraic manipulations, (4) can be simplified as,

$$\mathbf{D}_1\mathbf{y} = \bar{\mathbf{F}}\mathbf{a} + q^{-d}(\bar{\mathbf{R}}\mathbf{G}_w^{-1} - \bar{\mathbf{F}}\bar{\mathbf{G}}_w^{-1}\tilde{\mathbf{G}}\mathbf{K})\mathbf{y} \tag{6}$$

Since $E[\mathrm{tr}(\mathbf{y}(t)\mathbf{y}(t)^T)] = E[\mathrm{tr}(\mathbf{D}_1\mathbf{y}(t)\mathbf{y}(t)^T\mathbf{D}_1^T)]$ [9, Lemma 4.3.1] and $\bar{\mathbf{F}}$ is controller invariant, the second term in (6) can be set to zero to obtain the full multivariable MV control law. When the controller has structural constraints, this may not be possible since $\mathbf{K}$ has fewer degrees of freedom than the full multivariable controller.

To simplify notation in the remaining discussion, we define

$$\mathbf{A} = \bar{\mathbf{R}}\mathbf{G}_w^{-1}, \quad \mathbf{B} = \bar{\mathbf{F}}\bar{\mathbf{G}}_w^{-1}\tilde{\mathbf{G}}, \quad \mathbf{L} = \mathbf{A} - \mathbf{B}\mathbf{K} \tag{7}$$

Then using (6) and (7),

$$\begin{aligned} \mathbf{y} &= (\mathbf{D}_1 - q^{-d}\mathbf{L})^{-1}\bar{\mathbf{F}}\mathbf{a} \\ &= (\mathbf{I} - q^{-d}\mathbf{D}_1^T\mathbf{L})^{-1}\mathbf{D}_1^T\bar{\mathbf{F}}\mathbf{a} \end{aligned}$$

When the spectral radius of $\mathbf{D}_1^T\mathbf{L}(e^{j\omega})$ is less than 1 for all $\omega = [0\ ,\ 2\pi]$ or the closed loop system is stable, the series expansion of $(\mathbf{I} - q^{-d}\mathbf{D}_1^T\mathbf{L})^{-1}$ is convergent. Thus,

$$\mathbf{y} = \left(\sum_{i=0}^{\infty}(q^{-d}\mathbf{D}_1^T\mathbf{L})^i\right)\mathbf{D}_1^T\bar{\mathbf{F}}\mathbf{a} \tag{8}$$

As $\mathbf{a}(t)$ is a random noise sequence, $E[\mathbf{a}(t)\mathbf{a}^T(t+\tau)] = 0$ for all $\tau \neq 0$ and $\mathbf{D}_1$ is a unitary transfer matrix,

$$\begin{aligned} E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)] &= \|\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \|\mathbf{D}_1^T\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \cdots \\ &= \|\bar{\mathbf{F}}\|_2^2 + \|\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \cdots \end{aligned} \tag{9}$$

The higher order terms in (9) are non-linear in $\mathbf{K}$. Since $\|\bar{\mathbf{F}}\|_2^2$ is controller invariant, an approximate solution to the decentralized MV control problem is obtained by ignoring the higher order terms in (9) and finding the stationary point of $\|\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2$ with respect to the block diagonal $\mathbf{K}$. The resulting equations using this approach require an iterative procedure to be solved and in order to avoid this difficulty, we use the following result:

***Lemma** 1:* Let $\mathbf{X}, \mathbf{Y}$ be stable transfer matrices. Then,

$$\|\mathbf{X}\mathbf{Y}\|_2^2 \leq \|\mathbf{X}\|_2^2\|\mathbf{Y}\|_\infty^2$$

The proof of Lemma 1 is simple and is omitted for the sake of brevity. Using (9) and Lemma 1,

$$E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)] \leq \|\mathbf{F}\|_2^2 + \|\mathbf{L}\|_2^2\|\bar{\mathbf{F}}\|_\infty^2 + \cdots \tag{10}$$

With this simplification, the decentralized controller that provides an overestimate of the achievable output variance is obtained by solving the following optimization problem

$$\begin{aligned} &\min_{\mathbf{K}} \|\mathbf{L}\|_2^2 \\ \text{subject to} \quad &(\mathbf{1}_{nn} - \mathbf{J}) \circ \mathbf{K} = \mathbf{0} \end{aligned} \tag{11}$$

where $\mathbf{1}_{nn}$ is a matrix of ones and $\circ$ is the Hadamard product. $\mathbf{J}$ is a matrix representing the controller structure and is defined as

$$\mathbf{J}_{ij} = \begin{cases} 1 & \text{if} \quad \mathbf{K}_{ij} \neq 0 \\ 0 & \text{if} \quad \mathbf{K}_{ij} = 0 \end{cases} \tag{12}$$

The equality constraint in the optimization problem (11) accommodates the controller structure by ensuring that the off-block diagonal terms of the controller are zero.

III. DECENTRALIZED MV BENCHMARK

In this section, an explicit solution to the optimization problem given by (11) is provided. For these purposes, we present the following result,

***Lemma** 2:* Let $\mathbf{Y} = \mathbf{X}^T\mathbf{M}\mathbf{X} - \mathbf{N}^T\mathbf{X}$, where $\mathbf{X}$ is a block diagonal matrix. Then, the stationary point of $\mathrm{tr}(\mathbf{Y})$ with respect to $\mathbf{X}$ is found by solving

$$\mathbf{J} \circ \left[(\mathbf{M} + \mathbf{M}^T)\right]\mathbf{X} = \mathbf{J} \circ \mathbf{N} \tag{13}$$

where $\mathbf{J}$ represents the block diagonal structure of $\mathbf{X}$ and is defined similar to (12).

Proof: With $\mathbf{X} = \mathrm{diag}(\mathbf{X}_{11}, \cdots \mathbf{X}_{MM})$,

$$\mathrm{tr}(\mathbf{Y}) = \sum_{i=1}^{M}\mathrm{tr}(\mathbf{X}_{ii}^T\mathbf{M}_{ii}\mathbf{X}_{ii}) - \mathrm{tr}(\mathbf{N}_{ii})$$

The stationary point of $\mathrm{tr}(\mathbf{Y})$ with respect to $\mathbf{X}_{ii}$ is found by solving [14]

$$\frac{\partial[\mathrm{tr}(\mathbf{Y})]}{\partial\mathbf{X}_{ii}} = \left(\mathbf{M}_{ii} + \mathbf{M}_{ii}^T\right)\mathbf{X}_{ii} - \mathbf{N}_{ii} = \mathbf{0}$$

The result follows by considering the last expression for all i, $i = 1, \cdots, M$ together.

A. Simple interactor matrix

If the system has a simple interactor matrix, *i.e.* $\mathbf{D} = q^{-d}\mathbf{I}$, then $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}$, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$, where $\mathbf{G}_w = \mathbf{F} + q^{-d}\mathbf{R}$. Using Parseval's equality,

$$\|\mathbf{L}\|_2^2 = \sum_{i=0}^{\infty} \mathrm{tr}(\mathbf{L}_i^T \mathbf{L}_i) \tag{14}$$

where $\mathbf{L} = \mathbf{A} - \mathbf{B}\mathbf{K}$ as before and $\mathbf{L}_i$ is the i^{th} impulse response matrix of $\mathbf{L}$ defined as

$$\mathbf{L}_i = \mathbf{A}_i - \sum_{j=0}^{i}\sum_{k=0}^{i-j} \mathbf{B}_j \mathbf{K}_k \tag{15}$$

Then, the decentralized MV control law is obtained by finding the stationary point of $\|\mathbf{L}\|_2^2$ with respect to $\mathbf{K}_k$, $k = 1, 2 \ldots \infty$ subject to the structural constraint on the controller. For numerical reasons, however, it is necessary to approximate $\mathbf{A}, \mathbf{B}$ and $\mathbf{K}$ by finite impulse response models having order N. Using Lemma 2, the stationary point is found by solving,

$$\frac{\partial \|\mathbf{L}\|_2^2}{\partial \mathbf{K}_k} = \mathbf{J} \circ \left[\sum_{i=0}^{N-k} \mathbf{B}_i^T \mathbf{L}_{i+k} \right] = \mathbf{0} \tag{16}$$

To simplify notation in the further treatment, we define the following linear operator,

***Definition** 3:* Let $\mathbf{X}, \mathbf{Y}$ be defined such that $\dim(\mathbf{X}) = \dim(\mathbf{Y}_{ij})$ for all i, j. Then, the *block-wise Kronecker-Hadamard product* is defined as,

$$\mathbf{X} \oslash \mathbf{Y} = \begin{bmatrix} \mathbf{X} \circ \mathbf{Y}_{11} & \mathbf{X} \circ \mathbf{Y}_{12} & \cdots \\ \mathbf{X} \circ \mathbf{Y}_{21} & \mathbf{X} \circ \mathbf{Y}_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

A rearrangement of (16) gives,

$$\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right] \mathbf{K}_C = \mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{A}_C\right) \tag{17}$$

where $\mathbf{A}_C$ and $\mathbf{K}_C$ contain the impulse response matrices of $\mathbf{A}$ and $\mathbf{K}$ respectively, and $\mathbf{B}_H$ is a lower block triangular Hankel matrix. The $\mathbf{A}_C$, $\mathbf{K}_C$ and $\mathbf{B}_H$ are defined as

$$\begin{aligned} \mathbf{K}_C &= \begin{bmatrix} \mathbf{K}_0^T & \mathbf{K}_1^T & \mathbf{K}_2^T & \cdots \mathbf{K}_N^T \end{bmatrix}^T \\ \mathbf{A}_C &= \begin{bmatrix} \mathbf{A}_0^T & \mathbf{A}_1^T & \mathbf{A}_2^T & \cdots \mathbf{K}_N^T \end{bmatrix}^T \\ \mathbf{B}_H &= \begin{bmatrix} \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{B}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{B}_N & \mathbf{B}_{N-1} & \cdots & \cdots & \mathbf{B}_0 \end{bmatrix} \end{aligned} \tag{18}$$

When $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ is invertible, the suboptimal decentralized MV controller is given as,

$$\mathbf{K}_C = \left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]^{-1} \left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{A}_C\right)\right] \tag{19}$$

***Remark** 1:* Since $\mathbf{J}$ always has full rank, rank deficiency of $\mathbf{B}_H^T \mathbf{B}_H$ makes $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ singular. This happens when some of $\mathbf{B}_i$'s are singular. For a system with simple interactor matrix, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$ has no infinite zeros and thus $\mathbf{B}_i$ is nonsingular for all i.

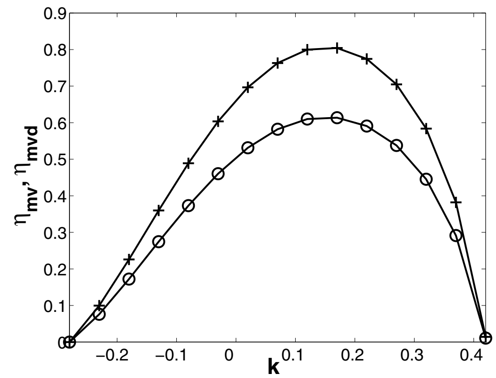

Fig. 3. Comparison of η_{mv} (o) and η_{mvd} (+) for Example 2. The controller structure limits the achievable output performance.

The earlier developments in this section are summarized by the following result:

***Proposition** 1:* Consider the system (3) with a simple interactor matrix. Define $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}, \mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$. Then, a suboptimal solution to finding a decentralized controller that minimizes $E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]$ is given by (19).

Let $\mathbf{y}_{mvd}$ be the output of the closed loop system under the optimal decentralized MV control law. Then, a decentralized performance index is defined as

$$\eta_{mvd} = \frac{\mathrm{E}[\mathrm{tr}(\mathbf{y}_{mvd}\mathbf{y}_{mvd}^T)]}{\mathrm{E}[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]} \tag{20}$$

The full multivariable performance index η_{mv} is defined similarly, where $\eta_{mv} \leq \eta_{mvd}$. Ideally, $0 \leq \eta_{mvd} \leq 1$, but when evaluated based on the suboptimal decentralized controller given by (19), η_{mvd} may exceed 1. In any case, a value of η_{mvd} close to zero always indicates poor performance.

In certain special cases, the decentralized controller given by (19) is optimal. For example, when $\mathbf{J} = \mathbf{1}_{nn}$, (19) reduces to $\mathbf{K}_C = \mathbf{B}_H^{-1}\mathbf{A}_C$. It is straightforward to verify that in this case, $\mathbf{K}_C$ corresponds to the optimal full multivariable MV control law of Huang and Shah [9]. Similarly, when $\mathbf{G}_w = \mathbf{I}$ or the system is affected by white noise, $\mathbf{F} = \mathbf{I}$ and $\mathbf{R} = \mathbf{0}$. Then $\mathbf{A} = \mathbf{0}$, which implies that $\mathbf{K}_C = \mathbf{0}$, which is optimal.

***Remark** 2:* When $\mathbf{F}$ commutes with $\mathbf{K}$, use of Lemma 1 to simplify (9) to (10) is not required. In this case, better estimates of η_{mvd} are obtained by redefining $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}\mathbf{F}$, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}\mathbf{F}$ and using Proposition 1 as before.

***Example** 2:* We revisit Example 1. The variation of η_{mv} and η_{mvd} with k is shown in Figure 3. For $k = 0.17$, $\eta_{mvd} \approx 0.82$, which is large compared to $\eta_{mv} \approx 0.6$. This justifies our earlier remark that the controller structure puts an inherent limitation on the achievable performance for this system and no significant performance improvement is possible by controller re-tuning.

For decentralized control, it is essential to choose the input-output pairings before the actual controller can be designed. For pairing selection based on achievable performance, Chen and McAvoy [15] simplified the resulting non-convex optimization problem by approximating the dynamic controller with a proportional controller. When the output performance is of primary interest, Proposition 1 can also be used for input-output pairing selection. For the system (1), the upper bound on achievable output performance for pairing on the diagonal and off-diagonal elements is 18.99 and 16.02 respectively. Based on this criterion, the latter alternative may be preferred.

B. General interactor matrix

When the system has a general interactor matrix, $\mathbf{B}$ is non-invertible due to presence of infinite zeros (see Remark 1) and some modifications are required. Let $\mathbf{D}_B$ be the unitary interaction matrix of $\mathbf{B}$ and $\tilde{\mathbf{B}} = \mathbf{D}_B\mathbf{B}$. Then

$$\begin{aligned}\|\mathbf{L}\|_2^2 &= \|\mathbf{A} - \mathbf{D}_B^{-1}\tilde{\mathbf{B}}\mathbf{K}\|_2^2 \\ &= \|\mathbf{D}_B\mathbf{A} - \tilde{\mathbf{B}}\mathbf{K}\|_2^2 = \|\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\|_2^2\end{aligned}$$

The suboptimal decentralized controller is obtained by following the same steps as before:

$$\mathbf{K}_C = \left[\mathbf{J} \oslash \left(\tilde{\mathbf{B}}_H^T\tilde{\mathbf{B}}_H\right)\right]^{-1}\left[\mathbf{J} \oslash \left(\tilde{\mathbf{B}}_H^T\tilde{\mathbf{A}}_C\right)\right] \qquad (21)$$

where $\tilde{\mathbf{A}}_C, \tilde{\mathbf{B}}_H$ are defined similar to (18). The earlier developments in this section are summarized by the following result:

***Proposition** 2:* Consider the system (3) with a general interactor matrix. Define $\tilde{\mathbf{A}} = \mathbf{D}_B\bar{\mathbf{R}}\mathbf{G}_w^{-1}$, $\tilde{\mathbf{B}} = \mathbf{D}_B\bar{\mathbf{F}}\bar{\mathbf{G}}_w^{-1}\tilde{\mathbf{G}}$, where $\mathbf{D}_B$ is the unitary interactor matrix of $\bar{\mathbf{F}}\bar{\mathbf{G}}_w^{-1}\tilde{\mathbf{G}}$. Then, a suboptimal solution to finding a decentralized controller that minimizes $E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]$ is given by (21).

In the previous example, controller structure posed significant limitations on the achievable performance. This is not always the case, as shown below:

***Example** 3:* Consider the following system adapted from Huang and Shah [9],

$$\mathbf{G} = \begin{bmatrix} \frac{q^{-1}}{1-0.4q^{-1}} & \frac{K_{12}q^{-2}}{1-0.1q^{-1}} \\ \frac{0.3q^{-1}}{1-0.1q^{-1}} & \frac{q^{-2}}{1-0.8q^{-1}} \end{bmatrix}$$

$$\mathbf{G}_w = \begin{bmatrix} \frac{1}{1-0.5q^{-1}} & \frac{-0.6}{1-0.5q^{-1}} \\ \frac{0.5}{1-0.5q^{-1}} & \frac{1}{1-0.5q^{-1}} \end{bmatrix}$$

where K_{12} controls the extent of interaction among the variables. The objective is to compare the performance of the following controller for different values of K_{12}.

$$\mathbf{K} = \begin{bmatrix} \frac{0.5-0.2q^{-1}}{1-0.5q^{-1}} & 0 \\ 0 & \frac{0.25-0.2q^{-1}}{(1-0.5q^{-1})(1+0.5q^{-1})} \end{bmatrix}$$

The η_{mvd}, η_{mv} for various K_{12} are shown in Figure 4. For each value of K_{12}, there exists a decentralized controller that closely matches the performance of the optimal full multivariable controller. Hence, the controller structure

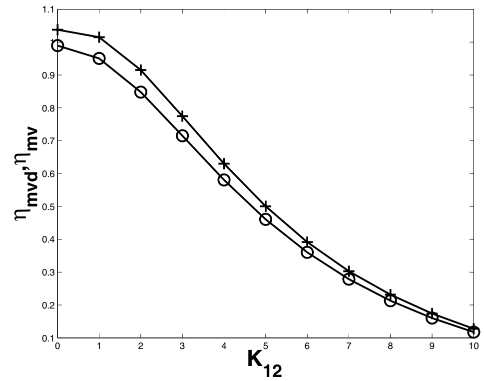

Fig. 4. Comparison of η_{mv} (o) and η_{mvd} (+) for Example 3. The controller structure poses no serious limitations.

poses no serious limitation on the achievable performance for this system. For a system with large interactions, when the individual loops of the decentralized controller are designed independently, the closed loop performance can deviate significantly from the design performance [16, Ch.10]. This example illustrates that large interactions, however, do not necessarily limit the achievable performance of decentralized controllers as compared to the full multivariable controllers.

IV. Achievable PID Performance

The suboptimal decentralized controller is expressed in terms of its impulse response matrices. By restricting the order of the controller or setting $\mathbf{K}_k = 0$ for all $k > p$, controllers with reduced complexity can be obtained. In this section, this approach is used to find an overestimate on achievable output variance using multi-loop PID controllers, which are expressed as,

$$\mathbf{K}_{\mathrm{PID}} = \frac{1}{\Delta}\sum_{i=0}^{2}\mathbf{C}_i q^{-i} = \frac{1}{\Delta}\mathbf{C}$$

where $\Delta = 1 - q^{-1}$ and $\mathbf{C}$ has the same block diagonal structure as the controller $\mathbf{K}$. By considering $1/\Delta$ as a part of $\tilde{\mathbf{G}}$ and minimizing $\|\mathbf{L}\|_2^2$ with respect to $\mathbf{C}$, an overestimate of the achievable PID performance can be derived. Then Propositions 1 and 2 can be used by limiting the column dimensions of $\mathbf{A_C}, \mathbf{B_H}$ to $3n$, where n is the dimension of the system $\mathbf{G}$. To ensure that the assumption of stability of $\mathbf{G}$ is satisfied, the integrator can be moved just inside the unit circle without affecting the result significantly. In general, controllers with reduced complexity having order p can be obtained by limiting the column dimensions of $\mathbf{A_C}$ and $\mathbf{B_H}$ to pn.

***Example** 4:* Consider the following system taken from Ko and Edgar [7],

$$y = \frac{q^{-6}}{1-0.8q^{-1}}u + \frac{1-0.2q^{-1}}{(1-0.3q^{-1})(1+0.4q^{-1})(1-0.5q^{-1})}a$$

Clearly the results presented earlier also hold for SISO systems. Based on these results, the achievable output variances under MV and PI control are 1.11 showing that the controller structure poses no limitations. However, when the disturbance model contains an additional integrator, the achievable output variances under MV and PI control are 11.95 and 17.86 respectively. The achievable performances differ by more than 50% revealing the effect of controller structure on achievable performance. Note that for both these cases, the achievable PI performance is close to the results obtained by Ko and Edgar [7], who used numerical search.

V. DISCUSSION

The results presented in this paper require that the system's model be fully known. This can be very demanding for online performance monitoring of industrial systems, especially in presence of changing operating conditions. We point out that even when the system's model is available, the exact solution to the general decentralized control problem is unknown. Sourlas and Manousiothakis [10] suggested that the optimal decentralized controller can possibly have infinite order. Under some minor assumptions, Blondel and Tsitsiklis [17] showed that the problem of system stabilization using fixed order decentralized controller is NP-hard. Thus, the estimation of exact or reasonably approximate decentralized MV benchmark, directly from data, is extremely difficult and possibly impossible.

The requirement of knowledge of the system's model can be partially relaxed by estimating $\mathbf{G}_w$ using regular operating data, as suggested by Ko and Edgar [7], [8]. Example 3 shows that the controller structure does not always limit the achievable performance. The identification of $\mathbf{G}$ should only be undertaken if large differences are seen between the actual output variance and MV benchmark for full multivariable controllers. Recently, Agrawal and Lakshminarayan [18] showed that by identifying the complementary sensitivity function from step test data, the MV benchmark for SISO systems under PID control can be calculated approximately using numerical search. The applicability of this promising approach to multi-loop control will be evaluated in future.

An apparent limitation of this work is that the suboptimal controller is expressed in terms of its impulse response matrices, whose determination is computationally inexpensive. Starting from a low value, the controller order can be gradually increased until convergence, but convergence can be extremely slow in some cases. This difficulty is overcome by recognizing that $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ is a sparse Toeplitz matrix and using available computationally efficient methods (*e.g.*, Brent *et al.* [19]) for its inversion.

The decentralized MV control law is based on an approximation of the closed loop expression and thus stability is not guaranteed. A possible approach to overcome this limitation is to reduce the gain of the decentralized controller until stability is achieved, however, such an approach increases the sub-optimality of the results.

VI. CONCLUSIONS

For performance assessment purposes, ignoring the controller structure can lead to incorrect conclusions regarding significant performance improvement through controller tuning. In this paper, we presented an approximate solution to the decentralized minimum variance control problem, which provides an overestimate of the achievable output variance without numerical search. The proposed method can easily handle the case of multi-loop PID controllers. The primary limitation of the proposed method is that complete knowledge of the system's model is required and some recommendations are provided to partially overcome this limitation.

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