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Observer Design for a Flexible Robot Arm with a Tip Load

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***Abstract*— In this paper, we consider the observer design for a flexible robot arm with a tip load. The robot arm is modeled as an Euler-Bernoulli beam. The beam is clamped to a motor at one end and attached to a force actuator at the other. Based on measurements at the boundaries, an exponentially stable observer is proposed. The existence, uniqueness and stability of solutions of the proposed observer are based on semigroup theory. Numerical simulation results are included to illustrate the performance of the proposed observer.**

I. Introduction

In recent years, the demand for high speed and low energy consuming systems has motivated the introduction of flexible parts in many mechanical system, e.g. robot arms with flexible links, spacecraft structures, etc. The problems of modeling and control for mechanical systems consisting of coupled absolutely rigid and elastic parts have thus become an important research area. The dynamics of this class of systems are often described by a combination of ordinary differential equations (ODEs), partial differential equations (PDEs), and a set of static boundary conditions. Due to the dynamic coupling between the rigid and flexible subsystems, the design of high-performance controllers for such systems is complicated.

The idea was first applied by Chen [4] to the systems described by wave equation (e.g. strings), and later extended to the Euler-Bernoulli beam equation and the Timoshenko beam equation by numerous authors, among others [1],[5],[6],[13],[14],[19],[20]. In particular, in [5], Chen et al. showed that a single actuator applied at the free end of the cantilever beam is sufficient to obtain uniformly stabilization of the deflection of the beam. In [14],[20] the orientation and stabilization of a beam attached to a rigid body were studied. Recently, Lynch and Wang [13] applied *flatness* in controller design for a hub-beam system with a tip payload. In [1], Aoustin et al. considered the motion planning and synthesis of a tracking controller of a flexible robot arm using *Mikusinski*'s operational calculus.

Observer design based on Lyapunov theory is well known and widely used for both linear systems and nonlinear systems. In [10],[18],[19] observer design for flexible-link robot described by ODEs is studied. Balas [2] considered observer design for linear flexible structures described by FEM. Demetriou [7], presented a method for construction of observer for linear second order lumped and distributed parameter systems using parameter-dependent Lyapunov functions. Kristiansen [9] applied *contraction* theory [11] in observer design for a class of linear distributed parameter systems. The damping forces were included in the last two cases. Thus, exponentially stable observers can easily be designed.

Here, as opposed to the work of [2],[10],[18],[19], observer design for a flexible-link robot is based on an *infinite-dimensional* model. Recently, the present authors [15] designed an exponentially stable observer for a motorized Euler-Bernoulli beam described by a combination of ODE, PDE and a set of static boundary conditions. The stability of the proposed observer was proven using *semigroup theory*; but in [15], the inertial forces of the load at the tip of the beam were omitted. Here, we remove this assumption and include the dynamics of the load at the tip in the observer design.

In this paper, we extend previous result on observer design for one-dimensional beam equation. The beam is clamped to a motor at one end, and attached to a force actuator at the other. Based on the measurements at the boundaries, an exponentially stable observer is designed. The existence, uniqueness and stability of solutions of the proposed observer are based on semigroup theory.

The paper is organized as follows. First, a model for a flexible robot arm is presented. Then, the observer design and analysis of the proposed observer are studied. Finally, some concluding remarks are given.

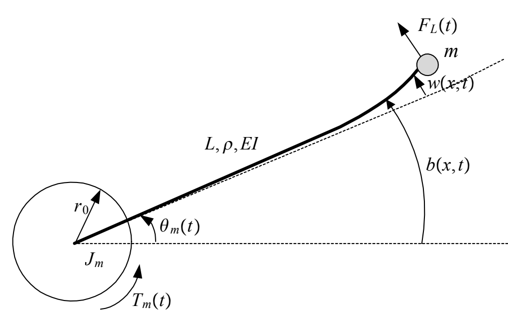

Fig. 1. Flexible robot arm with a tip load.

II. System Model

We consider a flexible beam clamped to a motor at one end and attached to a force actuator at the other (see Figure 1). The equations for the elastic motion of the system are given as ([8],[15]),

$$\rho b_{tt}(x,t) = -EIw_{xxxx}(x,t), \quad x \in]r_0, L[\quad (1)$$
$$mb_{tt}(L,t) = EIw_{xxx}(L,t) + F_L(t) \quad (2)$$
$$w(r_0,t) = w_x(r_0,t) = EIw_{xx}(L,t) = 0 \quad (3)$$

and the equation of motion for the hub is given by the angular momentum

$$\dot{h}(t) = T_m(t) \quad (4)$$

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where

$$b(x,t) = x\theta_m(t) + w(x,t), \quad x \in [r_0, L] \tag{5}$$

$$h(t) = J_m\dot{\theta}_m(t) + mLb_t(L,t) + \int_{r_0}^{L} \rho x b_t(x,t)\, dx \tag{6}$$

Here, $b(x,t)$ denotes the the arc length of the beam at point x and time t, $w(x,t)$ is the elastic displacement of the beam at x and time t, ρ is the mass per unit length of the beam, E is the modulus of elasticity of the beam, I is the area moment of inertia of the beam, r_0 is the clamping location of the beam, L is the length of the beam, m is the total mass of the load at the tip (the mass of the force actuator is included), θ_m is the angle of the motor, J_m is the mass moment of inertia of the motor, $T_m : \mathbb{R}^+ \to \mathbb{R}$ is the boundary control torque generates by the motor and $F_L : \mathbb{R}^+ \to \mathbb{R}$ is the boundary control force generated by the force actuator at the tip of the beam. The subscripts $(\cdot)_t$ and $(\cdot)_x$ denote the partial differential with respect to t and x, respectively. Throughout this paper, the time derivative is also often represented by a *dot*, e.g. $\dot{\theta}_m = d\theta_m/dt$.

Applying (1)-(3), (5)-(6) and integration by parts to (4), we get the equations of motion

$$\rho b_{tt} = -EIw_{xxxx}, \quad x \in]r_0, L[\tag{7}$$

$$mb_{tt} = EIw_{xxx} + F_L, \quad x = L \tag{8}$$

$$J_m\ddot{\theta}_m = -r_0 EI\, w_{xxx}|_{r_0} + EIw_{xx}|_{r_0} - LF_L + T_m \tag{9}$$

$$w|_{r_0} = w_x|_{r_0} = w_{xx}|_L = 0 \tag{10}$$

In this paper, we consider the following problem:

Problem: *Consider the system (7)-(10) with $m > 0$. Assume that the model (7)-(10) is perfectly known. Let the following measurements be available: $\theta_m(t)$, $w(L,t)$ and $EIw_{xxx}(L,t)$, $\forall t \geq 0$. Design an observer for this system.*

III. Observer Design

Let the measurements be denoted as follows: $y_1(t) = \theta_m(t)$, $y_2(t) = w(L,t)$ and $y_3(t) = EIw_{xxx}(L,t)$, $\forall t \geq 0$. Note that the measurement $y_3(t)$ represents the transverse internal force at the beam end. Utilizing the *coordinate error feedback* [11], we propose the observer

$$\dot{\bar{b}} = \dot{\hat{b}} - \frac{1}{m}\left[mh_d\left(Ly_1 + y_2\right) - h_p y_3\right] \cdot \delta_d\left(x - L\right), \; x \in]r_0, L] \tag{11}$$

$$\dot{\bar{\theta}} = \dot{\hat{\theta}} - \frac{H_d}{J_m} y_1 \tag{12}$$

$$\rho\ddot{\bar{b}} = -EI\hat{w}_{xxxx}, \quad x \in]r_0, L[\tag{13}$$

$$m\ddot{\bar{b}} = EI\hat{w}_{xxx} - mh_d\dot{\hat{b}} + h_p EI\dot{\hat{w}}_{xxx} + F_L\,, \quad x = L \tag{14}$$

$$J_m\ddot{\bar{\theta}} = -H_d\dot{\hat{\theta}} - H_p\left(\hat{\theta} - y_1\right) + EI\,\hat{w}_{xx}|_{r_0} - r_0 EI\,\hat{w}_{xxx}|_{r_0} - LF_L + T_m \tag{15}$$

$$\hat{w}|_{r_0} = \hat{w}_x|_{r_0} = \hat{w}_{xx}|_L = 0 \tag{16}$$

where h_p, h_d, H_p and H_d are positive observer gains; $\hat{b}$, $\hat{w}$ and $\hat{\theta}$ denote the estimates of b, w and θ_m, respectively, and $\delta_d(\cdot)$ is the *discrete* Dirac delta function, i.e. $\delta_d(0) = 1$, and $\delta_d(x) = 0$, $x \neq 0$. Note that the coordinate error feedback has similarities with *Luenberger's linear reduced-order observer* [12].

Applying (11)-(12) to (13)-(15) gives the observer dynamics

$$\rho\hat{b}_{tt} = -EI\hat{w}_{xxxx}, \quad x \in]r_0, L[\tag{17}$$

$$m\hat{b}_{tt} = EI\hat{w}_{xxx} - mh_d\tilde{b}_t + h_p EI\tilde{w}_{xxxt} + F_L\,, \quad x = L \tag{18}$$

$$J_m\ddot{\hat{\theta}} = -H_d\dot{\tilde{\theta}} - H_p\tilde{\theta} + EI\hat{w}_{xx}|_{r_0} - r_0 EI\hat{w}_{xxx}|_{r_0} - LF_L + T_m \tag{19}$$

$$\hat{w}|_{r_0} = \hat{w}_x|_{r_0} = \hat{w}_{xx}|_L = 0 \tag{20}$$

where $\tilde{b} = \hat{b} - b$, $\tilde{w} = \hat{w} - w$ and $\tilde{\theta} = \hat{\theta} - \theta_m$ denote the observer errors. Let

$$h_p = \frac{1}{h_d} \tag{21}$$

and define

$$\tilde{\nu}(t) = m\dot{\tilde{b}}\left(L,t\right) - h_p EI\tilde{w}_{xxx}\left(L,t\right) \tag{22}$$

Subtracting (17)-(20) by (7)-(10) gives the observer error dynamics

$$\rho\tilde{b}_{tt} = -EI\tilde{w}_{xxxx}\,, \quad x \in]r_0, L[\tag{23}$$

$$\dot{\tilde{\nu}} = -h_d\tilde{\nu} \tag{24}$$

$$J_m\ddot{\tilde{\theta}} = -H_d\dot{\tilde{\theta}} - H_p\tilde{\theta} - r_0\, EI\tilde{w}_{xxx}|_{r_0} + EI\tilde{w}_{xx}|_{r_0} \tag{25}$$

$$\tilde{w}|_{r_0} = \tilde{w}_x|_{r_0} = \tilde{w}_{xx}|_L = 0 \tag{26}$$

where $\tilde{\nu}$ is defined by (22).

Let $\mathbf{q} = (\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t, \tilde{\nu}) = (q_1, \ldots, q_5)$. The observer error dynamics (23)-(25) can thus be compactly written as

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}, \quad \mathbf{q}(0) \in H \tag{27}$$

where

$$\mathbf{A}\mathbf{q} = \begin{bmatrix} q_2 \\ -\frac{1}{J_m}(*) \\ q_4 \\ -\frac{EI}{\rho}q_{3,xxxx} + \frac{x}{J_m}(*) \\ -h_d q_5 \end{bmatrix}, \quad \forall \mathbf{q} \in D(\mathbf{A})$$

and

$$(*) = \left(H_d q_2 + H_p q_1 + r_0 EI\; q_{3,xxx}|_{r_0} - EI\; q_{3,xx}|_{r_0}\right)$$

Let $\Omega =]r_0, L[$. Define the spaces

$$H = \mathbb{R}^2 \times H_0^2(\Omega) \times L_2(\Omega) \times \mathbb{R}$$

$$D(\mathbf{A}) = \big\{\mathbf{q} \in \mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega) \times \mathbb{R} \big| \; q_{3,xx}|_L = 0 \; q_5 = m\left(Lq_2 + q_4\right) - h_p EI\; q_{3,xxx}|_L \big\}$$

where

$$L_2(\Omega) = \left\{ f \mid \int_\Omega |f(x)|^2 \, dx < \infty \right\}$$
$$H_0^k(\Omega) = \left\{ f \mid f, f', \ldots, f^{(k)} \in L_2(\Omega), \right. \left. f(r_0) = f'(r_0) = 0 \right\}$$

In H, we define the inner-product

$$\langle \mathbf{q}, \mathbf{z} \rangle_H = \int_\Omega \rho \left(x q_2 + q_4 \right) \left(x z_2 + z_4 \right) \, dx + J_m q_2 z_2 + \int_\Omega EI q_{3,xx} z_{3,xx} \, dx + H_p q_1 z_1 + \frac{1}{2m} q_5 z_5$$

where $\mathbf{q} = (q_1, \ldots, q_5) \in H$ and $\mathbf{z} = (z_1, \ldots, z_5) \in H$. The energy of (27) can be expressed as

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \langle \mathbf{q}, \mathbf{q} \rangle_H = \frac{1}{2} \|\mathbf{q}\|_H^2, \quad \forall \mathbf{q} \in H \\ &= \frac{1}{2} \int_\Omega \rho \tilde{b}_t^2 \, dx + \frac{1}{2} \int_\Omega EI \tilde{w}_{xx}^2 \, dx \\ &\quad + \frac{1}{2} H_p \tilde{\theta}^2 + \frac{1}{2} J_m \dot{\tilde{\theta}}^2 + \frac{1}{4m} \tilde{\nu}^2 \end{aligned} \tag{28}$$

It can be verified that $(H, \langle \cdot, \cdot \rangle_H)$ is a Hilbert space.

Assume that the control laws F_L and T_m are chosen such that the closed loop system of (7)-(10) is *well-posed*, i.e. the closed loop system has a unique solution, and the solution is bounded (see Remark 1). Hence, by showing the well-posedness of the observer error dynamics (23)-(26), we get the well-posedness of the observer (17)-(20).

Theorem 1: Consider the abstract problem (27). We have

I) the operator $\mathbf{A}$ generates a C_0-semigroup $\{e^{\mathbf{A}t}\}_{t \geq 0}$ of contractions on H,

II) the strong solution of (27) is exponentially stable for every $\mathbf{q}(0) \in D(\mathbf{A})$.

Proof: I) To prove this assertion, we apply the *Lumer-Phillips theorem* (see e.g. [16]). The time derivative of the energy function (28) along the solution trajectories of (27) is

$$\dot{\mathcal{E}}(t) = -H_d \dot{\tilde{\theta}}^2 - \frac{m h_d}{2} \tilde{b}_t(L)^2 - \frac{h_p}{2m} \left[EI \tilde{w}_{xxx}(L) \right]^2 \tag{29}$$

where integration by parts has successively been applied. Hence, the operator $\mathbf{A}$ is dissipative. To prove that the range of the operator $\lambda \mathbf{I} - \mathbf{A}$ is onto H for some $\lambda > 0$, we will first show that $\mathbf{A}$ is a compact operator. Let $\mathbf{g} = (g_1, \ldots, g_5) \in H$ be given. Consider the equation

$$\mathbf{A}\mathbf{q} = \mathbf{g} \tag{30}$$

i.e.

$$\begin{aligned} g_1 &= q_2 \\ -J_m g_2 &= H_d q_2 + H_p q_1 \\ &\quad + r_0 EI \, q_{3,xxx}|_{r_0} - EI \, q_{3,xx}|_{r_0} \\ g_3 &= q_4 \\ g_4 &= -\frac{EI}{\rho} q_{3,xxxx} - x g_2 \\ g_5 &= -h_d q_5 \end{aligned}$$

It can be verified that the unique solution of (30) is

$$q_1 = \frac{EI}{H_p} \left(-r_0 \, q_{3,xxx}|_{r_0} + q_{3,xx}|_{r_0} \right) - \frac{H_d}{H_p} g_1 - \frac{J_m}{H_p} g_2 \tag{31}$$
$$q_2 = g_1 \tag{32}$$
$$q_3(x) = -\frac{\rho}{EI} \int_{r_0}^{x} \int_{r_0}^{\xi_1} \int_{r_0}^{\xi_2} \int_{r_0}^{\xi_3} g_4(\xi_4) \, d\xi_4 d\xi_3 d\xi_2 d\xi_1 + \sum_{i=0}^{3} \mathsf{c}_i x^i - \frac{1}{5!} \frac{\rho g_2}{EI} x^5, \quad x \in [r_0, L] \tag{33}$$
$$q_4(x) = g_3(x), \quad x \in [r_0, L] \tag{34}$$
$$q_5 = -\frac{1}{h_d} g_5 \tag{35}$$

where $\mathsf{c}_0, \ldots, \mathsf{c}_3$ are uniquely determined by the boundary conditions (22), (26) and (35). Hence, the equation (30) has a unique $\mathbf{q} \in D(\mathbf{A})$. It follows that $\mathbf{A}^{-1}$ exists and maps H into $\mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega) \times \mathbb{R}$. Moreover, since $\mathbf{A}^{-1}$ maps every bounded set of H into bounded sets of $\mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega) \times \mathbb{R}$, and the embedding of the latter space into H is compact (see e.g. p. 14, [17]), it follows that $\mathbf{A}^{-1}$ is a compact operator. Consider now the equation,

$$(\lambda \mathbf{I} - \mathbf{A}) \mathbf{q} = \mathbf{A}(\lambda \mathbf{A}^{-1} - \mathbf{I}) \mathbf{q} = \mathbf{g} \tag{36}$$

for given $\lambda > 0$ and $\mathbf{g} \in H$. By *contraction mapping theorem*, it follows that (36) has a unique solution $\mathbf{q} \in D(\mathbf{A})$ for $0 < \lambda < ||\mathbf{A}^{-1}||^{-1}$. Hence, the operator $\lambda \mathbf{I} - \mathbf{A} : H \to H$ is onto for $0 < \lambda < ||\mathbf{A}^{-1}||^{-1}$. Furthermore, by (Th. 4.5, p. 15, [16]), it follows that $\lambda \mathbf{I} - \mathbf{A} : H \to H$ is onto for all $\lambda > 0$.

Since $(H, \langle \cdot, \cdot \rangle_H)$ is a Hilbert space, it follows from the argument above and (Th. 4.6, p. 16, [16]) that $D(\mathbf{A})$ is dense in H, i.e. $\overline{D(\mathbf{A})} = H$. Hence, $\mathbf{A}$ generates a C_0-semigroup $\left\{ e^{\mathbf{A}t} \right\}_{t \geq 0}$ of contractions on H.

II) To prove this assertion, we use a combination of the energy multiplier method and (Th. 4.1, p. 116, [16]). We define the following functional

$$\mathcal{V}(t) = 2(1 - \varepsilon) t \mathcal{E}(t) + \mathcal{U}(t) \tag{37}$$

where $\varepsilon \in]0, 1[$ is an arbitrary constant, $\mathcal{E}$ is given by (28), and

$$\mathcal{U}(t) = 2 \int_\Omega \rho x \tilde{b}_t \tilde{b}_x \, dx + 2 J_m \tilde{\theta} \dot{\tilde{\theta}} \tag{38}$$

In the sequel, the following inequalities are frequently used

$$ab \leq (\gamma a)^2 + \left(\frac{b}{\gamma} \right)^2, \quad \gamma \in \mathbb{R} \backslash \{0\} \tag{39}$$
$$(a + b)^2 \leq 2 \left(a^2 + b^2 \right) \tag{40}$$

for $\forall a, b \in \mathbb{R}$, and

$$f_x(x, t) \leq \left[L \int_{r_0}^{L} |f_{xx}|^2 \, dx \right]^{\frac{1}{2}}, \quad \forall x \in [r_0, L] \tag{41}$$

for $\forall f_x \in H_0^1(\Omega)$. The inequality (41) is a consequence of integration by parts and the boundary condition at $x = r_0$.

Applying the inequalities (39)-(41) to (38), it is straightforward to show that there exists a constant $C > 0$ such that the following holds

$$|\mathcal{U}(t)| \leq C\mathcal{E}(t), \quad \forall t \geq 0$$

One possible candidate is

$$C = 4\left(1 + L + \frac{2\rho L^2 + J_m}{H_p} + \frac{2\rho L^3}{EI}\right)$$

Hence, the following holds

$$[2(1-\varepsilon)t - C]\,\mathcal{E}(t) \leq \mathcal{V}(t) \leq [2(1-\varepsilon)t + C]\,\mathcal{E}(t) \tag{42}$$

for $t \geq 0$.

Next, differentiating (37) with respect to time along the solution trajectories of (27) gives

$$\frac{d}{dt}\mathcal{V}(t) = 2(1-\varepsilon)\mathcal{E}(t) + 2(1-\varepsilon)t\frac{d}{dt}\mathcal{E}(t) + \frac{d}{dt}\mathcal{U}(t)$$

where $\mathcal{E}$ and $d\mathcal{E}/dt$ are given by (28) and (29), respectively, and

$$\frac{d}{dt}\mathcal{U}(t) = \dot{\mathcal{U}}_1 + \dot{\mathcal{U}}_2 + \dot{\mathcal{U}}_3 + \dot{\mathcal{U}}_4$$

where

$$\begin{aligned}
\dot{\mathcal{U}}_1 &= 2\int_\Omega x\rho\tilde{b}_{tt}\tilde{b}_x\ dx \\
\dot{\mathcal{U}}_2 &= 2\int_\Omega \rho x\tilde{b}_t\tilde{b}_{xt}\ dx \\
\dot{\mathcal{U}}_3 &= 2J_m\ddot{\tilde{\theta}}\tilde{\theta} \\
\dot{\mathcal{U}}_4 &= 2J_m\dot{\tilde{\theta}}^2
\end{aligned}$$

Let us consider these terms separately.

$\underline{\dot{\mathcal{U}}_1 :}$

$$\begin{aligned}
\dot{\mathcal{U}}_1 &= -2\int_{r_0}^{L} x\tilde{b}_x EI\tilde{w}_{xxxx}\ dx \\
&= -2\left[x\tilde{b}_x EI\tilde{w}_{xxx}\right]_{x=r_0}^{L} + 2\left[\tilde{b}_x EI\tilde{w}_{xx}\right]_{x=r_0}^{L} \\
&\quad + \left[xEI\tilde{w}_{xx}^2\right]_{x=r_0}^{L} - 3\int_{r_0}^{L} EI\tilde{w}_{xx}^2\ dx \\
&= -2L\tilde{w}_x(L)\,EI\tilde{w}_{xxx}(L) - 2L\ \tilde{\theta}\ EI\tilde{w}_{xxx}(L) \\
&\quad +2EI\ \tilde{\theta}\ [r_0\tilde{w}_{xxx}(r_0) - \tilde{w}_{xx}(r_0)] \\
&\quad -r_0EI\tilde{w}_{xx}(r_0)^2 - 3\int_{r_0}^{L} EI\tilde{w}_{xx}^2\ dx
\end{aligned}$$

where integration by parts has successively been applied. Applying (39) gives

$$\begin{aligned}
\dot{\mathcal{U}}_1 &\leq 2L\left[(\gamma_1\ \tilde{w}_x|_L)^2 + \left(\frac{EI\ \tilde{w}_{xxx}|_L}{\gamma_1}\right)^2\right] \\
&\quad +2L\left[\left(\gamma_2\tilde{\theta}\right)^2 + \left(\frac{EI\ \tilde{w}_{xxx}|_L}{\gamma_2}\right)^2\right] \\
&\quad +2EI\ \tilde{\theta}\ [r_0\tilde{w}_{xxx}(r_0) - \tilde{w}_{xx}(r_0)] \\
&\quad -r_0EI\tilde{w}_{xx}(r_0)^2 - 3\int_{r_0}^{L} EI\tilde{w}_{xx}^2\ dx
\end{aligned}$$

for some $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$. By (41) we get

$$\begin{aligned}
\dot{\mathcal{U}}_1 &\leq 2L^2\gamma_1^2\int_{r_0}^{L}\tilde{w}_{xx}^2\ dx + 2L\ \left(\gamma_2\tilde{\theta}\right)^2 \\
&\quad + \left(\frac{2L}{\gamma_1^2} + \frac{2L}{\gamma_2^2}\right)\ [EI\tilde{w}_{xxx}(L)]^2 \\
&\quad +2EI\ \tilde{\theta}\ [r_0\tilde{w}_{xxx}(r_0) - \tilde{w}_{xx}(r_0)] \\
&\quad -r_0EI\tilde{w}_{xx}(r_0)^2 - 3\int_{r_0}^{L} EI\tilde{w}_{xx}^2\ dx
\end{aligned}$$

$\underline{\dot{\mathcal{U}}_2 :}$

$$\dot{\mathcal{U}}_2 = \rho L\tilde{b}_t(L)^2 - \rho r_0\tilde{b}_t(r_0)^2 - \int_{r_0}^{L}\rho\tilde{b}_t^2\ dx$$

$\underline{\dot{\mathcal{U}}_3 :}$

$$\begin{aligned}
\dot{\mathcal{U}}_3 &\leq 2EI\ \tilde{\theta}\ [-r_0\tilde{w}_{xxx}(r_0) + \tilde{w}_{xx}(r_0)] - 2H_p\tilde{\theta}^2 \\
&\quad +2H_d\ \left[\left(\gamma_3\tilde{\theta}\right)^2 + \left(\frac{\dot{\tilde{\theta}}}{\gamma_3}\right)^2\right], \forall\gamma_3 \in \mathbb{R}\backslash\{0\}
\end{aligned}$$

where (39) has been applied.

$\underline{\mathcal{E} :}$

$$\begin{aligned}
2\mathcal{E} &\leq \int_{r_0}^{L}\left(\rho\tilde{b}_t^2 + EI\tilde{w}_{xx}^2\right)\ dx + H_p\tilde{\theta}^2 + J_m\dot{\tilde{\theta}}^2 \\
&\quad +\frac{1}{m}\left(\left[m\tilde{b}_t(L)\right]^2 + [h_pEI\tilde{w}_{xxx}(L)]^2\right)
\end{aligned}$$

where (40) has been applied.

Hence, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) &\leq -\left[2 + \varepsilon - \frac{2L^2}{EI}\gamma_1^2\right]\int_{r_0}^{L} EI\tilde{w}_{xx}^2\ dx \\
&\quad -\left[(1+\varepsilon)H_p - 2L\gamma_2^2 - 2H_d\gamma_3^2\right]\ \tilde{\theta}^2 \\
&\quad -\left[2(1-\varepsilon)H_dt - (3-\varepsilon)J_m - \frac{2H_d}{\gamma_3^2}\right]\ \dot{\tilde{\theta}}^2 \\
&\quad -\left[(1-\varepsilon)mh_dt - (1-\varepsilon)m - \rho L\right]\ \tilde{b}_t(L)^2 \\
&\quad -\left[(1-\varepsilon)\frac{h_p}{m}t\right. \\
&\qquad \left. -\frac{(1-\varepsilon)h_p^2}{m} - \frac{2L}{\gamma_1^2} - \frac{2L}{\gamma_2^2}\right](EI\tilde{w}_{xxx}|_L)^2 \\
&\quad -r_0EI\tilde{w}_{xx}(r_0)^2 - \rho r_0\tilde{b}_t(r_0)^2 - \varepsilon\int_{r_0}^{L}\rho\tilde{b}_t^2 dx
\end{aligned}$$

Let $\varepsilon \in\ ]0,1[$ be fixed. By choosing $\gamma_1, \gamma_2, \gamma_2 \in \mathbb{R}\backslash\{0\}$ sufficiently small, the first two terms of $d\mathcal{V}/dt$ become negative for $\forall t \geq 0$. Hence, the following inequality holds

$$\frac{d}{dt}\mathcal{V} \leq 0, \quad t \geq t_1 \tag{43}$$

for sufficiently large time t_1,

$$t_1 = \max\left\{\frac{3J_m + \frac{2H_d}{\gamma_3^2}}{2(1-\varepsilon)H_d}, \frac{m + \rho L}{(1-\varepsilon)mh_d}, \frac{h_p^2 + \frac{2mL}{\gamma_1^2} + \frac{2mL}{\gamma_2^2}}{(1-\varepsilon)h_d}\right\}$$

Combination of (42) and (43) gives

$$\mathcal{E}(t) \leq \frac{\mathcal{V}(t)}{2(1-\varepsilon)t - C}, \quad t \geq t_{\max}$$

where

$$t_{\max} = \max\left\{t_1, \frac{C}{2(1-\varepsilon)}\right\}$$

From (29), it follows that $\mathcal{E}(t) \leq \mathcal{E}(0), \forall t \geq 0$. By (42) we get

$$\mathcal{V}(t) \leq C\mathcal{E}(0), \quad \forall t \geq 0$$

i.e.

$$\mathcal{E}(t) \leq \frac{C}{2(1-\varepsilon)t - C}\mathcal{E}(0), \quad t \geq t_{\max}$$

Since $\mathcal{E}(t) = \frac{1}{2}||\mathbf{q}(t)||_H^2$, it follows that $||\mathbf{q}(t)||_H < \infty$ and decays as $O\left(1/\sqrt{t}\right)$ for sufficiently large time. Hence,

$$\int_0^\infty \|\mathbf{q}(t)\|_H^p \ dt = \int_0^\infty \left\|e^{\mathbf{A}t}\mathbf{q}(0)\right\|_H^p \ dt < \infty$$

for $\forall p > 2$ and $\forall \mathbf{q}(0) \in D(\mathbf{A})$. By density of $D(\mathbf{A})$ in H, the following also holds

$$\int_0^\infty \|\mathbf{q}(t)\|_H^p \ dt < \infty \ , \quad \forall \mathbf{q}(0) \in H$$

for $\forall p > 2$. According to (Th. 4.1, p. 116, [16]), there exist constants $M \geq 1$ and $\mu > 0$ such that

$$\|\mathbf{q}(t)\|_H \leq Me^{-\mu t}\|\mathbf{q}(0)\|_H, \quad \forall \mathbf{q}(0) \in H$$

for $t \geq 0$. Hence, the observer error tends exponentially to zero. ∎

Remark 1: Assume that $\dot{\theta}(t)$ and $w_t(L,t)$ are measurable. The total energy of (7)-(10) can be expressed as

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}\int_\Omega \rho b_t^2 \ dx \\ &+ \frac{1}{2}m\left(b_t|_L\right)^2 + \frac{1}{2}\int_\Omega EIw_{xx}^2 \ dx\end{aligned}$$

The time derivative of $\mathcal{E}$ along the solution trajectories of (7)-(10) is

$$\frac{d}{dt}\mathcal{E}(t) = w_t(L,t)F_L(t) + \dot{\theta}_m(t)T_m(t)$$

This shows that the system (7)-(10) is passive. The system (7)-(10) can thus be stabilized by any passive controllers, e.g. PD-controllers. Note also that the control laws do not influence the convergence of the observer. This means that the observer error tends exponentially to zero for any stabilizing control laws.

IV. SIMULATION

To stabilize the system (7)-(10), we apply the following control laws

$$F_L(t) = -k_d w_t(L,t) \quad (44)$$

$$T_m(t) = J_m\ddot{\theta}_d + K_d(\dot{\theta}_d - \dot{\theta}_m) + K_p(\theta_d - \theta_m) \quad (45)$$

where k_d, K_p, K_d are positive control gains, and θ_d, $\dot{\theta}_d$, $\ddot{\theta}_d$ are given desired reference trajectories. To simulate the system (7)-(10) with the feedback control laws (44)-(45) and the proposed observer (11)-(16), the *finite-element method* with *Hermitian* basis functions has been applied (see e.g. [3],[8]). The beam was divided into 5 elements.

The system parameters used in the simulations are: $L = 1$ [m], $\rho = 2.43$ [kg/m], $E = 70 \times 10^9$ [N/m^2], $I = 6.75 \times 10^{-8}$ [m^4], $r_0 = 0.1$ [m], $J_m = 0.5$ [kgm^2], $m = 0.25$ [kg]. The controller gains and the observer gains used in simulations are: $K_p = 100$, $K_d = 100$, $k_d = 10$, $H_p = 40$, $H_d = 40$, $h_d = 10$. The initial conditions for the plant and the observer are: $\theta_m(0) = -15^0$, $\hat{\theta}(0) = 15^0$, $\dot{\hat{\theta}}(0) = \dot{\theta}_m(0) = 0$, and $w(x,0) = w_t(x,0) = \hat{w}(x,0) = \hat{w}_t(x,0) = 0$, $x \in [r_0, L]$. We turned on the observer at time $t = 2$ seconds.

The reference trajectory θ_d, the angle of the motor θ_m and the estimate of θ_m are shown in Figure 2. The velocity of the motor $\dot{\theta}_m$ and its observed value $\dot{\hat{\theta}}$ are shown in Figure 3. The 2-norm of the observer error vector $(\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t)$ is shown in Figure 4. We observe that the vibrations are damped out quickly, and the observer values converge exponentially to the plant, as expected.

The simulation results for a sine reference signal, with amplitude 1 and frequency 1 [rad/sec] are shown in Figures 5-7. Again, the observer converges exponentially to the plant.

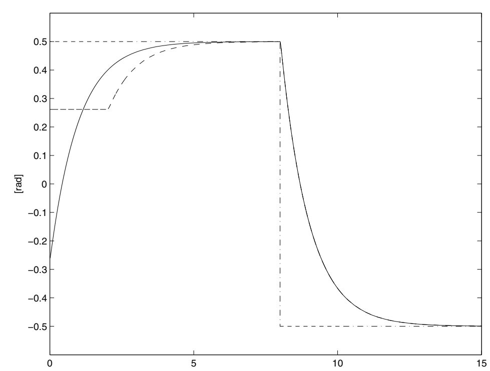

Fig. 2. θ_m [—], $\hat{\theta}$ [- -] and θ_d [· - ·].

V. CONCLUSIONS

In this paper, we considered the observer design for a flexible robot arm. The system is modeled as a motorized Euler-Bernoulli beam with a tip load. Based on measurements at the boundaries, an exponentially stable observer is proposed. The existence, uniqueness and stability of solutions of the proposed observer are based on semigroup theory. The simulation results are in agreement with the theoretical results.

VI. ACKNOWLEDGEMENTS

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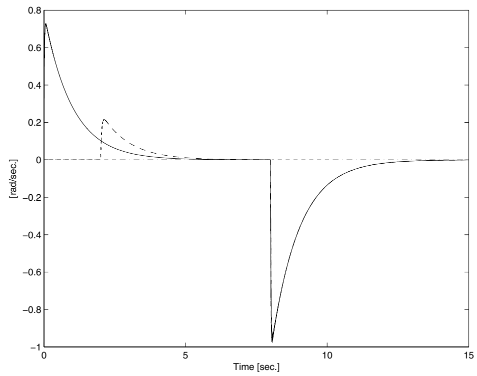

Fig. 3. $\dot{\theta}_m$ [—], $\dot{\hat{\theta}}$ [- -] and $\dot{\theta}_d$ [· - ·].

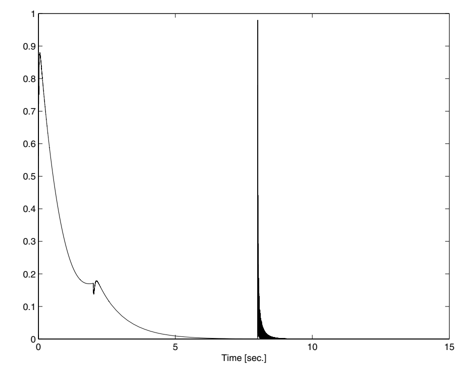

Fig. 4. 2-norm of the observer error vector $(\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t)$.

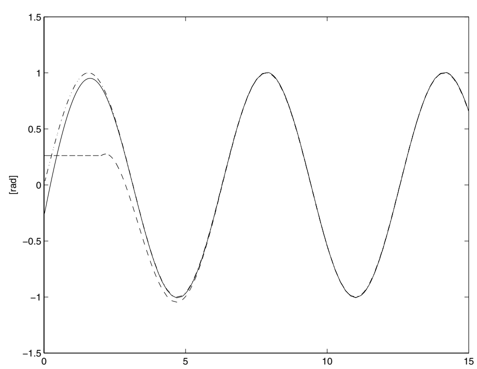

Fig. 5. θ_m [—], $\hat{\theta}$ [- -] and θ_d [· - ·].

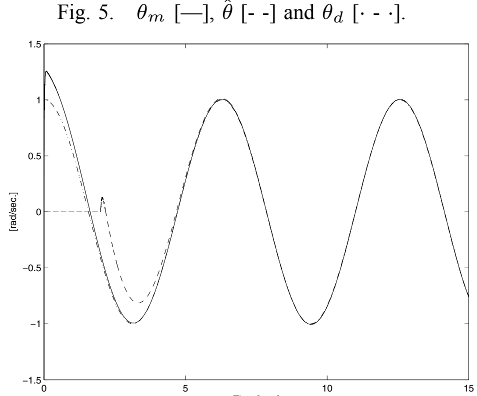

Fig. 6. $\dot{\theta}_m$ [—], $\dot{\hat{\theta}}$ [- -] and $\dot{\theta}_d$ [· - ·].

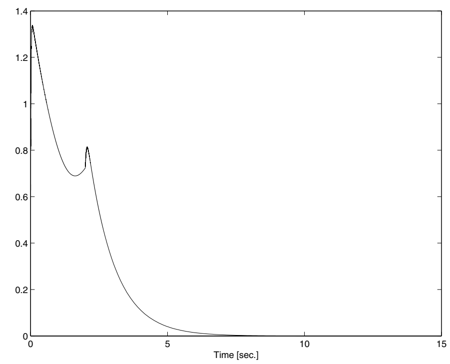

Fig. 7. 2-norm of the observer error vector $(\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t)$.

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