Zero Optimized Tracking for Linear Continuous-Time Systems

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Abstract— **Optimal additional zero locations of continuous-time systems with p free zeros, some fixed zeros and repeated, fixed poles tracking a reference impulse and step response are derived in this paper based on general closed-form impulse and step responses.**

Keywords: Linear Continuous–Time Systems, Zero Optimizing Controllers, Tracking.

I. Introduction

Closed-form transfer function responses for continuous-time systems are of considerable interest in the area of control systems and in filter design. Closed-form transfer function responses have been derived in [4] and [7], opening up many new interesting applications, e.g., solving for optimal zero locations by minimizing transient responses [4]; minimizing the step response deviation from given reference step responses [6] and [8]; and approaching the model reduction problem [5] by minimizing the difference in the impulse response of the original and the reduced–order model, keeping a subset of the original eigenvalues and a desired relative degree.

It is well known, that continuous–time transfer function responses are strongly affected, not only by the eigenvalues or poles, but the numerator coefficients, or equivalently, the system's zeros, as well. In general, the zeros of a continuous–time system are determined by properties of the plant as well as the location of sensors and actuators. Much interest has been shown in the general shaping of system responses, e.g., in the extrema–free problem. The influence of zero locations on the number of extrema in the step response of systems possessing real poles and real zeros was discussed in [3], where lower and upper bounds on the number of extrema are given. The necessary and sufficient conditions for a third–order transfer function such that the non–overshooting and the monotone nondecreasing step responses are ensured, were presented in [14]. While most of these results lend themselves to analysis as opposed to synthesis, sufficient conditions that can be used for the synthesis of compensators for realizing non–overshooting and monotonically increasing responses for minimum phase SISO systems were presented in [13].

Pole placement has been much discussed in the literature and methods for optimal pole placement using standard state feedback, e.g., the linear quadratic regulator, are well known. Zero placement is also a very relevant design issue, as evident for example in publications on zero placement of linear multivariable systems [2], [15], [16] and [17]. Controllers that affect zeros can be designed, although zeros are not affected by state feedback in SISO (Single Input Single Output) systems. One example of such a controller is the well known PID controller, [10] and [11]. In a similar manner, stable zeros can be affected by simple inverse compensation. Further, such a controller using dynamic output feedback and dynamic feedforward, can be designed to place the poles as well as to move the (stable) zeros of a system effectively by cancellation, see, e.g., [1].

In this paper, the work in [8] and [12] is reformulated and extended to include the case of repeated eigenvalues. Linear continuous–time system responses are summarized in Section II, [9] and [18]. Optimal zero locations tracking a reference transient response are derived in Section III. Examples of an open-loop tracking-controller are given in Section IV.

II. Linear Continuous–Time System Responses

Consider the standard transfer function given by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{(s+\lambda_1)^{d_1}(s+\lambda_2)^{d_2}\cdots(s+\lambda_\nu)^{d_\nu}}. \quad (1)$$

It is assumed that the system's eigenvalues $-\lambda_1$, $-\lambda_2$, $\ldots$, $-\lambda_\nu$ are distinct and repeated d_1, d_2, $\ldots$, d_ν times, respectively, and furthermore it is assumed that the system is causal, i.e., $m < n$. The impulse responses are of the generic form [9], [18]

$$y_I(t) = BH\mathcal{E}(t), \qquad t > 0 \quad (2)$$

where

$$B = \begin{bmatrix} b_m & b_{m-1} & \cdots & b_0 \end{bmatrix} \quad (3)$$

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contains the numerator coefficients and

$$H = \begin{bmatrix} h_{01} & h_{02} & \cdots & h_{0\nu} \\ h_{11} & h_{12} & \cdots & h_{1\nu} \\ \vdots & \vdots & & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{m\nu} \end{bmatrix} \tag{4}$$

is an $(m+1) \times n$ matrix. The first line in (4) is given by

$$H_0 = \begin{bmatrix} h_{01} & h_{02} & \cdots & h_{0\nu} \end{bmatrix}, \tag{5}$$

where each

$$h_{0i} = \begin{bmatrix} \kappa_{i1} & \kappa_{i2} & \cdots & \kappa_{id_i} \end{bmatrix} \tag{6}$$

contains the partial fraction coefficients of a unity numerator Laplace transform given by

$$\begin{aligned} Y_b(s) &= \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n} \\ &= \frac{1}{(s+\lambda_1)^{d_1}(s+\lambda_2)^{d_2}\cdots(s+\lambda_\nu)^{d_\nu}}. \end{aligned} \tag{7}$$

The unity numerator partial fraction coefficients are easily computed by

$$\kappa_{id_i} = \prod_{q=1, q\neq i}^{\nu} (-\lambda_i + \lambda_q)^{-d_q} \tag{8}$$

and for $j = 1, 2, 3, \ldots, d_i - 1$

$$\kappa_{ij} = \frac{1}{d_i - j} \sum_{q=1}^{d_i - j} \kappa_{i(j+q)} (-1)^q \sum_{p=1, p\neq i}^{\nu} \frac{d_p}{(-\lambda_i + \lambda_p)^q}, \tag{9}$$

Then,

$$h_{ki} = h_{(k-1)i} W_i, \qquad k = 1, 2, \ldots, m, \tag{10}$$

where

$$W_i = \begin{bmatrix} -\lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & -\lambda_i & \ddots & & \vdots \\ 0 & 1 & -\lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\lambda_i \end{bmatrix} \tag{11}$$

is a $d_i \times d_i$ matrix. Finally, all time dependent factors are contained in the $n \times 1$ vector, effectively forming an independent time basis vector

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \\ \vdots \\ \mathcal{E}_\nu(t) \end{bmatrix} \tag{12}$$

where

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{-\lambda_i t} \\ t e^{-\lambda_i t} \\ \vdots \\ \frac{t^{(d_i - 1)}}{(d_i - 1)!} e^{-\lambda_i t} \end{bmatrix}. \tag{13}$$

It should be emphasized that (2) is a general closed–form solution for linear continuous–time system responses corresponding to a general transfer function of the form (1). There are no restrictions, the eigenvalues can be real and/or complex, repeated and/or not and stable and/or unstable. It should also be noted that

$$\mu = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1d_1} & \cdots & \mu_{\nu 1} & \cdots & \mu_{\nu d_\nu} \end{bmatrix} = BH \tag{14}$$

is a new easily computable recursive form of partial fraction expansion coefficients for the general transfer function of the form (1), given by the well known expression

$$\mu_{ij} = \frac{1}{(d_i - j)!} \frac{d^{d_i - j}}{ds^{d_i - j}} \left[(s + \lambda_i)^{d_i} \frac{Y(s)}{U(s)} \right] \Bigg|_{s = -\lambda_i}. \tag{15}$$

Further, the new form (2) reduces to an earlier form in the case of nonrepeated eigenvalues published in [4] and is given by

$$y_I(t) = \tilde{\mathcal{B}} \tilde{\Lambda} \tilde{\mathcal{E}}(t), \quad t > 0 \tag{16}$$

where

$$\tilde{\mathcal{B}} = [b_m \quad -b_{m-1} \quad b_{m-2} \quad \cdots \quad (-1)^m b_0],$$

$$\tilde{\Lambda} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_\nu \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_\nu^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_\nu^m \end{bmatrix} \text{ and}$$

$$\tilde{\mathcal{E}}(t) = \begin{bmatrix} \frac{e^{-\lambda_1 t}}{\prod_{i=2}^{\nu}(-\lambda_1 + \lambda_i)} \\ \frac{e^{-\lambda_2 t}}{\prod_{i=1, i\neq 2}^{\nu}(-\lambda_2 + \lambda_i)} \\ \vdots \\ \frac{e^{-\lambda_k t}}{\prod_{i=1, i\neq k}^{\nu}(-\lambda_k + \lambda_i)} \\ \vdots \\ \frac{e^{-\lambda_\nu t}}{\prod_{i=1}^{\nu-1}(-\lambda_\nu + \lambda_i)} \end{bmatrix}.$$

We can calculate the step response, $y_S(t)$ at time $t \geq 0$, by simply integrating the impulse response using $y_S(0^-) = 0$ and

$$\begin{aligned} \int_0^t \frac{t'^p}{p!} e^{-\lambda_i t'} dt' &= -\frac{1}{\lambda_i^{p+1}} e^{-\lambda_i t'} - \frac{1}{\lambda_i^p} t' e^{-\lambda_i t'} \\ &\quad - \cdots - \frac{1}{\lambda_i^2} \frac{t'^{p-1}}{(p-1)!} e^{-\lambda_i t'} - \frac{1}{\lambda_i} \frac{t'^p}{p!} e^{-\lambda_i t'} \Bigg|_0^t. \end{aligned}$$

Then,

$$\int_0^t \mathcal{E}_i(t') dt' = V_i \mathcal{E}_i(t') \big|_0^t,$$

where

$$V_i = \begin{bmatrix} -\frac{1}{\lambda_i} & 0 & \cdots & 0 \\ -\frac{1}{\lambda_i^2} & -\frac{1}{\lambda_i} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -\frac{1}{\lambda_i^{d_i}} & -\frac{1}{\lambda_i^{d_i - 1}} & \cdots & -\frac{1}{\lambda_i} \end{bmatrix}. \tag{17}$$

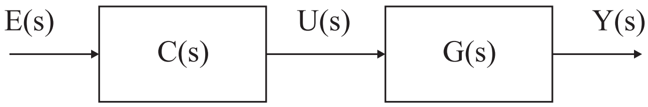

Fig. 1. The open-loop system.

Then the step response is

$$\begin{aligned} y_S(t) &= \int_0^t BH\mathcal{E}(t')dt' \\ &= BHV\left(\mathcal{E}(t) - \mathcal{E}(0)\right) \\ &= BHV\mathcal{E}(t) + \frac{b_m}{a_n} \end{aligned} \tag{18}$$

where

$$V = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_\nu \end{bmatrix}. \tag{19}$$

Next we calculate the ramp response, by integrating the step response

$$\begin{aligned} y_R(t) &= \int_0^t BHV\mathcal{E}(t') + \frac{b_m}{a_n}dt' \\ &= BHV^2\left(\mathcal{E}(t) - \mathcal{E}(0)\right) + \frac{b_m}{a_n}t, \end{aligned} \tag{20}$$

etc. for higher order responses.

III. Optimal Additional Zero Locations Tracking a Reference Impulse response

Consider the open-loop transfer function given in Eq. (1) and let $G(s) = \frac{Y(s)}{U(s)}$. We would like to design a controller of the form

$$C(s) = c_0 s^p + c_1 s^{p-1} + \cdots + c_p, \tag{21}$$

see Fig. 1. It should be noted that this controller is by nature noncausal, as is in fact the well know PID controller. In a closed-loop setup, the controller can be realized using poles to limit the high frequency response, as is often done in a practical setup of a PID controller.

The open-loop transfer function of the controlled system has the form

$$\frac{Y(s)}{E(s)} = C(s)G(s) \tag{22}$$

$$= \frac{(c_0 s^p + c_1 s^{p-1} + \cdots + c_p)(b_0 s^m + b_1 s^{m-1} + \cdots + b_m)}{s^n + a_1 s^{n-1} + \cdots + a_n}.$$

Now defining C as

$$C = [c_p \;\; c_{p-1} \;\; c_{p-2} \;\; \cdots \;\; c_0] \tag{23}$$

and $\mathcal{B}$ as a $(p+1) \times (p+m+1)$ convolution matrix given by, [10] and [12]

$$\mathcal{B} = \begin{bmatrix} b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_0 & 0 \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_m & b_{m-1} & \cdots & b_0 \end{bmatrix}. \tag{24}$$

Then the numerator coefficients of the controlled system can be written as the product $C\mathcal{B}$ and the transfer function can be written as

$$\frac{Y(s)}{E(s)} = \frac{C\mathcal{B} \times \begin{bmatrix} 1 & s & \cdots & s^{p+m-1} & s^{p+m} \end{bmatrix}^T}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}. \tag{25}$$

Thus, the impulse response for the open-loop controlled system is given by

$$y_{Ic}(t) = C\mathcal{B}H\mathcal{E}(t). \tag{26}$$

where

$$H = \begin{bmatrix} h_{01} & h_{02} & \cdots & h_{0\nu} \\ h_{11} & h_{12} & \cdots & h_{1\nu} \\ \vdots & \vdots & & \vdots \\ h_{(m+p)1} & h_{(m+p)2} & \cdots & h_{(m+p)\nu} \end{bmatrix}. \tag{27}$$

Now consider the design of the controller $C(s)$ such that the system tracks a desired reference impulse response. We can do this by minimizing the integral of the square error between the reference impulse response and the impulse response of the controlled system.

Assume that the reference open-loop transfer function is given by

$$\begin{aligned} \frac{Y_r(s)}{U_r(s)} &= \frac{b_{r0}s^{m_r} + b_{r1}s^{m_r-1} + \cdots + b_{rm_r}}{s^{n_r} + a_{r1}s^{n_r-1} + \cdots + a_{rn_r}} \\ &= \frac{b_{r0}s^{m_r} + b_{r1}s^{m_r-1} + \cdots + b_{rm_r}}{(s+\lambda_{r1})^{d_{r1}}(s+\lambda_{r2})^{d_{r2}}\cdots(s+\lambda_{r\nu_r})^{d_{r\nu_r}}} \end{aligned} \tag{28}$$

with the eigenvalues $-\lambda_{r1}, \ldots, -\lambda_{r\nu_r}$ repeated $d_{r1}, \ldots, d_{r\nu_r}$ times, respectively. We can write the impulse response of this reference system similar to Eq. (2), i.e.,

$$y_{Ir}(t) = B_r H_r \mathcal{E}_r(t). \tag{29}$$

We then define a cost function measuring the controlled impulse response deviation from the reference impulse response as

$$\begin{aligned} J &= \int_0^\infty \left(y_{Ir}(t) - y_{Ic}(t)\right)^2 dt \\ &= \int_0^\infty \left(B_r H_r \mathcal{E}_r(t) - C\mathcal{B}H\mathcal{E}(t)\right)^2 dt. \end{aligned} \tag{30}$$

Differentiating the cost function with respect to C and setting the result equal to zero gives

$$\begin{aligned}\frac{\partial J}{\partial C} &= \int_0^\infty \frac{\partial}{\partial C}\left(B_r H_r \mathcal{E}_r(t) - C\mathcal{B}H\mathcal{E}(t)\right)^2 dt \\ &= \int_0^\infty \frac{\partial}{\partial C}\Big(\left(B_r H_r \mathcal{E}_r(t)\right)^2 \\ &\quad -2B_r H_r \mathcal{E}_r(t) C\mathcal{B}H\mathcal{E}(t) + \left(C\mathcal{B}H\mathcal{E}(t)\right)^2\Big) dt \\ &= \int_0^\infty \Big(-2B_r H_r \mathcal{E}_r(t)\left(\mathcal{B}H\mathcal{E}(t)\right)^T \\ &\quad + 2C\mathcal{B}H\mathcal{E}(t)\left(\mathcal{B}H\mathcal{E}(t)\right)^T\Big) dt \\ &= -2B_r H_r \int_0^\infty \mathcal{E}_r(t)\mathcal{E}(t)^T dt\,(\mathcal{B}H)^T \\ &\quad +2C\mathcal{B}H \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^T dt\,(\mathcal{B}H)^T \\ &= -2\mathcal{D} + 2C\mathcal{A} \;=\; 0 \end{aligned} \tag{31}$$

where we have defined

$$\mathcal{D} = B_r H_r \int_0^\infty \mathcal{E}_r(t)\mathcal{E}(t)^T dt\,(\mathcal{B}H)^T \tag{32}$$

and

$$\mathcal{A} = \mathcal{B}H \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^T dt\,(\mathcal{B}H)^T. \tag{33}$$

The fact that the matrix $\mathcal{A}$ is invertible, is easily seen as the matrix $\mathcal{B}$ as in (24) has the same rank as the number of rows in $\mathcal{B}$; the matrix H as in (27) has the same rank as the number of columns in H; and the matrix $\int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^T dt$ has the same rank as the number of elements in $\mathcal{E}(t)$ (see (12) and (13)), since all the element functions of $\mathcal{E}(t)$ are linearly independent on $[0,\infty)$. This gives us the simple closed-form solution

$$C = \mathcal{D}\mathcal{A}^{-1}. \tag{34}$$

As given in (12) and (13), $\mathcal{E}(t)$ can be written as

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \\ \vdots \\ \mathcal{E}_\nu(t) \end{bmatrix} \tag{35}$$

where

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{-\lambda_i t} \\ te^{-\lambda_{i_p} t} \\ \vdots \\ \frac{t^{(d_i-1)}}{(d_i-1)!}e^{-\lambda_i t} \end{bmatrix}, \tag{36}$$

and similarly

$$\mathcal{E}_r(t) = \begin{bmatrix} \mathcal{E}_{r1}(t) \\ \mathcal{E}_{r2}(t) \\ \vdots \\ \mathcal{E}_{r\nu_r}(t) \end{bmatrix} \tag{37}$$

where

$$\mathcal{E}_{ri_r}(t) = \begin{bmatrix} e^{-\lambda_{ri_r} t} \\ te^{-\lambda_{ri_r} t} \\ \vdots \\ \frac{t^{(d_{ri_r}-1)}}{(d_{ri_r}-1)!}e^{-\lambda_{ri_r} t} \end{bmatrix}. \tag{38}$$

Calculating the (ρ,σ)–th element of the (k,j)–th subblock of $\int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^T dt$, i.e., of the matrix $\int_0^\infty \mathcal{E}_k(t)\mathcal{E}_j(t)^T dt$ is given by

$$\left[\int_0^\infty \mathcal{E}_k(t)\mathcal{E}_j(t)^T dt\right]_{\rho,\sigma} = \frac{\begin{pmatrix}\rho+\sigma-2 \\ \rho-1\end{pmatrix}}{(\lambda_k+\lambda_j)^{\rho+\sigma-1}}. \tag{39}$$

Similarly, the (ρ,σ)–th element of the (k,j)–th subblock of $\int_0^\infty \mathcal{E}_r(t)\mathcal{E}(t)^T dt$, i.e., of the matrix $\int_0^\infty \mathcal{E}_{rk}(t)\mathcal{E}_j(t)^T dt$ is given by

$$\left[\int_0^\infty \mathcal{E}_{rk}(t)\mathcal{E}_j(t)^T dt\right]_{\rho,\sigma} = \frac{\begin{pmatrix}\rho+\sigma-2 \\ \rho-1\end{pmatrix}}{(\lambda_{rk}+\lambda_j)^{\rho+\sigma-1}}. \tag{40}$$

In order to track the transient behavior of a given step response a simple modification is needed, i.e., Eq. (30) becomes

$$\begin{aligned} J_S &= \int_0^\infty \left(y_{Sr}(t) - y_{Sc}(t)\right)^2 dt \\ &= \int_0^\infty \left(B_r H_r V_r \mathcal{E}_r(t) - C\mathcal{B}HV\mathcal{E}(t)\right)^2 dt \end{aligned} \tag{41}$$

where the V-matrices are given in Eq. (19). The $\mathcal{D}_S$ and $\mathcal{A}_S$ matrices in Eqs. (32) and (33) then become

$$\mathcal{D}_S = B_r H_r V_r \int_0^\infty \mathcal{E}_r(t)\mathcal{E}(t)^T dt\,(\mathcal{B}HV)^T \tag{42}$$

and

$$\mathcal{A}_S = \mathcal{B}HV \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^T dt\,(\mathcal{B}HV)^T. \tag{43}$$

The optimal zeros are then given as before by

$$C = \mathcal{D}_S\mathcal{A}_S^{-1}. \tag{44}$$

Similarly we can track any higher-order response by using the V-matrix in a different power, effectively minimizing the transient deviation between the reference system and the controlled system. It should be noted, that the V-matrix is of full rank in any power, as can be seen in (17) and (19). Therefore the former argument on the invertability of the $\mathcal{A}$ matrices stands for tracking of higher-order responses as well.

IV. Open-loop Examples

Let us consider three examples for systems with repeated eigenvalues. The first one simply demonstrates the trivial case where pole cancellations are the optimal solution. In the second and the third example, an extreme system including an eightfold pole is considered, tracking a desired impulse response as well as a desired step response.

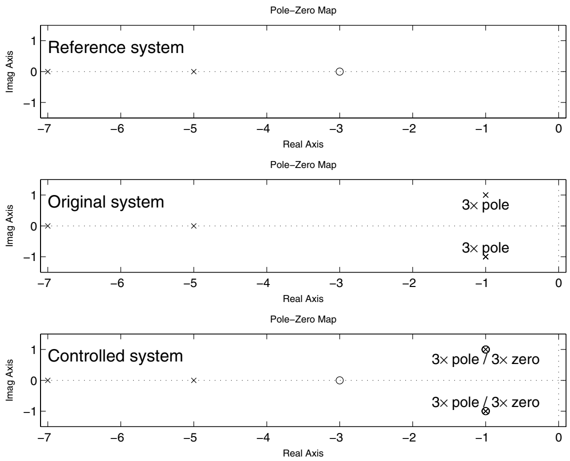

Fig. 2. Optimal zero locations for Example 1.

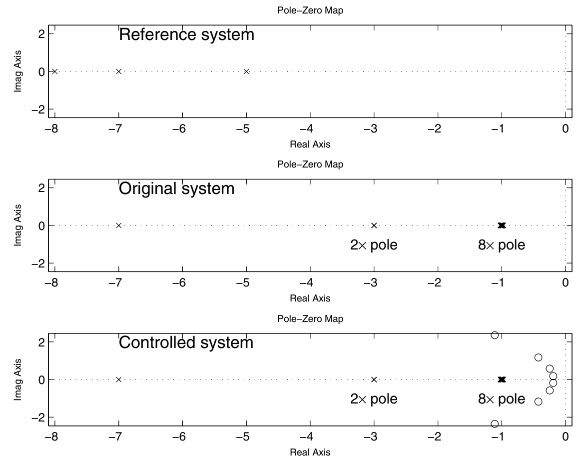

Fig. 3. Optimal zero locations for Example 2.

Example 1: Assume that the original system has eight poles, two distinct poles at -5 and -7 and a triple pole-pair at $-1 \pm i$ and no zeros. The reference system has a zero at -3 and the poles at -5 and -7, thus the obvious solution is simply to cancel out the repeated complex pole pairs and add a zero at -3. The optimal zeros are computed resulting in[1]

$$\begin{aligned} C(s) &= 0.04167s^7 + 0.375s^6 + 1.5s^5 + 3.583s^4 \\ &\quad +5.5s^3 + 5.5s^2 + 3.333s + 1. \end{aligned}$$

The resulting pole zero plots are shown in Fig. 2. As expected the closed-form solution simply cancels unwanted poles and adds the needed zeros to mimic the original system completely.

Example 2: The original system contains no zeros, an eightfold pole at -1, a double pole at -3 and a single pole at -7. The reference system has no zeros and distinct poles at -5, -7 and -8. Here we wish to track the impulse response of the reference system. The optimal zeros are computed resulting in[1]

$$\begin{aligned} C(s) = {} & 3.508s^8 + 13.96s^7 + 47.87s^6 + 72.25s^5 \\ & +91.63s^4 + 61.36s^3 + 31.59s^2 + 7.903s + 1. \end{aligned}$$

The results are shown in Fig. 3 (pole/zero plot) and Fig. 5 (impulse and step responses). As we can see the resulting impulse response shows excellent tracking and even though the step response could also be considered a good tracking performance, the next example shows an excellent step response tracking for the same systems, where the step response deviation itself is minimized.

Example 3: Here we use exactly the same systems as in Ex. 2, but now we track the step response of the reference system. The optimal zeros are computed resulting in[1]

$$\begin{aligned} C(s) &= 3.185s^8 + 14.84s^7 + 45.52s^6 + 76.4s^5 \\ &\quad +88.41s^4 + 64.71s^3 + 30.7s^2 + 8.322s + 1, \end{aligned}$$

see Fig. 4. The impulse and the step responses are plotted in Fig. 6. By carefully examining the impulse response we see a slight deterioration. Naturally, the step response has improved by minimizing the step response deviation.

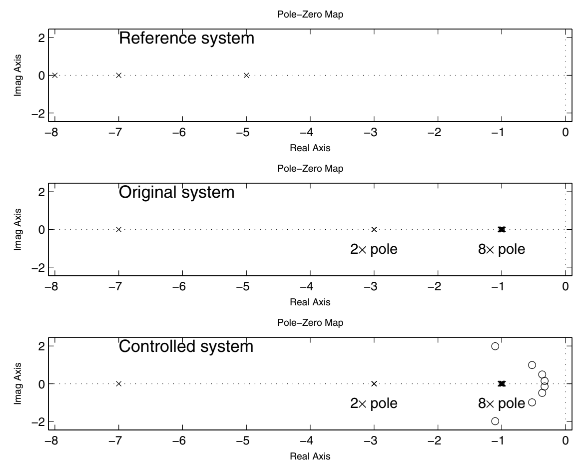

Fig. 4. Optimal zero locations for Example 3.

V. Conclusions

Optimal additional zero locations of continuous-time systems with p free zeros and some fixed zeros and fixed poles tracking a reference impulse and step responses, were derived in this paper, based on general closed-form impulse and step responses. Further, earlier work was reformulated and expanded to include the case of repeated eigenvalues. In the general case of p free zeros, the impulse and step response deviation from a given open-loop reference transient response was minimized, resulting in an explicit and easily computable solution for the free part of the

[1] The original- and the reference system were scaled to have a unity DC-gain.

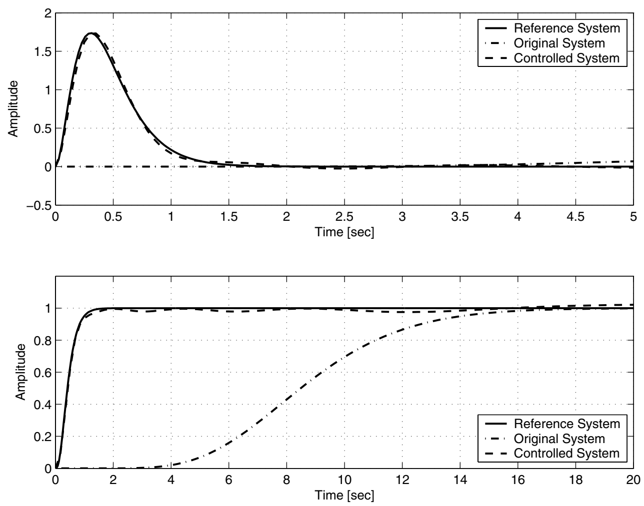

Fig. 5. Optimal impulse and step responses for Example 2.

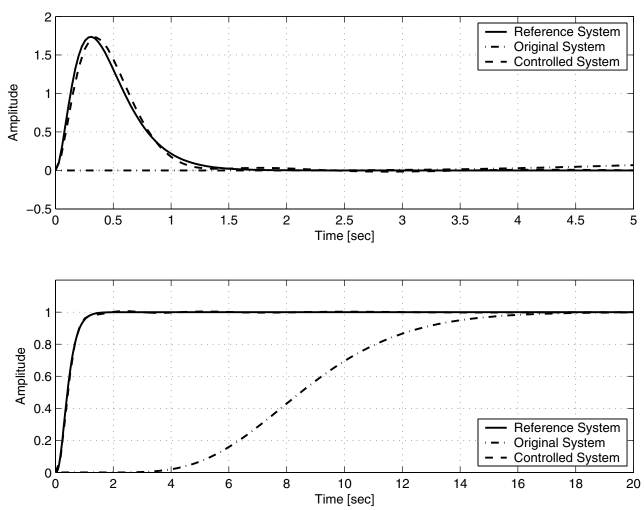

Fig. 6. Optimal impulse and step responses for Example 3.

transfer function numerator coefficients. Thus, the results obtained are simple and easily applicable to a large class of systems.

VI. ACKNOWLEDGMENTS

This work was supported by the University of Iceland Research Fund.

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