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Fuzzy Control System Designs using Redundancy of Descriptor Representation:A Fuzzy Lyapunov Function Approach

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Abstract— **This paper presents fuzzy control system designs using redundancy of descriptor representation. A wider class of Takagi-Sugeno fuzzy controllers using the redundancy is employed to derive stabilization conditions for both common Lyapunov functions and fuzzy Lyapunov functions. We show that the fuzzy Lyapunov function approach is less conservative than the common Lyapunov function approach. A design example also illustrates the utility of the fuzzy Lyapunov function approach using redundancy of descriptor representation.**

I. Introduction

Nonlinear control based on the Takagi-Sugeno fuzzy model [1] has received a lot of attention over the last decade (e.g., see [2]- [11]). An advantage of the fuzzy model-based control [12] is to provide a natural, simple and effective design approach although other nonlinear control techniques [13] require special and rather involved knowledge. In addition, it is known that any smooth nonlinear control systems can be approximated by the Takagi-Sugeno fuzzy models (with liner rule consequence) [14].

Recently, piecewise Lyapunov function approaches have received increasing attention as they attempt to relax the conservativeness of stability and stabilization problems. However, stabilization conditions for fuzzy Lyapunov functions [15] and piecewise Lyapunov functions [16] become BMIs in general. In [15], the well-known completing square technique was introduced to convert the BMIs into LMIs. The conversion causes conservative results in general. Hence, the converted LMIs do not completely contain the LMIs for the common quadratic Lyapunov function although the fuzzy Lyapunov function contains the common quadratic Lyapunov function as a special case. In this paper, we derive LMI design conditions (that contains the LMIs for the common quadratic Lyapunov function as a special case) using redundancy of descriptor representation. The redundancy also provides us the possibility of designing a fuzzy controller for systems with input nonlinearities. A fuzzy descriptor system design has been already discussed in [17]. The design in [17] did not fully take an advantages of redundancy of descriptor representation.

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This paper is organized as follows. Section II recalls the previous results with respect to fuzzy model and stability conditions. Section III introduces a wider class of Takagi-Sugeno fuzzy controllers and derives stabilization conditions based on common quadratic Lyapunov functions. Section IV presents stability conditions based on fuzzy Lyapunov functions. We show that the fuzzy Lyapunov function approach is less conservative than the common Lyapunov approach. Section V illustrates a design example to demonstrate the utility of the fuzzy Lyapunov function approach using redundancy of descriptor representation.

II. Fuzzy model and stability conditions

Consider the following nonlinear systems:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$ is the state vector, $\boldsymbol{u}(t) = [u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T$ is the input vector. Based on the sector nonlinearity concept [12], we can exactly represent (1) with the Takagi-Sugeno fuzzy model (2) (globally or at least semi-globally).

Model Rule i**:** $If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip}$

$$then\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \quad i = 1, 2, \cdots, r, \tag{2}$$

where $z_j(t)$ $(j = 1, 2, \cdots, p)$ is the premise variable. The membership function associated with the *i*th $Model\ Rule$ and *j*th premise variable component is denoted by M_{ij}. r denotes the number of $Model\ Rules$. Each $z_j(t)$ is a measurable time-varying quantity that may be states, inputs, measurable external variables and/or time. It has been tacitly assumed in fuzzy model-based design that each $z_j(t)$ does not depend on the inputs $\boldsymbol{u(t)}$. However, the fuzzy control designs using redundancy of descriptor representation permit that each $z_j(t)$ depends on the inputs $\boldsymbol{u(t)}$. This is an advantage of fuzzy control designs using redundancy of descriptor representation.

The defuzzification process of the model (2) can be represented as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}, \tag{3}$$

where $\boldsymbol{z}(t) = [z_1(t) \cdots z_p(t)]$. From the properties of membership functions, the following relations hold.

$$h_i(\boldsymbol{z}(t)) = \frac{w_i(\boldsymbol{z}(t))}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \geq 0, \quad \sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1,$$

where $w_i(\boldsymbol{z}(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t))$. The parallel distributed compensation provides the following control rules for the fuzzy model (2):

Control Rule i:

$$If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip}$$

$$then\ \boldsymbol{u}(t) = -\boldsymbol{F}_i\boldsymbol{x}(t) \quad i = 1, 2, \cdots, r \tag{4}$$

The overall fuzzy controller can be calculated by

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i\boldsymbol{x}(t). \tag{5}$$

A sufficient condition [12] for ensuring the stability of the feedback system consisting of (3) and (5) is given as follows;

$$\boldsymbol{X} > \boldsymbol{0} \tag{6}$$

$$-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{M}_i^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_i > \boldsymbol{0} \tag{7}$$

$$-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}_j^T - \boldsymbol{A}_j\boldsymbol{X} + \boldsymbol{M}_j^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_j$$

$$+\boldsymbol{M}_i^T\boldsymbol{B}_j^T + \boldsymbol{B}_j\boldsymbol{M}_i \geq \boldsymbol{0} \quad i < j \tag{8}$$

where $\boldsymbol{M}_i = \boldsymbol{F}_i\boldsymbol{X}$. We can obtain feedback gains stabilizing (3) by solving the LMIs (6) - (8). However, it should be emphasized that the fuzzy controller (5) can not be applied in general when $\boldsymbol{z}(t)$ depend on $\boldsymbol{u}(t)$ since the premise variables $\boldsymbol{z}(t)$ depend on $\boldsymbol{u}(t)$, i.e., since the control inputs $\boldsymbol{u}(t)$ to be calculated are also contained in the right side hand of (5). In this case, even if we have a feasible solution for the LMIs (6)-(8), it is difficult to calculate the control input $\boldsymbol{u}(t)$ using (5). Section III will give an answer of the problem.

In [15], we defined a fuzzy Lyapunov function and derived stabilization conditions via the fuzzy Lyapunov function, where we required $h_i(\boldsymbol{z})$ to be C^1 functions. It should be noted that the assumption is satisfied for fuzzy models constructed from smooth (at least C^1) nonlinear systems by using a sector nonlinearity approach [12]. The sector nonlinearity approach can construct a global or semi-global fuzzy model that exactly represent the dynamics of a nonlinear system. The candidate fuzzy Lyapunov function for the Takagi-Sugeno fuzzy system (2) is defined as

$$V(\boldsymbol{x}(t)) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{x}^T(t)\boldsymbol{P}_i\boldsymbol{x}(t), \tag{9}$$

where $\boldsymbol{P}_i$ is a positive definite matrix. This candidate Lyapunov function satisfies (1) V is C^1, (2) $V(\boldsymbol{0}) = 0$ and $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$ and (3) $\|\boldsymbol{x}\| \to \infty \Rightarrow V(\boldsymbol{x}) \to \infty$. The fuzzy Lyapunov function shares the same membership functions with the Takagi-Sugeno fuzzy model of a system. Hence, the fuzzy Lyapunov function reduces to the common Lyapunov function when $\boldsymbol{P} = \boldsymbol{P}_i$ for all i. Unfortunately, stabilization conditions (for the control system consisting of (3) and (5)) based on the fuzzy Lyapunov functions become BMIs. In [15], the well-known completing square technique was introduced to convert the BMIs into LMIs. The conversion causes conservative results in general. Hence, the converted LMIs do not completely contain the LMIs for the common quadratic Lyapunov function although the fuzzy Lyapunov function contains the common quadratic Lyapunov function as a special case. In this paper, we derive LMI design conditions (that contains the LMIs for the common quadratic Lyapunov function as a special case) using redundancy of descriptor representation.

III. Design based on Common Lyapunov Function

A. New fuzzy controller using redundancy of descriptor representation

We propose a fuzzy controller (10) using redundancy of descriptor representation.

$$\boldsymbol{E}\dot{\boldsymbol{u}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{K}_i\boldsymbol{u}(t) + \boldsymbol{F}_i\boldsymbol{x}(t)\} \tag{10}$$

The controller (10) reduces to fuzzy dynamic state feedback controller [18] when $\boldsymbol{E} = \boldsymbol{I}$. It reduces to the state feedback controller (5) when $\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{K}_i = \boldsymbol{I}$. Thus, the controller (10) is a more general form containing some types of controllers.

From (3) and (10), we have the following descriptor representation.

$$\boldsymbol{E}^*\dot{\hat{\boldsymbol{x}}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{A}_i^*\hat{\boldsymbol{x}}(t), \tag{11}$$

where

$$\boldsymbol{E}^* = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}, \boldsymbol{A}_i^* = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{F}_i & \boldsymbol{0} & \boldsymbol{K}_i \end{bmatrix},$$

$$\hat{\boldsymbol{x}}(t) = [\boldsymbol{x}^T(t)\ \boldsymbol{x}^T(t)\ \boldsymbol{u}^T(t)]^T.$$

B. Stabilization conditions

Theorem 1 gives a sufficient stability condition for (11).

Theorem 1: If there exists matrix $\boldsymbol{X}^*$ satisfying (12) and (13), the control system (11) is stable.

$$\boldsymbol{X}^{*T}\boldsymbol{E}^* = \boldsymbol{E}^{*T}\boldsymbol{X}^* \geq \boldsymbol{0} \tag{12}$$

$$\boldsymbol{A}_i^{*T}\boldsymbol{X}^* + \boldsymbol{X}^{*T}\boldsymbol{A}_i^* < \boldsymbol{0} \quad i = 1, 2, \cdots, r \tag{13}$$

(proof) The proof is omitted due to lack of space.

Remark 1: The number of stabilization conditions (6)-(8) to design the fuzzy controller (5) is $(r^2 + r)/2 + 1$. On the other hand, the number of stabilization conditions (12) and (13) to design the fuzzy controller (10) is $r + 1$. The reason why the number of conditions reduces is because the cross terms with respect to $\boldsymbol{F}_i$ and $\boldsymbol{B}_i$ do not exist, i.e., because

$\boldsymbol{F}_i$ and $\boldsymbol{B}_i$ appear separately in the matrices $\boldsymbol{A}_i^*$. This is an advantage of redundancy of descriptor representation. The number of stabilization conditions can be drastically reduced when r is larger, i.e., when the system becomes more complicated. This fact shows that the design condition (12) and (13) are useful for complicated systems.

Note that the matrix $\boldsymbol{A}^*$ contains the feedback gains $\boldsymbol{F}_i$ and $\boldsymbol{K}_i$. Therefore, the term $\boldsymbol{A}^{*T}\boldsymbol{X}^*$ are not linear, that is, the condition (13) is not an LMI in general. To overcome the difficulty, we define $\boldsymbol{X}^*$ as

$$\boldsymbol{X}^* = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \boldsymbol{X}_{23} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{X}_{33} \end{bmatrix}. \tag{14}$$

For the above $\boldsymbol{X}^*$, the condition (13) becomes an LMI with respect to the feedback gains and the variables in $\boldsymbol{X}^*$. Then, the conditions in Theorem 1 can be converted into LMIs.

Corollary 1: If we use (14) as a common $\boldsymbol{X}^*$, then the conditions (13) and (12) reduce to

$$\begin{bmatrix} \boldsymbol{X}_{21} + \boldsymbol{X}_{21}^T & * & * \\ \boldsymbol{A}_i^T\boldsymbol{X} - \boldsymbol{X}_{21} + \boldsymbol{X}_{22}^T & -\boldsymbol{X}_{22} - \boldsymbol{X}_{22}^T & * \\ \boldsymbol{B}_i^T\boldsymbol{X} + \boldsymbol{M}_i + \boldsymbol{X}_{23}^T & -\boldsymbol{X}_{23}^T & \boldsymbol{N}_i + \boldsymbol{N}_i^T \end{bmatrix} < \boldsymbol{0} \quad i = 1, 2, \cdots, r, \tag{15}$$

$\boldsymbol{X} = \boldsymbol{X}^T > \boldsymbol{0}$ and $\boldsymbol{X}_{33}^T\boldsymbol{E} = \boldsymbol{E}^T\boldsymbol{X}_{33} > \boldsymbol{0}$, respectively, where $\boldsymbol{M}_i = \boldsymbol{X}_{33}^T\boldsymbol{F}_i$ and $\boldsymbol{N}_i = \boldsymbol{X}_{33}^T\boldsymbol{K}_i$. The symbol '$*$' denotes the transposed elements (matrices) for symmetric positions.

When $\boldsymbol{K}_i = \boldsymbol{I}$ and $\boldsymbol{E} = \boldsymbol{0}$, the LMI conditions in Corollary 1 can be simplified as follows.

Corollary 2: Assume that we use (14) as a common $\boldsymbol{X}^*$. When $\boldsymbol{K}_i = \boldsymbol{I}$ and $\boldsymbol{E} = \boldsymbol{0}$, i.e., when (10) reduces to (5), the conditions (13) and (12) reduce to

$$\begin{bmatrix} \boldsymbol{X}_{21} + \boldsymbol{X}_{21}^T & * & * \\ \boldsymbol{A}_i^T\boldsymbol{X} - \boldsymbol{X}_{21} + \boldsymbol{X}_{22}^T & -\boldsymbol{X}_{22} - \boldsymbol{X}_{22}^T & * \\ \boldsymbol{B}_i^T\boldsymbol{X} + \boldsymbol{M}_i + \boldsymbol{X}_{23}^T & -\boldsymbol{X}_{23}^T & \boldsymbol{X}_{33} + \boldsymbol{X}_{33}^T \end{bmatrix} < \boldsymbol{0} \tag{16}$$

and $\boldsymbol{X} = \boldsymbol{X}^T > \boldsymbol{0}$, respectively, where $\boldsymbol{M}_i = \boldsymbol{X}_{33}^T\boldsymbol{F}_i$.

IV. Design based on Fuzzy Lyapunov Function

We have already proposed a fuzzy Lyapunov function [15] for the ordinary fuzzy system. This paper extends the result to fuzzy descriptor systems. The fuzzy Lyapunov function provides us more relaxed stability results. As in [15], we require $h_i(\boldsymbol{z})$ to be C^1 functions.

Theorem 2: Assume that $\dot{h}_k(\boldsymbol{z}(t)) \leq \phi_k$ for $k = 1, 2, \cdots, r-1$. Then, the control system (11) is stable if there exists $\boldsymbol{X}_i^*$ satisfying

$$\boldsymbol{X}_i^{*T}\boldsymbol{E}^* = \boldsymbol{E}^{*T}\boldsymbol{X}_i^* \geq \boldsymbol{0} \quad i = 1, 2, \cdots, r \tag{17}$$

$$\boldsymbol{E}^{*T}(\boldsymbol{X}_k^* - \boldsymbol{X}_r^*) \geq \boldsymbol{0} \quad k = 1, 2, \cdots, r-1 \tag{18}$$

$$\frac{1}{2}(\boldsymbol{A}_j^{*T}\boldsymbol{X}_i^* + \boldsymbol{X}_i^{*T}\boldsymbol{A}_j^* + \boldsymbol{A}_i^{*T}\boldsymbol{X}_j^* + \boldsymbol{X}_j^{*T}\boldsymbol{A}_i^*) + \sum_{k=1}^{r-1}\phi_k\boldsymbol{E}^{*T}(\boldsymbol{X}_k^* - \boldsymbol{X}_r^*) < \boldsymbol{0} \quad i \leq j. \tag{19}$$

(proof) The proof is omitted due to lack of space.

Remark 2: The condition (13) implies the condition (19). In other word, the condition (19) reduces to the condition (13) when $\boldsymbol{X}^* = \boldsymbol{X}_i^*$ for all i. The condition (18) always holds when $\boldsymbol{X}^* = \boldsymbol{X}_i^*$ for all i. Therefore, Theorem 2 is less conservative than Theorem 1.

Remark 3: In Theorem 2, we assumed that $\dot{h}_k(\boldsymbol{z}(t)) \leq \phi_k$. However, in real system designs, it is not easy to find ϕ_k satisfying the assumption. A way of solving the problem was addressed in [15].

The conditions in Theorem 2 can be converted into the LMIs if we use

$$\boldsymbol{X}_i^* = \begin{bmatrix} \boldsymbol{X}_i & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{X}_{21i} & \boldsymbol{X}_{22i} & \boldsymbol{X}_{23i} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{X}_{33} \end{bmatrix} \tag{20}$$

as $\boldsymbol{X}_i^*$.

Corollary 3: If we use the matrix (20), then the conditions (19), (17) and (18) reduce to

$$\frac{1}{2}\begin{bmatrix} \boldsymbol{X}_{21i} + \boldsymbol{X}_{21i}^T + \boldsymbol{X}_{21j} + \boldsymbol{X}_{21j}^T & * & * \\ \boldsymbol{A}_i^T\boldsymbol{X}_j + \boldsymbol{A}_j^T\boldsymbol{X}_i - \boldsymbol{X}_{21i} + \boldsymbol{X}_{22i}^T - \boldsymbol{X}_{21j} + \boldsymbol{X}_{22j}^T & -\boldsymbol{X}_{22i} - \boldsymbol{X}_{22i}^T - \boldsymbol{X}_{22j} - \boldsymbol{X}_{22j}^T & * \\ \boldsymbol{B}_i^T\boldsymbol{X}_j + \boldsymbol{B}_j^T\boldsymbol{X}_i + \boldsymbol{M}_i + \boldsymbol{M}_j + \boldsymbol{X}_{23i}^T + \boldsymbol{X}_{23j}^T & -\boldsymbol{X}_{23i}^T - \boldsymbol{X}_{23j}^T & \begin{pmatrix} \boldsymbol{N}_i + \boldsymbol{N}_i^T \\ +\boldsymbol{N}_j + \boldsymbol{N}_j^T \end{pmatrix} \end{bmatrix}$$
$$+\sum_{k=1}^{r-1}\phi_k\begin{bmatrix} \boldsymbol{X}_k - \boldsymbol{X}_r & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < \boldsymbol{0} \quad i \leq j, \tag{21}$$

$$\boldsymbol{X}_i = \boldsymbol{X}_i^T \geq \boldsymbol{0} \quad i = 1, 2, \cdots, r \tag{22}$$

$$\boldsymbol{X}_{33}^T\boldsymbol{E} = \boldsymbol{E}^T\boldsymbol{X}_{33} \geq \boldsymbol{0} \tag{23}$$

$$\boldsymbol{X}_k - \boldsymbol{X}_r \geq \boldsymbol{0} \quad k = 1, 2, \cdots, r-1, \tag{24}$$

respectively, where $\boldsymbol{M}_i = \boldsymbol{X}_{33}^T\boldsymbol{F}_i$ and $\boldsymbol{N}_i = \boldsymbol{X}_{33}^T\boldsymbol{K}_i$.

Remark 4: If we use $\boldsymbol{X}_{33i}$ instead of $\boldsymbol{X}_{33}$, $\boldsymbol{M}_i$ and $\boldsymbol{N}_i$ should be $\boldsymbol{M}_{ij} = \boldsymbol{X}_{33j}^T\boldsymbol{F}_i$ and $\boldsymbol{N}_{ij} = \boldsymbol{X}_{33j}^T\boldsymbol{K}_i$. For the case, even if the LMIs in Corollary 3 are feasible, $\boldsymbol{F}_i$ and $\boldsymbol{K}_i$ can not be uniquely determined from $\boldsymbol{M}_{ij} = \boldsymbol{X}_{33j}^T\boldsymbol{F}_i$ and $\boldsymbol{N}_{ij} = \boldsymbol{X}_{33j}^T\boldsymbol{K}_i$.

Remark 5: In [15], we derived stabilizing conditions for the ordinary Takagi-Sugeno fuzzy models using fuzzy Lyapunov functions. Unfortunately, the conditions were not LMIs. Therefore, in [15], the well-known completing square technique was introduced to convert into LMIs. However, the conversion causes conservative results. On the one hand, Corollary 3 directly provides LMI conditions since $\boldsymbol{F}_i$ and $\boldsymbol{B}_i$ appear separately in the matrices $\boldsymbol{A}_i^*$. This is an advantage of redundancy of descriptor representation.

When $\boldsymbol{K}_i = \boldsymbol{I}$ and $\boldsymbol{E} = \boldsymbol{0}$, the LMI conditions can be simplified as follows.

Corollary 4: Assume that we use the matrix (20). When $\boldsymbol{K}_i = \boldsymbol{I}$ and $\boldsymbol{E} = \boldsymbol{0}$, i.e., when (10) reduces to (5), the conditions (19), (17) and (18) reduce to

$$\frac{1}{2}\begin{bmatrix} \boldsymbol{X}_{21i} + \boldsymbol{X}_{21i}^T + \boldsymbol{X}_{21j} + \boldsymbol{X}_{21j}^T & * & * \\ \boldsymbol{A}_i^T\boldsymbol{X}_j + \boldsymbol{A}_j^T\boldsymbol{X}_i - \boldsymbol{X}_{21i} + \boldsymbol{X}_{22i}^T - \boldsymbol{X}_{21j} + \boldsymbol{X}_{22j}^T & -\boldsymbol{X}_{22i} - \boldsymbol{X}_{22i}^T - \boldsymbol{X}_{22j} - \boldsymbol{X}_{22j}^T & * \\ \boldsymbol{B}_i^T\boldsymbol{X}_j + \boldsymbol{B}_j^T\boldsymbol{X}_i + \boldsymbol{M}_i + \boldsymbol{M}_j + \boldsymbol{X}_{23i}^T + \boldsymbol{X}_{23j}^T & -\boldsymbol{X}_{23i}^T - \boldsymbol{X}_{23j}^T & 2(\boldsymbol{X}_{33} + \boldsymbol{X}_{33}^T) \end{bmatrix}$$

$$+\sum_{k=1}^{r-1}\phi_k\begin{bmatrix} \boldsymbol{X}_k - \boldsymbol{X}_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < \boldsymbol{0} \quad i \le j, \tag{25}$$

$$\boldsymbol{X}_i = \boldsymbol{X}_i^T \ge \boldsymbol{0} \quad i = 1, 2, \cdots, r, \tag{26}$$

$$\boldsymbol{X}_k - \boldsymbol{X}_r \ge \boldsymbol{0} \quad k = 1, 2, \cdots, r-1, \tag{27}$$

respectively, where $\boldsymbol{M}_i = \boldsymbol{X}_{33}^T\boldsymbol{F}_i$.

V. Design Example

Consider the pendulum system (28) used in [19].

$$\begin{array}{l} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -a\sin x_1(t) - bx_2(t) + cu(t), \end{array} \tag{28}$$

where $a = \frac{Mgl}{Ml^2+I}$, $b = \frac{\mu}{Ml^2+I}$ and $c = \frac{1}{Ml^2+I}$. M, I, μ and l denote mass, inertia, friction and distance to the center of gravity of the pendulum, respectively. $u(t)$ denotes the motor torque. In this simulation, $l = 1.0$, $g = 9.8$, $I = \frac{Ml^2}{3}$. Under $x_1(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the system (28) can be converted into the following fuzzy model [12]:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(z(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_iu(t)\}, \tag{29}$$

where $z(t) = \frac{\sin(x_1(t))}{x_1(t)}$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -a \times \frac{2}{\pi} & -b \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ -a \times 1 & -b \end{bmatrix}$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 & c \end{bmatrix}^T, \ \boldsymbol{B}_2 = \begin{bmatrix} 0 & c \end{bmatrix}^T.$$

The membership functions are obtained as

$$h_1(z(t)) = \frac{1 - z(t)}{1 - \frac{2}{\pi}}, \ h_2(z(t)) = \frac{z(t) - \frac{2}{\pi}}{1 - \frac{2}{\pi}}.$$

We compare the fuzzy Lyapunov approach (Corollary 4) with the common Lyapunov approach (Corollary 2). First, we design stable controllers for several combinations of M and μ using Corollary 2. Figure 1 shows the feasible area for the combinations, where the dotted area denotes the feasible area. The common $\boldsymbol{X}$ for $M = 0.1$ and $\mu = 0.1$ is obtained as

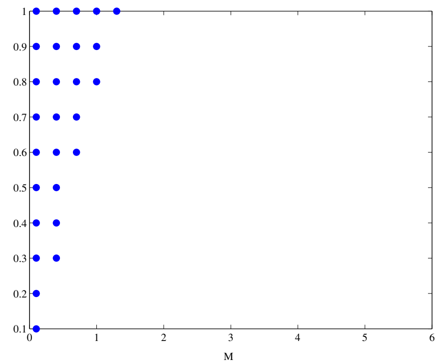

Fig. 1. Feasible area for common Lyapunov function.

$$\boldsymbol{X} = \begin{bmatrix} 1.7191 & 0.0092 \\ 0.0092 & 0.2937 \end{bmatrix}.$$

Next, we design stable controllers using Corollary 4. In Corollary 4, we need to select the values of ϕ_1 and ϕ_2. Since r=2, it is enough to consider only ϕ_1. $\dot{h}_1(z(t))$ and $\dot{h}_2(z(t))$ are obtained as

$$\dot{h}_1(z(t)) = \frac{\{-x_1(t)\cos(x_1(t)) + \sin(x_1(t))\}\dot{x}_1(t)}{(1 - \frac{2}{\pi})x_1^2(t)},$$
$$\dot{h}_2(z(t)) = \frac{\{x_1(t)\cos(x_1(t)) - \sin(x_1(t))\}\dot{x}_1(t)}{(1 - \frac{2}{\pi})x_1^2(t)}.$$

Note that $\dot{h}_1(z(t))$ is a function of $x_2(t) = \dot{x}_1(t)$. We consider the range of $x_2(t)$ as $x_2(t) \in [-q \ q]$. Then, the maximum value of $\dot{h}_1(z(t))$ is dependent of q. We use the maximum value of $\dot{h}_1(z(t))$ for $x_1(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $x_2(t) \in [-q \ q]$ as ϕ_1. We design stable controllers for several combinations of d in addition to (M, μ) using Corollary 4. Figures 2-7 show the feasible area for each range selection of $x_2(t)$, where the values of ϕ_1 for the given q are shown.

For $q = 0.5$, 1.0, 2.0, 3.0, 6.0 and 100, we obtain $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ ($M = 0.1$ and $\mu = 0.1$) as

$$\boldsymbol{X}_1 = \begin{bmatrix} 80.866 & 4.1755 \\ 4.1755 & 14.250 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 98.108 & 3.4829 \\ 3.4829 & 15.076 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \begin{bmatrix} 671.12 & 35.412 \\ 35.412 & 113.42 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 762.51 & 31.520 \\ 31.520 & 121.25 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \begin{bmatrix} 7.4810 & 0.3872 \\ 0.3872 & 1.2566 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 8.1189 & 0.3531 \\ 0.3531 & 1.3417 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \begin{bmatrix} 3794.6 & 202.70 \\ 202.70 & 635.40 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 3990.0 & 189.20 \\ 189.20 & 660.93 \end{bmatrix}$$

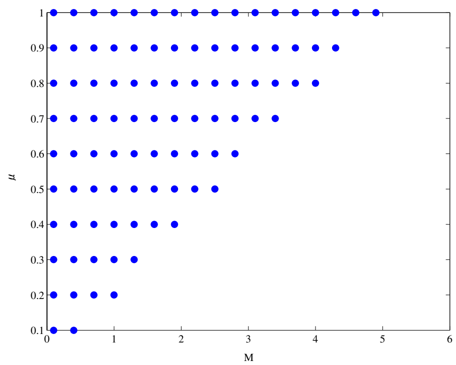

Fig. 2. Feasible area for fuzzy Lyapunov function ($q = 0.5$ & $\phi_1 = 0.5576$).

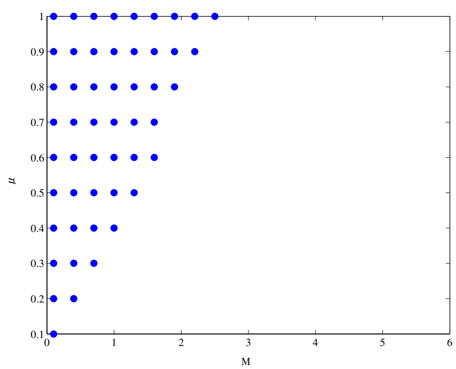

Fig. 3. Feasible area for fuzzy Lyapunov function ($q = 1.0$ & $\phi_1 = 1.1153$).

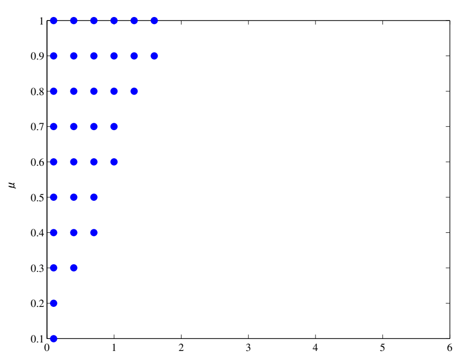

Fig. 4. Feasible area for fuzzy Lyapunov function ($q = 2.0$ & $\phi_1 = 2.2306$).

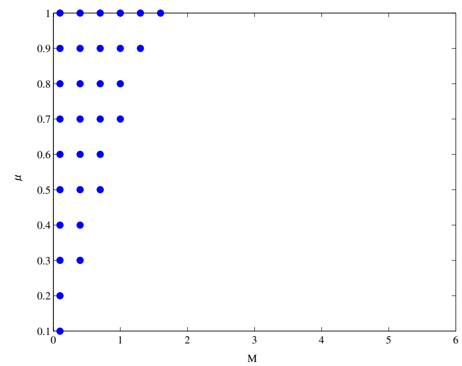

Fig. 5. Feasible area for fuzzy Lyapunov function ($q = 3.0$ & $\phi_1 = 3.3460$).

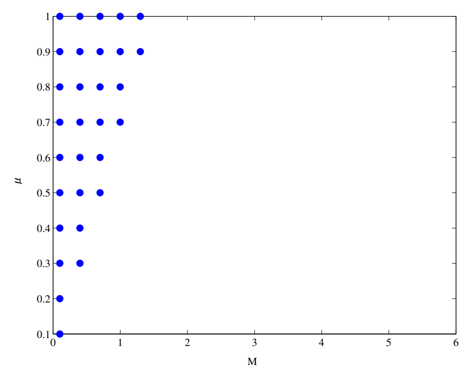

Fig. 6. Feasible area for fuzzy Lyapunov function ($q = 6.0$ & $\phi_1 = 6.6920$).

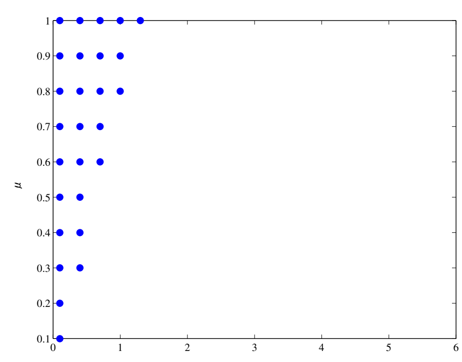

Fig. 7. Feasible area for fuzzy Lyapunov function ($q = 100$ & $\phi_1 = 111.53$).

$$\boldsymbol{X}_1 = \begin{bmatrix} 199.32 & 10.177 \\ 10.177 & 33.254 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 204.39 & 9.7224 \\ 9.7224 & 34.138 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \begin{bmatrix} 140.52 & 6.9660 \\ 6.9660 & 23.760 \end{bmatrix}, \ \boldsymbol{X}_2 = \begin{bmatrix} 140.75 & 6.9491 \\ 6.9491 & 23.792 \end{bmatrix}$$

respectively. It can be seen from the figures that the feasible areas for the fuzzy Lyapunov functions are wider than that of the common Lyapunov function when ϕ_1 is small. Even when ϕ_1 is quite large, the feasible areas for the fuzzy Lyapunov functions approach that of the common Lyapunov function. In fact, $\boldsymbol{X}_1$ is almost same as $\boldsymbol{X}_2$ when $q = 100$. If $\boldsymbol{X}^* = \boldsymbol{X}_i^*$ for all i, that is, if we consider the common Lyapunov function case, Theorem 2 reduces to Theorem 1. Thus, the fuzzy Lyapunov approach provides less conservative results. In fact, the LMIs in Corollary 4 for $a = 1$, $b = 0.1$, $c = 1$ and $\phi_1 = 0.17$ are feasible. Figure 8 shows the simulation result for $\boldsymbol{x}(0) = [1 \ 0]$. The designed fuzzy controller stabilizes the system (28). The LMIs in Corollary 2 for $a = 1$, $b = 0.1$ and $c = 1$ are infeasible. Therefore, any feedback gains can not be obtained by Corollary 2.

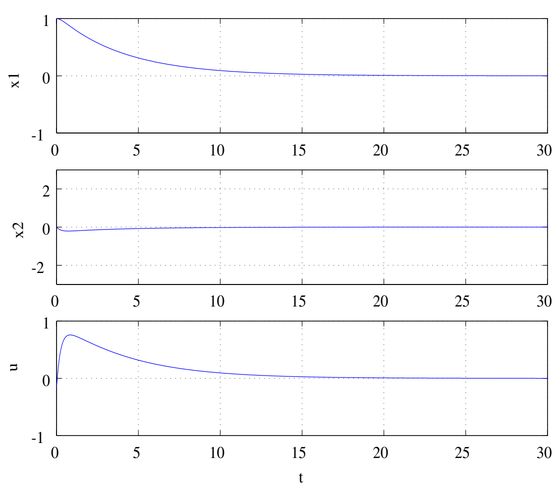

Fig. 8. Control result (Corollary 4).

Remark 6: To solve the LMIs in Corollary 4, ϕ_k should be selected so as to satisfy $\dot{h}_k(\boldsymbol{z}(t)) \leq \phi_k$. In this example, we can find ϕ_1 from $\dot{h}_1(z(t))$. Even if it is difficult to find ϕ_k, we can use a large value as ϕ_k in practice. Of course, an extreme large value could cause conservative results. However, the conditions in Corollary 4 at least guarantee less conservative results than those in Corollary 1.

VI. Conclusion

This paper has presented fuzzy control system designs using redundancy of descriptor representation. A wider class of Takagi-Sugeno fuzzy controllers using the redundancy has been employed to derive stabilization conditions for both common Lyapunov functions and fuzzy Lyapunov functions. We show that the fuzzy Lyapunov function approach is less conservative than the common Lyapunov function approach. A design example has illustrated the utility of the fuzzy Lyapunov function approach using redundancy of descriptor representation.

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