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A Fast Time-Optimal Control Synthesis Algorithm for a Class of Linear Systems

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***Abstract*—The paper deals with a new approach for synthesis of time-optimal control for a class of linear systems. It is based on the decomposition of the time-optimal control problem into a class of decreasing order problems, and the properties and relations between problems within this class. First, the problems' state-space properties are analyzed, and then the optimal control is obtained by using a multi-step procedure avoiding the switching hyper surface description. The emphasis in this paper is on the optimal control synthesis stage of the approach proposed. A property of the considered class of problems is studied which enables development of a fast algorithm for synthesis of time-optimal control without using the switching hyper surface.**

I. Introduction

The linear time-optimal control problem has a half-a-century history. Fundamental theoretical results have been obtained and a great number of papers have been published in this field. However, in the last decade the interest towards this problem considerably declines. It may be stated that despite the more than 40-year intensive research, the synthesis of time-optimal control for high order systems is still an open problem. An approach to go further in the solution of the time-optimal synthesis problem is to refine the well-known state-space method, removing the factors that restrict its application to low order systems only. Some new state-space properties of a class of linear systems make possible to develop an efficient time-optimal synthesis approach requiring no description of the switching hyper surface [1]-[3]. In this paper a property of the considered class of problems is studied which makes it possible to develop a fast time-optimal control synthesis algorithm without having to use the switching hyper surface.

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The following time-optimal synthesis problem for a linear system of order $\boldsymbol{k}$ is considered. The system is described by

$$\begin{aligned}
&\dot{\boldsymbol{x}}_k = A_k \boldsymbol{x}_k + B_k u_k,\\
&\boldsymbol{x}_k = [x_1 \quad x_2 \quad . \quad . \quad . \quad x_k]^{\mathrm{T}}, \ \boldsymbol{x}_k \in R^k,\\
&A_k = \mathrm{diag}(\lambda_1, \quad \lambda_2, \quad . \quad . \quad . \quad , \lambda_k),\\
&\qquad \lambda_i \in R, \ \lambda_i \le 0, \ {\scriptstyle i,j=\overline{1,k}}, \ \lambda_i \ne \lambda_j \ \text{if} \ {\scriptstyle i \ne j},\\
&B_k = [b_1 \quad b_2 \quad . \quad . \quad . \quad b_k]^{\mathrm{T}}, \ b_i \in R, \ b_i \ne 0, \ {\scriptstyle i=\overline{1,k}},\\
&\qquad \overline{1,k} = 1, 2, \ \ldots, k.
\end{aligned} \tag{1}$$

The initial and the target states of the system are

$$\boldsymbol{x}_k(0) = [x_{10} \quad x_{20} \quad . \quad . \quad . \quad x_{k0}]^{\mathrm{T}} \tag{2}$$

and

$$\boldsymbol{x}_k(t_{kf}) = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k}]^{\mathrm{T}} \tag{3}$$

where t_{kf} is unspecified. The admissible control $u_k(t)$ is a piecewise continuous function that takes its values from the range

$$-u_0 \le u_k(t) \le u_0, \ u_0 = const > 0\,. \tag{4}$$

We suppose that $u_k(t)$ is continuous on the boundary of the set of allowed values (4) and in the points of discontinuity τ we have

$$u(\tau) = u(\tau + 0)\,. \tag{5}$$

The problem is to find an admissible control $u_k = u_k(\boldsymbol{x}_k)$ that transfers the system (1) from its initial state (2) to the target state (3) in minimum time, i.e. minimizing the performance index

$$J_k = \int_0^{t_{kf}} dt = t_{kf}\,. \tag{6}$$

We shall refer to this problem as **Problem *A(k)*** and to the set {Problem *A(n)*, Problem *A(n-1)*, ..., Problem *A(1)*}, $n \ge 2$, as **class of problems *A(n), A(n-1), …, A(1)***.

The following relations exist between the systems of Problem *A(k)* and Problem *A(k-1)*, $k = \overline{n, 2}$:

$$A_k = \left[\begin{array}{c:c} A_{k-1} & 0_{((k-1)\times 1)} \\ \hdashline 0_{(1\times(k-1))} & \lambda_k \end{array}\right], \ B_k = \left[\begin{array}{c} B_{k-1} \\ \hdashline b_k \end{array}\right], \ \boldsymbol{x}_k(0) = \left[\begin{array}{c} \boldsymbol{x}_{k-1}(0) \\ \hdashline x_{k0} \end{array}\right]. \tag{7}$$

For Problem *A(k)*, $k=\overline{n,1}$, denote:

$u_k^o(t)$ - the optimal control which is a piecewise constant function taking the values $+u_0$ or $-u_0$ and having at most $(k-1)$ discontinuities [4]-[7];

t_{kf}^o - the minimum of the performance index;

L_{kk-1} - the set of all state space points for which the optimal control has no more than $(k-2)$ discontinuities;

S_k - the switching hyper surface. Note that S_k is time-invariant and includes the state space origin. As it is well known, the switching hyper surface S_k is identical with the set L_{kk-1} [5, ch. 14].

II. Preliminary Results

In this section we present some preliminary results proved in [1], [2], along with the idea of the proposed approach.

Let $k\ge 2$. Suppose we are in the initial point $\boldsymbol{x}_k(0)$ of the Problem *A(k)* state-space and the obviously easier Problem *A(k-1)* has been solved, i.e. we have the optimal control $u_{k-1}^o(t)$ and the minimum of the performance index t_{k-1f}^o of Problem *A(k-1)*. Applying the optimal control $u_{k-1}^o(t)$ of Problem *A(k-1)* to the system of Problem *A(k)* with initial state $\boldsymbol{x}_k(0)$ we obtain the trajectory

$$\boldsymbol{x}_k(t)=e^{A_k t}\boldsymbol{x}_k(0)+\int_0^t e^{A_k(t-\tau)}B_k u_{k-1}^o(\tau)d\tau,\ t\in[0,\ t_{k-1f}^o]. \quad (8)$$

The following result is valid for this trajectory [1], [2].

Theorem 1. *The state trajectory of system (1) starting from the initial point* $\boldsymbol{x}_k(0)$ *and generated by the optimal control* $u_{k-1}^o(t)$, $t\in[0,\ t_{k-1f}^o]$, *either entirely lies on the switching hyper surface* S_k, *or is above or below* S_k, *nowhere intersecting it.*

According to this theorem all points of trajectory (8) have the same relation to the switching hyper surface S_k of Problem *A(k)*, including the initial point $\boldsymbol{x}_k(0)$ and the final point

$$\boldsymbol{x}_k(t_{k-1f}^o)=e^{A_k t_{k-1f}^o}\boldsymbol{x}_k(0)+\int_0^{t_{k-1f}^o} e^{A_k(t_{k-1f}^o-\tau)}B_k u_{k-1}^o(\tau)d\tau. \quad (9)$$

It is shown in [1], [2] that

$$\boldsymbol{x}_k(t_{k-1f}^o)\in Ox_k, \quad (10)$$

and its last, *k*th coordinate denoted by x_{kw} is given by

$$x_{kw}=e^{\lambda_k t_{k-1f}^o}x_{k0}+\int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau,\ k=\overline{n,2}. \quad (11)$$

Another property of the class *A(n), A(n-1), ..., A(1)* is also studied in [1], [2], which makes possible the synthesis of optimal control for Problem *A(k)*, $k=\overline{n,2}$.

Theorem 2. *There exists no piecewise constant control* $u(t)$ *with an amplitude* u_0 *and* ***k*** *non zero intervals of constancy,* $1\le k\le(n-1)$, *transferring the system*

$$\dot{x}_i=\lambda_i x_i+b_i u,\ \lambda_i\in R,\ b_i\in R,\ i,j=\overline{1,n},\quad b_i\ne 0,\ \lambda_i\ne\lambda_j \text{ when } i\ne j \quad (12)$$

from any point of any axis $Ox_1, Ox_2, \dots, Ox_n$ *in the system state space to the origin O, and vice-versa – from the origin O to a point of any axis* $Ox_1, Ox_2, \dots, Ox_n$ *in the state space.*

From this theorem and the properties of the switching hyper surface S_k it follows

Corollary 1 [1], [2]. *The unique time optimal control that transfers the system of Problem A(k), where* $n\ge k\ge 2$, *from every point of the positive or negative part of any state space axis* $Ox_1, Ox_2, \dots, Ox_k$ *to the origin O, has exactly* ***k*** *non zero intervals of constancy, and the positive, respectively the negative, part of any axis* $Ox_1, Ox_2, \dots, Ox_k$ *is above or below the switching hyper surface* S_k.

In accordance with this corollary the term $x_{k+}\in\{-1,\ +1\}$, $k=\overline{2,n}$, is introduced in [1], [2] to indicate the relation of the axis Ox_k to the switching hyper surface S_k and the optimal control values for the points of the positive and negative semi-axis Ox_k. Thus for

$$\boldsymbol{x}_k(0)=[x_{10}\quad x_{20}\quad .\ .\ .\quad x_{k0}]^{\mathrm{T}}= =[\underbrace{0\quad 0\quad .\ .\ .\quad 0}_{k-1}\ x_{k0}]^{\mathrm{T}},\quad x_{k0}:\operatorname{sign}(x_{k0})=x_{k+}.$$

we have $u_k^o(0)=u_0$.

The time-optimal synthesis problem for the initial point $\boldsymbol{x}_k(0)$ can be solved based on the solution of problem *A(k-1)* and the relation of the final point (9) of trajectory (8) to the switching hyper surface S_k [1], [2].

Theorem 3. *If the solution of Problem A(k-1),* $k=\overline{n,2}$, *is found, then the optimal control of Problem A(k) for initial state* $\boldsymbol{x}_k(0)$ *can be determined as*

$$u_k^o(0)=u_k(\boldsymbol{x}_k(0))=\begin{cases}+u_0 & \text{if } x_{k+}x_{kw}>0\\ u_{k-1}^o(0) & \text{if } x_{k+}x_{kw}=0\\ -u_0 & \text{if } x_{k+}x_{kw}<0\end{cases}, \quad (13)$$

where x_{kw} *is given by (11).*

Based on this theorem, the following time-optimal synthesis algorithm is proposed [1], [2].

A. Basic Algorithm for synthesis of optimal control for the initial state of Problem A(k), $k=\overline{n,2}$

Step 1. Solve Problem *A(k-1)* to find $u_{k-1}^o(t)$ and t_{k-1f}^o;

Step 2. Compute x_{kw} from (11);

Step 3. Determine $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$ according to (13).

From Theorem 3 it also follows

Corollary 2. If $x_{kw}=0$, the solution of Problem $A(k$-1) is also the solution of Problem *A(k)*, i.e. $u_{k-1}^{o}(t)=u_{k}^{o}(t)$, $t_{k-1f}^{o}=t_{kf}^{o}$, and vice-versa: if Problem *A(k)* and Problem *A(k-1)* have the same solution, i.e. $u_{k-1}^{o}(t)=u_{k}^{o}(t)$, $t_{k-1f}^{o}=t_{kf}^{o}$, then $x_{kw}=0$.

Corollary 3. Depending on the value of x_{kw}, there are three possibilities:

- all $\boldsymbol{x}_k(0)$ for which $x_{kw}=0$ lie on the switching hyper surface S_k;
- all $\boldsymbol{x}_k(0)$ corresponding to $x_{kw}>0$ are above or below S_k and the optimal control for these points is $x_{k+}u_0$;
- all $\boldsymbol{x}_k(0)$ for which $x_{kw}<0$ are also above or below S_k, but in opposite to the area for $x_{kw}>0$, and the corresponding optimal control is $(-1)x_{k+}u_0$.

For $x_{kw}\neq 0$, the trajectory with initial point $\boldsymbol{x}_k(0)$ is considered, generated by $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$. For the points of this trajectory, a ***New Problem A(k)*** is consecutively defined in the same way as Problem *A(k)* but taking as initial state the current trajectory point. The corresponding new sub-problem *A(k-1)* is then solved and the value of x_{kw} is computed. The movement along the trajectory continues until obtaining $x_{kw}=0$.

In the next section we shall show that under some conditions the solution of the new problem *A(k-1)* and the computation of x_{kw} can be avoided for a part of the optimal trajectory. This makes possible to develop a faster algorithm for synthesis of time-optimal control for Problem *A(k)*.

III. Main Result

Denote by $u_{k-1}'^{o}(t)\big|_{\boldsymbol{x}_{kt}}$ and $\min J_{k-1}'\big|_{\boldsymbol{x}_{kt}}=t_{k-1f}'^{o}\big|_{\boldsymbol{x}_{kt}}$ the optimal control and the minimum of the performance index of New Problem *A(k-1)* for the current point $\boldsymbol{x}_{kt}$ of the trajectory of system (1) starting from $\boldsymbol{x}_k(0)$ generated by the control $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$, $2\le k\le n$.

We shall prove the following result.

Theorem 4. *Let the initial state* $\boldsymbol{x}_k(0)$ *of Problem A(k),* $2\le k\le n$*, not belong to the switching hyper surface* S_k*. Consider the part of the optimal trajectory not lying on* S_k*, i.e. the trajectory with initial point* $\boldsymbol{x}_k(0)$ *generated by* $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$ *for* $t\in[0,\ t_{k1}^o)$*. If there exists a point*

$$\boldsymbol{x}_k(t_1)=e^{A_k t_1}\boldsymbol{x}_k(0)+\int_0^{t_1} e^{A_k(t_1-\tau)}B_k u_k^o(0)d\tau,\ \ t_1\in[0,\ t_{k1}^o)$$

such that

$$u_{k-1}'^{o}(0)\big|_{\boldsymbol{x}_k(t_1)}=u_k^o(0) \tag{14}$$

then:

1. *the part of the trajectory of system (1) with initial point* $\boldsymbol{x}_k(t_1)$ *generated by the control* $u_{k-1}'^{o}(t)\big|_{\boldsymbol{x}_k(t_1)}$ *for* $t\in[0,\ t_{k-11}'^{1}]$*, where* $t_{k-11}'^{1}$ *is the length of the first constancy interval of* $u_{k-1}'^{o}(t)\big|_{\boldsymbol{x}_k(t_1)}$*, is also a part of the optimal trajectory for* $\boldsymbol{x}_k(0)$*, which does not lie on* S_k*;*
2. *for the optimal trajectory points* $\boldsymbol{x}_k^{1+}$ *not lying on* S_k *and situated after the considered common trajectory part, it is valid*
   $u_{k-1}'^{o}(0)\big|_{\boldsymbol{x}_k^{1+}}=-u_k^o(0)$ *when* $u_{k-1}'^{o}(0)\big|_{\boldsymbol{x}_k^{1+}}\neq 0$,
   i.e. the initial optimal control in New Problem A(k-1) for $\boldsymbol{x}_k^{1+}$ *has an opposite value to the optimal control* $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$*, except in the case*
   $u_{k-1}'^{o}(0)\big|_{\boldsymbol{x}_k^{1+}}=0$.

Proof. Consider the part of the optimal trajectory of system (1) situated out of the switching hyper surface S_k, i.e. the part of the trajectory of (1) starting from $\boldsymbol{x}_k(0)$ and generated by the control $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$ for $t\in[0,t_{k1}^o)$. If a New Problem *A(k)* is formulated for every point $\boldsymbol{x}_k^p$ of this trajectory part, then according to Theorem 1 and Corollaries 2 and 3, the trajectory of (1) starting from $\boldsymbol{x}_k^p$ and generated by the optimal control $u_{k-1}'^{o}(t)\big|_{\boldsymbol{x}_k^p}$, $t\in[0,\ t_{k-1f}'^{o}\big|_{\boldsymbol{x}_k^p}]$, does not lie on the switching hyper surface S_k and nowhere intersects S_k.

Suppose, there exists a point $\boldsymbol{x}_k(t_1)$ of the considered trajectory part, defined as

$$\boldsymbol{x}_k(t_1)=e^{A_k t_1}\boldsymbol{x}_k(0)+\int_0^{t_1} e^{A_k(t_1-\tau)}B_k u_k^o(0)d\tau,\ \ t_1\in[0,t_{k1}^o)$$

so that

$$u_{k-1}'^{o}(0)\big|_{\boldsymbol{x}_k(t_1)}=u_k^o(0)\cdot$$

Then the trajectory of system (1) starting form $\boldsymbol{x}_k(t_1)$ and generated by $u_{k-1}'^{o}(t)\big|_{\boldsymbol{x}_k(t_1)}$, $t\in[0,\ t_{k-1f}'^{o}\big|_{\boldsymbol{x}_k^p}]$, is situated out of and nowhere intersects the switching hyper surface S_k, and its first part for $t\in[0,\ t_{k-11}'^{1}]$ is generated by the control $u_k^o(0)$. It follows from the theorem for existence and uniqueness of a normal system [8] that this first trajectory part is also a part of the trajectory of (1) with initial state $\boldsymbol{x}_k(0)$ generated by the control $u_k^o(0)=u_k(\boldsymbol{x}_k(0))$ for $t\in[0,t_{k1}^o)$. This completes the first part of the theorem proof.

Consider now the points $\boldsymbol{x}_k^{1+}$ of the trajectory of (1) generated by $u_k^o(0) = u_k(\boldsymbol{x}_k(0))$ for $t \in [0, t_{k1}^o)$, which are situated after the considered common trajectory part. Let the end of this common part correspond to $t = t_2$, i.e.

$$\boldsymbol{x}_k(t_1) = e^{A_k t_1}\boldsymbol{x}_k(0) + \int_0^{t_1} e^{A_k(t_1-\tau)} B_k u_k^o(0) d\tau$$

$$\boldsymbol{x}_k(t_2) = e^{A_k t_2}\boldsymbol{x}_k(0) + \int_0^{t_2} e^{A_k(t_2-\tau)} B_k u_k^o(0) d\tau$$

$$0 \le t_1 < t_2 = t_1 + t'^1_{k-11},\ t_2 \in (0, t_{k1}^o),\ 2 \le k \le n$$

and

$$\{\boldsymbol{x}_k^{1+}\} = \{\boldsymbol{x}_k(t):\ \boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^{t} e^{A_k(t-\tau)} B_k u_k^o(0) d\tau,\ t \in (t_2, t_{k1}^o)\}\cdot$$

Then the optimal control $u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)}$ in the New Problem *A(k-1)* for the point $\boldsymbol{x}_k(t_2)$ is

$$u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)} = u'^o_{k-1}(t + t'^1_{k-11})\big|_{\boldsymbol{x}_k(t_1)}\ . \tag{15}$$

The control $u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_1)}$ is a piecewise constant function with no more than $(k-1)$ non-zero intervals of constancy, where $2 \le k \le n$. Therefore, the optimal control $u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)}$, $2 \le k \le n$, in the New Problem *A(k-1)* for the point $\boldsymbol{x}_k(t_2)$ is a piecewise constant function with no more than $(k-2)$ non-zero constancy intervals. This means that the point $\boldsymbol{x}_{k-1}(t_2)$ in the state space of Problem *A(k-1),* defined as

$$\boldsymbol{x}_{k-1}(t_2) : \boldsymbol{x}_k(t_2) = \begin{bmatrix} \boldsymbol{x}_{k-1}(t_2) \\ x_k(t_2) \end{bmatrix},\ 2 \le k \le n\,,$$

or

$$\boldsymbol{x}_{k-1}(t_2) = e^{A_{k-1} t_2}\boldsymbol{x}_{k-1}(0) + \int_0^{t_2} e^{A_{k-1}(t_2-\tau)} B_{k-1} u_k^o(0) d\tau,\quad 2 \le k \le n\,,$$

is a point of the switching hyper surface S_{k-1}, i.e.

$$\boldsymbol{x}_{k-1}(t_2) \in S_{k-1}\ ,\ 2 \le k \le n\,.$$

Then the following two alternative cases are possible for $\boldsymbol{x}_{k-1}(t_2)$:

1. $\boldsymbol{x}_{k-1}(t_2) \in S_{k-1},\ \boldsymbol{x}_{k-1}(t_2) \ne [\underbrace{0\ \ 0\ \ ...\ \ 0}_{k-1}]^{\mathrm{T}},\ 2 \le k \le n;$
2. $\boldsymbol{x}_{k-1}(t_2) \in S_{k-1},\ \boldsymbol{x}_{k-1}(t_2) = [\underbrace{0\ \ 0\ \ ...\ \ 0}_{k-1}]^{\mathrm{T}},\ 2 \le k \le n.$

In the first case $u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)}$ is a piecewise constant function with no more than $(k-2) \ge 1$ non-zero intervals of constancy, where $2 \le k \le n$. Since equations (14) and (15) are valid, it follows

$$u'^o_{k-1}(0)\big|_{\boldsymbol{x}_k(t_2)} = -u_k^o(0) \tag{16}$$

Suppose there exists a point $\boldsymbol{x}_k(t_3) \in \{\boldsymbol{x}_k^{1+}\}$ such that

$$\begin{aligned} &u'^o_{k-1}(0)\big|_{\boldsymbol{x}_k(t_3)} = u_k^o(0) \\ &\boldsymbol{x}_k(t_3) = e^{A_k t_3}\boldsymbol{x}_k(0) + \int_0^{t_3} e^{A_k(t_3-\tau)} B_k u_k^o(0) d\tau \end{aligned} \tag{17}$$

and denote by t'^3_{k-11} the length of the first constancy interval of $u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_3)}$. Then

$$u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_3)} = u_k^o(0)\ \text{ when }\ t \in [0, t'^3_{k-11}).$$

Therefore, for the points of the trajectory of (1) generated by $u_k^o(0) = u_k(\boldsymbol{x}_k(0))$, which belong to the part from $\boldsymbol{x}_k(0)$ to $\boldsymbol{x}_k(t_3)$, it is valid

$$u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_3-\tilde{t})} = \begin{cases} u_k^o(0) & \text{when } t \in [0, \tilde{t} + t'^3_{k-11}) \\ u'^o_{k-1}(t-\tilde{t})\big|_{\boldsymbol{x}_k(t_3)} & \text{when } t \ge \tilde{t} + t'^3_{k-11} \end{cases}$$

where

$$\boldsymbol{x}_k(t_3-\tilde{t}) = e^{A_k(t_3-\tilde{t})}\boldsymbol{x}_k(0) + \int_0^{(t_3-\tilde{t})} e^{A_k[(t_3-\tilde{t})-\tau]} B_k u_k^o(0) d\tau,\quad \tilde{t} \in [0, t_3].$$

Taking into account that $t_3 \in (t_2, t_{k1}^o)$, for $\tilde{t} = t_3 - t_2$ we get

$$u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_3-t_3+t_2)} = u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)} = \begin{cases} u_k^o(0) & \text{when } t \in [0, t_3 - t_2 + t'^3_{k-11}) \\ u'^o_{k-1}(t - t_3 + t_2)\big|_{\boldsymbol{x}_k(t_3)} & \text{when } t \ge t_3 - t_2 + t'^3_{k-11} \end{cases} \tag{18}$$

where

$$\boldsymbol{x}_k(t_3 - t_3 + t_2) = e^{A_k t_2}\boldsymbol{x}_k(0) + \int_0^{t_2} e^{A_k(t_2-\tau)} B_k u_k^o(0) d\tau.$$

For $t = 0$ it follows from (18)

$$u'^o_{k-1}(0)\big|_{\boldsymbol{x}_k(t_2)} = u_k^o(0)$$

where

$$\boldsymbol{x}_k(t_2) = e^{A_k t_2}\boldsymbol{x}_k(0) + \int_0^{t_2} e^{A_k(t_2-\tau)} B_k u_k^o(0) d\tau$$

which contradicts (16). Hence the assumption (17) is not true and thus the second part of Theorem 4 in case 1 is proved.

In case 2, the initial point for the New Problem *A(k-1)* coincides with the terminal point (the origin of the state space of Problem *A(k-1)*) and therefore

$$u'^o_{k-1}(t)\big|_{\boldsymbol{x}_k(t_2)} = 0. \tag{19}$$

If we suppose there exists a point $\boldsymbol{x}_k(t_3) \in \{\boldsymbol{x}_k^{1+}\}$ satisfying

$$\begin{aligned} &u'^o_{k-1}(0)\big|_{\boldsymbol{x}_k(t_3)} = u_k^o(0) \\ &\boldsymbol{x}_k(t_3) = e^{A_k t_3}\boldsymbol{x}_k(0) + \int_0^{t_3} e^{A_k(t_3-\tau)} B_k u_k^o(0) d\tau \end{aligned} \tag{20}$$

then similarly to case 1 we get

$$u'^{o}_{k-1}(0)\Big|_{x_k(t_2)} = u^o_k(0)$$

which contradicts (19). Hence the assumption (20) is not true and thus the second part of the theorem in case 2 is proved. This completes the proof of Theorem 4.

This result makes possible to develop a fast time-optimal control synthesis algorithm, modifying the basic synthesis algorithm in the following way.

Suppose Problem *A(k-1)* is solved and the optimal control $u^o_k(0) = u_k(\boldsymbol{x}_k(0))$ is obtained. If a zero value of x_{kw} is found, then the computed solution of Problem *A(k-1)* is a solution of Problem *A(k)*, i.e.

$$u^o_{k-1}(t) = u^o_k(t)\,,\ t^o_{k-1f} = t^o_{kf}\,.$$

If $x_{kw} \neq 0$, the optimal control $u^o_k(0) = u_k(\boldsymbol{x}_k(0))$ is applied to system (1) and for the generated trajectory points New Problem *A(k)* is consecutively defined. The corresponding New Problem *A(k-1)* is then solved and x_{kw} is computed. The movement along the trajectory continues until $x_{kw} = 0$ is reached. If, during this movement, in some point

$$\boldsymbol{x}_k(t_1) = e^{A_k t_1}\boldsymbol{x}_k(0) + \int_0^{t_1} e^{A_k(t_1-\tau)} B_k u^o_k(0)\,d\tau$$

one obtains

$$u'^{o}_{k-1}(0)\Big|_{x_k(t_1)} = u^o_k(0)$$

then a jump is made along the trajectory from $\boldsymbol{x}_k(t_1)$ to the point

$$\boldsymbol{x}_k(t_2) = e^{A_k t_2}\boldsymbol{x}_k(0) + \int_0^{t_2} e^{A_k(t_2-\tau)} B_k u^o_k(0)\,d\tau$$

$$0 \le t_1 < t_2 = t_1 + t'^{1}_{k-11},\ t_2 \in (0, t^o_{k1}),\ 2 \le k \le n$$

where t'^{1}_{k-11} is the length of the first constancy interval of $u'^{o}_{k-1}(t)\Big|_{x_k(t_1)}$.

Thus the solving of New Problem *A(k-1)* is avoided for the points between $\boldsymbol{x}_k(t_1)$ and $\boldsymbol{x}_k(t_2)$, and the movement along the trajectory continues further $\boldsymbol{x}_k(t_2)$ until reaching $x_{kw} = 0$.

IV. Numerical Example

Let $n = 7$ and the Problem *A(7)* data be

$$A_7 = \mathrm{diag}(0, -2, -4, -6, -8, -10, -12),\quad u_0 = 1,$$

$$B_7 = \begin{bmatrix}1 & -6 & 15 & -20 & 15 & -6 & 1\end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{x}_7(0) = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^{\mathrm{T}}.$$

In the class of problems *A(7), A(6), ..., A(1)*, Problem *A(1)* is solved analytically. For problems *A(k)*, $k = \overline{n,2}$, an 2ε hyper cube is set around the state space origin. It is assumed that the terminal state for a given problem can be any point in the corresponding ε area. An admissible control is considered as an approximated problem solution if it is the optimal control making possible to reach a terminal state within the ε area.

Denote by $u^{\tilde{o}}_k(t)$ and $t^{\tilde{o}}_{kf}$, $k = \overline{n,1}$, the approximated solutions for the class of problems *A(n), A(n-1), ..., A(1)*. The controls $u^{\tilde{o}}_k(t)$ are piecewise constant functions with no more than *k* non-zero constancy intervals with lengths $t^{\tilde{o}}_{ki}$ and signs $s^{\tilde{o}}_{ki}$, respectively.

After an axes initialization [2], an approximated problem solution is computed using the proposed fast algorithm for time-optimal control synthesis. The results obtained are presented in Table 1 and Fig. 1 and 2. In Fig. 2 the time-optimal system output

$$y = C\boldsymbol{x_7},\ C = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

is also shown.

V. Concluding Remarks

In this paper a new approach to the time-optimal control synthesis problem for a class of linear systems is presented. In contrast to the existing time-optimal control synthesis methods, the new approach does not require the description of the switching hyper surface and thus enables the synthesis of time-optimal control for high order systems of the given class.

The presented approach is based on the state-space properties of the considered class of problems and consists of two main stages. The first one comprises the state-space analysis called axes initialization while at the second one the optimal control is obtained. Both stages use a multi-step time-optimal control synthesis procedure for the problems of the considered class.

This paper is focused on the second stage of the synthesis procedure and presents a fast algorithm for synthesis of time-optimal control for the considered class of systems. The algorithm makes possible to avoid the solution of the corresponding optimal control sub-problems for a part of the optimal trajectory and thus enables efficient design and implementation of time-optimal control for high order systems.

TABLE 1

i	$t^{\tilde{o}}_{7i}$ [s]	$s^{\tilde{o}}_{ki}u_0$	$\boldsymbol{x}_7(t^{\tilde{o}}_{7f})$
1	1.457	-1.0	-4.441 e-16
2	0.710	1.0	-1.643 e-04
3	0.414	-1.0	1.280 e-04
4	0.264	1.0	-1.858 e-04
5	0.166	-1.0	-2.049 e-04
6	0.091	1.0	-1.625 e-04
7	0.029	-1.0	-0.952 e-04
$t^{\tilde{o}}_{7f}$ = 3.132 [s]			ε = 2.500 e-04

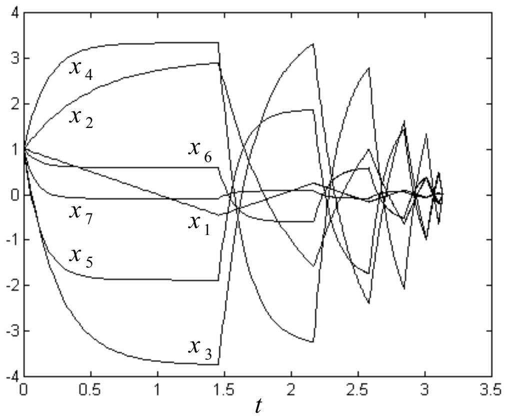

Fig. 1

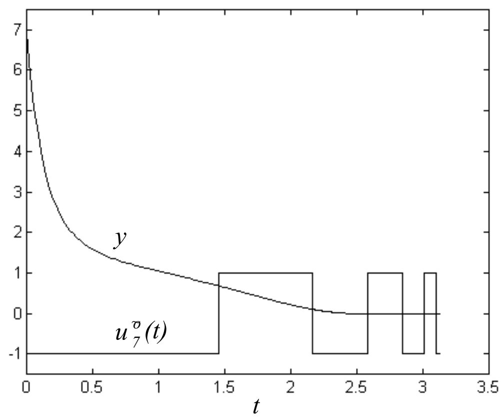

Fig. 2

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