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State-Feedback Optimal Controllers for Deterministic Nonlinear Systems

Chang-Hee Won*,

***Abstract*— A full-state feedback optimal control problem is solved for a general deterministic nonlinear system. The solution method is based on transforming Hamilton-Jacobi equation into an algebraic equation using the pseudo-inverse. Then we interpret the value function in terms of the control Lyapunov function and provide the stabilizing controller and the stability margins. We also derive an optimal controller for a nonlinear system which requires a solution of the state dependent Riccati equation. Simple examples demonstrate each method.**

I. Introduction

Linear optimal control theory is a well developed area, however, nonlinear optimal control theory is not as well developed as the linear version. One of the dif£culties of the nonlinear optimal control is in solving the Hamilton-Jacobi (HJ) partial differential equation. In this paper, we will convert the HJ partial differential equation problem into £nding a pseudo-inverse and solving an algebraic equation problem. Recently, a simple inversion method was proposed by Aliyu [1] for a nonlinear H_∞ control problem. We will utilize some of his concepts in this paper. Moreover, interest in control Lyapunov function generated renewed interest in nonlinear optimal control. In 2000, Primbs *et al.* solved a nonlinear control problem using a new class of receding horizon control and control Lyapunov function [9]. This control scheme relates pointwise min-norm control with optimal control, but it does not solve nonlinear optimal control problem. In 2004, Curtis and Beard solved a nonlinear control problem using the satis£cing sets and control Lyapunov function [4]. This framework completely characterizes all universal formulas, but once again it is not optimal control in the traditional sense. Our paper utilizes some of the concepts introduced in the above papers to solve traditional optimal control problem for a deterministic nonlinear system.

In this paper, we derive the optimal full-state-feedback solution of a deterministic af£ne nonlinear system. Then we compare the solutions with the inversion method of Aliyu [1], and we relate the value function with the control Lyapunov function. Finally, we derive the state dependent Riccati Equation (SDRE) controller for a nonlinear system.

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*C. Won is with the Department of Electrical Engineering, University of North Dakota, Grand Forks, ND 58202-7165, Phone (701)777-3368, won@und.edu

II. Hamilton-Jacobi Equation and Optimal Controller

Consider the following system,

$$dx(t) = f(x(t), u(t))dt = g(x(t))dt + B(x(t))u(t)dt, \quad (1)$$

where $g : \mathbb{R}^n \to \mathbb{R}^n, B : \mathbb{R}^n \to \mathbb{R}^{n\times m}, x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. Assume that g and B are continuously differentiable in all arguments. Furthermore, consider the following cost function

$$J(x, k) = \int_{t_0}^{t_F} l(x) + k'(x)R(x)k(x)d\tau, \quad (2)$$

where $l(x)$ is continuously differentiable, $l(x) \geq 0$, and $R(x) > 0$. We also assume that $[g(x), l(x)]$ is zero-state detectable. The optimization problem is to design a state-feedback controller, $k(x)$, that will minimize the cost function,

$$V_1^*(x) = \inf_k J(x, k). \quad (3)$$

Now, we optimize the Hamilton-Jacobi equation to £nd the optimal controller,

$$0 \;=\; \inf_k \left\{ (g + Bk)' \frac{\partial V_1^*}{\partial x} + l + k'Rk \right\}, \quad (4)$$

where we suppressed the argument x. The minimizing controller is obtained as

$$k^* = -\frac{1}{2} R^{-1}(x) B'(x) \frac{\partial V_1^*(x)}{\partial x}, \quad (5)$$

where $\frac{\partial V_1^*(x)}{\partial x}$ is a column vector. Second order condition, $R(x) > 0$, is satis£ed also. Therefore, the minimum is guaranteed, and the controller is a candidate for an optimal controller.

Instead of £nding V_1^*, we will determine an explicit expression for $\frac{\partial V_1^*(x)}{\partial x}$. Assuming that there exists a smooth, $V_1^* : \mathbb{R}^n \to \mathbb{R}^+$, substituting Eq. (5) into Eq. (4), and suppressing the argument x, we obtain a Hamilton Jacobi (HJ) equation

$$\frac{1}{4} \frac{\partial V_1^*}{\partial x}' BR^{-1}B' \frac{\partial V_1^*}{\partial x} - g' \frac{\partial V_1^*}{\partial x} - l = 0. \quad (6)$$

The above HJ equation implies the following HJ inequality for smooth $V_1^* \geq 0$

$$\frac{1}{4} \frac{\partial V_1^*}{\partial x}' BR^{-1}B' \frac{\partial V_1^*}{\partial x} - g' \frac{\partial V_1^*}{\partial x} - l \leq 0. \quad (7)$$

To £nd the optimal controller we introduce the slightly modi£ed version of the lemma by Liu and Leake [8], [10].

Our version gives two solutions and uses pseudo-inverse. Let $x \in \mathbb{R}^n$ be a real n-vector, $z(x)$ and $y(x)$ be real r-vector functions, and $\alpha(x)$ be a real function de£ned on $\mathbb{R}^n$.

Lemma 2.1: Let $y(x)$ and $\alpha(x)$ be given. Then $z(x)$ satis£es the condition

$$z'(x)z(x) + 2z'(x)y(x) + \alpha(x) = 0 \tag{8}$$

if and only if $|y(x)|^2 \geq \alpha(x)$. In such a case, the set of all solutions to (8) is represented by

$$z(x) = -y(x) \pm \beta(x)a(x) \tag{9}$$

where $\beta(x) = \left(|y(x)|^2 - \alpha(x)\right)^{\frac{1}{2}}$ and $a(x)$ is an arbitrary unit vector.

Proof. The suf£ciency follows by direct evaluation. To show the conditions are necessary, note that $|y|^2 < \alpha$ implies that

$$|z + y|^2 < 0,$$

which is a contradiction. Let $\pm w = z + y$, then (8) implies that $w'w = \beta^2$. Take $a = \dfrac{w}{|w|}$, we have $w = \pm\beta a$. Consequently, (9) follows. □

Lemma 2.2: (Liu and Leake's Lemma). Let X be a non-negative de£nite symmetric-real matrix. Then $z(x)$ satis£es the condition

$$z'(x)Xz(x) + 2z'(x)y(x) + \alpha(x) = 0 \tag{10}$$

if and only if

$$y'(x)X^{+}y(x) \geq \alpha(x), \tag{11}$$

where X^{+} is the pseudo-inverse of X. In this case, the set of all solutions to (10) is represented by

$$z(x) = -X^{+}y(x) \pm \sqrt{H^{+}}a(x)\beta(x) \tag{12}$$

where

$$\beta(x) = \left(y'(x)X^{+}y(x) - \alpha(x)\right)^{\frac{1}{2}}, \tag{13}$$

H is a non-singular matrix such that $X = H'H$, and $a(x)$ is an arbitrary unit vector.

Proof. The existence of such an H is well-known. The proof of this lemma follows from Lemma 2.1 by a change of variables $\hat{z} = Hz$ and $\hat{y} = (H^{+})'y$. □

In the next theorem the notation $\otimes$ is used for Kronecker product and I_n is used for $n \times n$ identity matrix. Now, we state the main theorem of this section.

Theorem 2.1: Assume V_1^* is smooth. Let

$$P(x) = B(x)R^{-1}(x)B'(x) \tag{14}$$

and

$$\rho(x) = \sqrt{P(x)}a(x)\sqrt{g'(x)P^{+}(x)g(x) + l(x)}. \tag{15}$$

For an af£ne nonlinear system (1), the optimal controller that minimizes the cost function (2) is given by

$$k^*(x) \quad = \quad -R^{-1}(x)B'(x)P^{+}(x)\left[g(x) \pm \rho(x)\right], \tag{16}$$

if the following symmetry and non-negative de£nite requirements are satis£ed.

$$\begin{aligned} &P^{+}\left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right) + \frac{\partial P^{+}}{\partial x'}(I_n \otimes (g \pm \rho)) \\ &\quad = \left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right)' P^{+} + (I_n \otimes (g \pm \rho))' \left(\frac{\partial P^{+}}{\partial x'}\right)' \end{aligned} \tag{17}$$

and

$$P^{+}\left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right) + \frac{\partial P^{+}}{\partial x'}(I_n \otimes (g \pm \rho)) \geq 0. \tag{18}$$

Proof. Utilizing the Liu and Leake's Lemma on Eq. (6) with $z = \dfrac{\partial V_1^*}{\partial x}, X = (BR^{-1}B'/4), y = -g/2$, and $\alpha = -l$, we have

$$\begin{aligned} \frac{\partial V_1^*}{\partial x} &= -\frac{1}{2}\left(\frac{BR^{-1}B'}{4}\right)^{+}(-g) \pm \sqrt{\left(\frac{BR^{-1}B'}{4}\right)^{+}}a \\ &\quad \sqrt{\frac{1}{4}g'\left(\frac{1}{4}BR^{-1}B'\right)^{+}g + l} \\ &= 2(BR^{-1}B')^{+}g \pm 2\sqrt{(BR^{-1}B')^{+}}a \\ &\quad \sqrt{g'\left(BR^{-1}B'\right)^{+}g + l}. \end{aligned} \tag{19}$$

From Eq. (5), the optimal controller is given as

$$\begin{aligned} k^* &= -R^{-1}B'(BR^{-1}B')^{+}g \mp R^{-1}B'\sqrt{(BR^{-1}B')^{+}}a \\ &\quad \sqrt{g'\left(BR^{-1}B'\right)^{+}g + l} \\ &= -R^{-1}B'(BR^{-1}B')^{+}\Big[g \pm \sqrt{BR^{-1}B'}a \\ &\quad \sqrt{g'\left(BR^{-1}B'\right)^{+}g + l}\Big], \end{aligned} \tag{20}$$

where the condition (11) is satis£ed because $g'(BR^{-1}B')^{+}g + l \geq 0$.

In order to solve the HJ equation (6), we have to solve for ρ with two requirements. The £rst requirement is that V_1^* is symmetric and the second requirement is that V_1^* is positive de£nite. Using Eqs. (14) and (15), Eq. (19) can be written as

$$\frac{\partial V_1^*(x)}{\partial x} = 2P^{+}(x)(g(x) \pm \rho(x)), \tag{21}$$

and the second partial derivative of the value function is given as

$$\begin{aligned} \frac{\partial^2 V_1(x)}{\partial x \partial x'} &= 2P^{+}(x)\left(\frac{\partial g(x)}{\partial x'} \pm \frac{\partial \rho(x)}{\partial x'}\right) \\ &\quad + 2\frac{\partial P^{+}(x)}{\partial x'}\left(I_n \otimes (g(x) \pm \rho(x))\right). \end{aligned} \tag{22}$$

And $\dfrac{\partial^2 V_1}{\partial x \partial x'} = \left(\dfrac{\partial^2 V_1}{\partial x \partial x'}\right)'$ imply that V_1^* is a scalar function and symmetric [1]. Thus, from Eq. (22), the symmetry requirement is given as (17) and the non-negative de£nite requirement is given as (18). □

Remark. Note that the value function, $V_1^*(x)$ does not have to be explicitly determined.

Remark. The symmetry requirement, (17) is satis£ed for all scalar cases. Furthermore, if $P^+(x)$ is a constant matrix then we obtain a simpler symmetry requirement,

$$P^+\left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right) = \left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right)' P^+.$$

Remark. Note that the above optimal controller (16) is in the same form as the H_∞ optimal controller obtain by Aliyu using the inversion method [1]. Also note that $a(x)$ is a free parameter whose only constraint is that its magnitude is one.

Remark. Note that V_1^* serves as a Lypunov function, thus with the usual zero-state detectability condition on $(g(x), l(x))$ [2], guarantees the asymptotic stability of the optimal feedback system.

Corollary 2.1: Assume that $l(x) \geq 0$ and $R(x) > 0$ are bounded for all x. Then an in£nite horizon cost function,

$$J(x,k) = \lim_{t_F \to \infty} \int_{t_0}^{t_F} x'(\tau)Q(x)x(\tau) + k'(x)R(x)k(x)d\tau, \tag{23}$$

converges to the minimal value and the optimal controller is determined to be

$$k^*(x) \quad = \quad -R^{-1}(x)B'(x)P^+(x)\left[g(x) \pm \rho(x)\right], \tag{24}$$

which is the same controller as in Eq. (16).

Proof. Optimal controller can be derived as in the £nite time horizon case. We need to show that the cost function converges to a limiting value. Eq. (23) converges to a £nite number as $t_F \to \infty$ because we assumed that $l(x)$ and $R(x)$ are bounded. This number is an upper bound for the minimal value of the criterion. Moreover, Eq. (23) is monotonically nondecreasing function of t_F because $l(x) \geq 0$ and $R(x) > 0$ for all x. This proves that the minimal value of J has a limit as $t_F \to \infty$ because it has a upper bound and it is monotonically decreasing. Thus the optimal controller (16) is a solution for the in£nite horizon case. □.

Example 1. As an example, we will consider a version of Freeman and Kototovic's example given in [5], [6]. Consider the following system

$$dx(t) = [x(t) - x^3(t)]dt + u(t)dt$$

with a cost function

$$J = \int_{t_0}^{t_F} \frac{1}{2}(x^2 + u^2)dt,$$

Now, £nd the solutions of minimal £rst cumulant control problem. In this example, we take $g(x) = x - x^3, B = 1, R = 1/2$, and $l(x) = x^2/2$. We determine that $P = 2$ and $\rho(x) = a(x)\sqrt{x^6 - 2x^4 + 2x^2}$. Substituting these values into the optimal controller equation (24), we have

$$k^* = -x + x^3 - \sqrt{x^6 - 2x^4 + 2x^2}, \tag{25}$$

where $a(x) = 1$ for the scalar case. This is the same optimal controller obtained using pointwise min-norm control law in [6]. The £rst requirement (17) is satis£ed because this is a scalar example. The second requirement (18) is numerically determined to be satis£ed for $0 < x < 0.74066$. Figure 1 shows that how the state approaches zero exponentially using the optimal control (25). The initial states were chosen as $0.9, 0.5$, and 0.1. The $a(x)$ was chosen to be one.

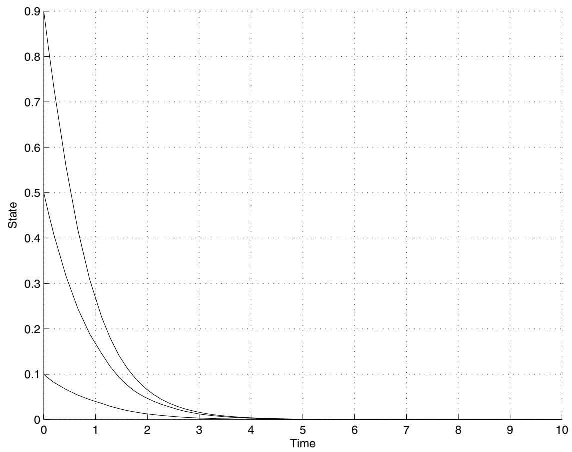

Fig. 1. The state, x, as a function of time, t.

Example 2. Consider the following second order nonlinear system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^3 + x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{26}$$

with the state weighting matrix, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the control weighting matrix, $R = 4/3$. Using Eqs. (14) and (15), we have

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 3/4 \end{bmatrix} \quad P^+ = \begin{bmatrix} 0 & 0 \\ 0 & 4/3 \end{bmatrix}$$

and

$$\rho(x) = \begin{bmatrix} 0 \\ \frac{\sqrt{7}}{2} a_2(x) x_2 \end{bmatrix}$$

where $a(x) = [a_1(x)a_2(x)]'$ is the arbitrary unit vector. Then using Eq. (16) and (21) we £nd

$$\frac{\partial V_1^*}{\partial x} = \begin{bmatrix} 0 \\ \left(1 \pm \frac{\sqrt{7}}{2} a_2(x)\right) x_2 \end{bmatrix}$$

and

$$k^* = -\left(1 \pm \frac{\sqrt{7}}{2} a_2(x)\right) x_2.$$

It is straight forward to verify that the £rst requirement, Eq. (17) is satis£ed. also the second requirement is found to be

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{4}{3} \pm 2\frac{\sqrt{2}}{3} a_2(x) \end{bmatrix} \geq 0.$$

The second requirement is satis£ed for an appropriate $a_2(x)$. Figure 2 is the phase portrait of the system for various initial values of x_1 and x_2. In plotting Figure 2, we used $a_2(x) = 1$ and the second requirement was satis£ed. To compare with the pointwise min-norm controller, Figure 3 shows the phase portrait of the same system using the pointwise min-norm controller.

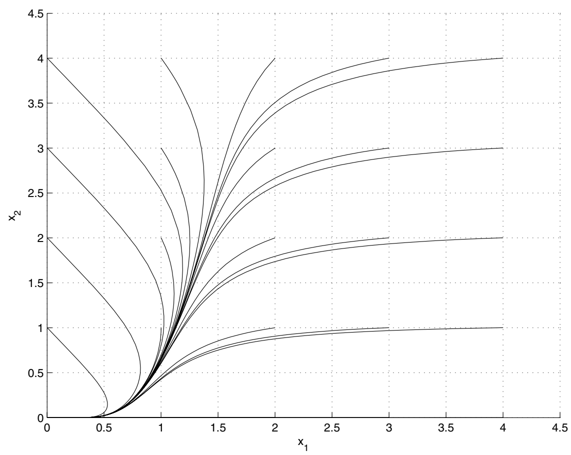

Fig. 2. The phase portrait of Example 2 using an optimal controller

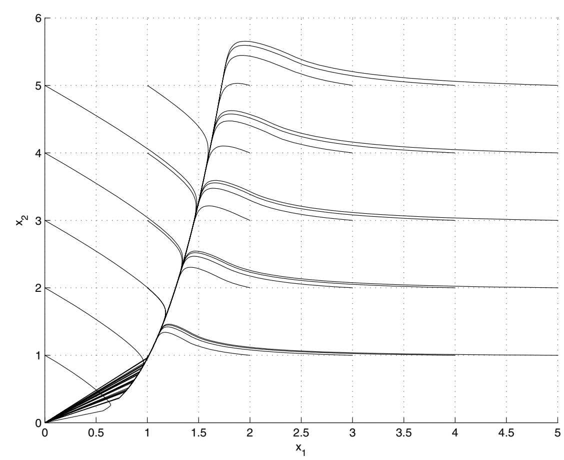

Fig. 3. The phase portrait of Example 2 using a min-norm controller

A. Solutions via Inversion Method

Now we will £nd the solutions to the nonlinear deterministic optimization problem using the inversion method of [1] and show that it is equivalent to our solution. Let

$$P(x) = B(x)R^{-1}(x)B'(x).$$

Also, let M and N be differential manifolds which are locally Euclidean and compact. TM is the tangent bundle of M with dimensions $2n$.

Theorem 2.2: For the HJ equation (6), we assume that there exists a vector £eld $\rho : N \to TM$, such that

$$0 = -\rho'(x)P^{+}(x)\rho(x) + g'(x)P^{+}(x)g(x) + l(x) \quad (27)$$

where the matrix P^{+} is the pseudo-inverse of P. Then

$$\frac{\partial V_1^*(x)}{\partial x} = 2P^{+}(x)(g(x) \pm \rho(x)), \quad (28)$$

satis£es the HJ equation (6).

Proof. Direct substitution of Equation (28) into Equation (6) gives the discriminant equation (27). □

Remark. Equation (27) is called the discriminant equation. This equation is a £rst order quasi-linear partial differential equation that can be solved by the classic method of characteristics. This discriminant equation corresponds to the Riccati equation in linear quadratic regulator problem. In order to £nd the optimal controller, we have to solve the discriminant equation with two requirements given in Eqs. (17) and (18).

The optimal controller is given by the following equation,

$$k^* = -R^{-1}(x)B'(x)P^{+}(x)(g(x) \pm \rho(x)),$$

which is equivalent to Eq. (16).

B. Interpretation via Control Lyapunov Function

If the control problem is to £nd a feedback control law, $k(x)$ such that the closed loop system

$$\dot{x} = f(x, k(x)) \quad (29)$$

is globally asymptotically stable at the equilibrium point $x = 0$, then we can choose a function $V_1(x)$ as a Lyapunov function, and £nd $k(x)$ to guarantee that for all $x \in \mathbb{R}^n$ such that

$$\frac{\partial V_1(x)}{\partial x}'(g(x) + B(x)k(x)) \leq -W(x). \quad (30)$$

A system for which an appropriate choice of $V_1(x)$ and $W(x)$ exists is said to possess a control Lyapunov function (clf). Following [7] we de£ne the control Lyapunov function, $V_1(x)$, of system (29) to be smooth, positive de£nite, and radially unbounded if

$$\inf_u \left\{ \frac{\partial V_1}{\partial x} f(x, u) \right\} < 0, \quad (31)$$

for all x not equal to zero.

In order to relate the clf inequality to the HJ equation of the optimal control, we introduce the following lemma by Curtis and Beard with a slight modi£cation [4].

Lemma 2.3: If $S = S' > 0$ then the set of solutions to the quadratic inequality

$$\zeta' S\zeta + d'\zeta + c \leq 0 \quad (32)$$

where $\zeta \in \mathbb{R}^n$ is nonempty if and only if

$$\frac{1}{4}d'S^{-1}d - c \geq 0 \quad (33)$$

and is given by

$$\zeta = -\frac{1}{2}S^{-1}d + S^{-1/2}v\sqrt{\frac{1}{4}d'S^{-1}d - c} \quad (34)$$

where $v \in B(\mathbb{R}^n) = \{\zeta \in \mathbb{R}^n : \|\zeta\| \leq 1\}$.

Proof. See [4], and modify the inequalities.

Now, we state the theorem that gives all stabilizing controller.

Theorem 2.3: Assume that $V_1(x)$ is smooth, positive de£nite, and radially unbounded function. Assume also $W(x) = l(x) + k'R(x)k$. For the nonlinear system (1), all state feedback stabilizing controllers are given by

$$k = -\frac{1}{2}R^{-1}B'\frac{\partial V_1}{\partial x} + R^{-1/2}v \sqrt{\frac{1}{4}\frac{\partial V_1}{\partial x}' BR^{-1}B'\frac{\partial V_1}{\partial x} - g'\frac{\partial V_1}{\partial x} - l} \quad (35)$$

if and only if

$$\frac{1}{4}\frac{\partial V_1}{\partial x}' BR^{-1}B'\frac{\partial V_1}{\partial x} - g'\frac{\partial V_1}{\partial x} - l \geq 0. \quad (36)$$

Proof. From (30), we obtain the control Lyapunov function inequality as

$$\frac{\partial V_1}{\partial x}'(g + Bk) + l + k'Rk \leq 0. \quad (37)$$

Then we let $\zeta = k, S = R, d = B'\frac{\partial V_1}{\partial x}$, and $c = g'\frac{\partial V_1}{\partial x} + l$ and use Curtis and Beard's Lemma 2.3 on Eq. (37), to obtain all the stabilizing controller. □

Remark. Equation (7) states that HJ equation for the optimal value function is less than or equal to zero. Equation (36) states that HJ-type equation for a clf function, V_1, has to be greater than or equal to zero. Thus, for the optimal value function to be a clf, the HJ equation has to be equal to zero.

Remark. Following [4], we can also show that k in Eq. (35) is inversely optimal for some $\tilde{R}(x)$ and $l(x)$.

Here we discuss the robustness property of the optimal controller. We follow the de£nition of stability margin given in [4]: an asymptotically stabilizing control law, $u = k(x)$, has stability margins (m_1, m_2) where $-1 \leq m_1 < m_2 \leq \infty$, if for every $\alpha \in (m_1, m_2), u = (1 + \alpha)k(x)$, also asymptotically stabilizes the system.

Theorem 2.4: If $k(x)$ is stabilizing control of (35) and

$$\frac{\partial V_1}{\partial x}' B(x)R^{-1/2}(x)v(x) \leq 0$$

then it has stability margin of $(-1/2, \infty)$.

Proof. We closely follow the proof given for the robust satis£cing control in [4, Theorem 14]. From (30), we obtain the control Lyapunov function inequality as

$$\frac{\partial V_1}{\partial x}'(g + Bk) \leq -l - k'Rk.$$

Add $\alpha\frac{\partial V_1}{\partial x}' Bk$ to both sides to obtain

$$\frac{\partial V_1}{\partial x}' g + (1+\alpha)\frac{\partial V_1}{\partial x}' Bk \leq -l - k'Rk + \alpha\frac{\partial V_1}{\partial x}' Bk. \quad (38)$$

Non-positiveness of the right hand side guarantees asymptotic stability. Eq. (35) is rewritten as

$$k = -\frac{1}{2}R^{-1}B'\frac{\partial V_1}{\partial x} + R^{-1/2}v\sigma.$$

Substituting this k into Eq. (38), and after some algebraic manipulation we obtain,

$$-l - k'Rk + \alpha\frac{\partial V_1}{\partial x}' Bk = -l - \sigma^2 v'v - \frac{1}{2}(\frac{1}{2} + \alpha) \frac{\partial V_1}{\partial x}' BR^{-1}B'\frac{\partial V_1}{\partial x} + (1 + \alpha)\frac{\partial V_1}{\partial x}' BR^{-1/2}v\sigma.$$

The £rst and second term on the right side of the equality are nonpositive. The third term on the right side are nonpositive if $\alpha \in (-1/2, \infty)$, The last term on the right side are nonpositive if $\alpha \in (-1/2, \infty)$ and $\frac{\partial V_1}{\partial x}' B(x)R^{-1/2}(x)v(x) \leq 0$. □

III. State Dependent Riccati Equation

Here we will derive the State Dependent Riccati Equation (SDRE) for a value function that is in quadratic form using Kronecker products. We £nd the partial differential equations from the necessary conditions of optimality.

Theorem 3.1: Assume that $l(x) = x'Qx$ and $V_1^*(x)$ is a symmetric nonnegative de£nite matrix,

$$V_1^*(x) = x'\mathcal{V}_1(x)x. \quad (39)$$

For the nonlinear system given by (1) with $g(x) = A(x)x(t)$, the optimal controller that minimizes the cost function (2) with $l(x) = x'Q(x)x$, is given by (suppressing the argument x)

$$k^* = -\frac{1}{2}R^{-1}B'\frac{\partial V_1}{\partial x} = -\frac{1}{2}R^{-1}B'\left(2\mathcal{V}_1 x + (I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x\right). \quad (40)$$

Provided that the following conditions are satis£ed:

$$0 = A'\mathcal{V}_1 + \mathcal{V}_1 A + Q - \mathcal{V}_1 BR^{-1}B'\mathcal{V}_1, \quad (41)$$

$$\sqrt{R^{-1}}B'(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x} = \sqrt{R^{-1}}B'\left[\frac{\partial \mathcal{V}_1}{\partial x} \otimes x'\right] = 0, \quad (42)$$

and

$$A(x)'(I_n \otimes x')\frac{\partial \mathcal{V}_1(x)}{\partial x} = 0. \quad (43)$$

Proof. Suppressing arguments for simplicity, we obtain the following HJ equation using Eqs. (4) and (5)

$$0 = x'A'\frac{\partial V_1^*}{\partial x} - \frac{1}{4}\left(\frac{\partial V_1^*}{\partial x}\right)' BR^{-1}B'\frac{\partial V_1^*}{\partial x} + x'Qx. \quad (44)$$

This is the necessary partial differential equations for the optimal control problem. Because $V_1(x)$ is assumed to be symmetric nonnegative de£nite matrix, we have

$$\frac{\partial V_1^*}{\partial x} = 2\mathcal{V}_1(x)x + (I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x, \quad (45)$$

where $\otimes$ is the Kronecker product. Substitute the above equations into Equation (44) to obtain

$$\begin{aligned} 0 &= x'Qx - x'\mathcal{V}_1 BR^{-1}B'\mathcal{V}_1 x \\ &\quad - \left[(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x\right]' BR^{-1}B'\mathcal{V}_1 x \\ &\quad -\frac{1}{4}\left[(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x\right]' BR^{-1}B' \left[(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x\right] \\ &\quad + x'A'2\mathcal{V}_1 x + x'A'\left[(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}x\right]. \end{aligned} \tag{46}$$

Rewriting the above equation, we get

$$\begin{aligned} 0 &= x'\left(Q - \mathcal{V}_1 BR^{-1}B'\mathcal{V}_1 + 2A'\mathcal{V}_1\right)x \\ &\quad + x'\left[-\frac{\partial \mathcal{V}_1}{\partial x}'(I_n \otimes x')'BR^{-1}B'\mathcal{V}_1\right. \\ &\quad -\frac{1}{4}\frac{\partial \mathcal{V}_1}{\partial x}'(I_n \otimes x)BR^{-1}B'(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x} \\ &\quad \left. + A'(I_n \otimes x')\frac{\partial \mathcal{V}_1}{\partial x}\right]x. \end{aligned} \tag{47}$$

Thus, we obtain the three conditions in Eqs. (41), (42), and (43). And the optimal controller is given by Eq. (40). □

Remark. Note that for the special case when $A(x)$ and $B(x)$ are in the phase-variable form of the state equation, the condition (42) is satis£ed. Furthermore, both conditions (42) and (43) are satis£ed if $\dfrac{\partial \mathcal{V}_1}{\partial x_i} = 0$ for $i = 1, \cdots, n$.

Example. Consider Example 2 in Section II. Just rewriting the system equation we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{48}$$

Substituting the same Q and R matrices as in Example 2, and using Eq. (41) we obtain,

$$\mathcal{V}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{4 \mp 2\sqrt{7}}{3} \end{bmatrix}.$$

And from Eq. (40) the optimal controller is determined to be

$$k^* = -\left(1 \pm \frac{\sqrt{7}}{2}\right)x_2.$$

This is the same controller as Section II, Example 2. Because $\mathcal{V}_1$ is a constant in this example, the state dependent Riccati equation (41) was the only requirement.

IV. Conclusions

For a general nonlinear system, we present a method to determine the optimal controller using pseudo-inverse method. Then we relate this method with the inversion method of Aliyu. Furthermore, the value function is related to the control Lyapunov function and this gives the stabilizing controller with good stability margin. State dependent Riccati equation optimal controller is derived for a general nonlinear system. Simple examples illustrate the developed optimal controller determination procedures.

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