A Duality-based LPV Approach to Polynomial State Feedback Design

Christian Ebenbauer, Tobias Raff, Frank Allgöwer
Institute for Systems Theory in Engineering (IST),
University of Stuttgart, 70550 Stuttgart, Germany
{ce,raff,allgower}@ist.uni-stuttgart.de

Abstract— **In this paper, a state feedback design for the class of polynomial control systems is proposed. The proposed design solves the stabilization problem by considering polynomial control systems as linear parameter varying (LPV) control systems. With the help of duality theory and the sum of squares decomposition, a stabilizing state feedback can be computed efficiently, if a certain semidefinite program is feasible. The main advantages of the proposed state feedback design are that no special requirements on the system structure are imposed and that the results are of global nature.**

I. INTRODUCTION

Nonlinear feedback design has been passed through a remarkable phase over the last two decades. Many descriptive concepts for nonlinear feedback analysis turned into constructive design strategies [7]. However, nonlinear feedback design is far away from being solved. Many existing feedback designs impose very particular assumptions on the control system which are hard to verify or simply not satisfied in real-world applications e.g. special requirements on the system structure or only local validity of the designed feedback. These requirements are often the main stumbling blocks for applying these strategies. Additionally, there are only a few computational tools available for designing nonlinear feedbacks. Recently, polynomial control systems have gained a lot of interest [6], [12], [13]. Polynomial control systems are control systems where the maps in the control system description are polynomial maps. This class of control systems includes the class of linear control system, and many nonlinear control problems can be formulated or approximated by polynomial control systems. In combination with semidefinite programming, in particular with the help of the sum of squares decomposition, many problems in polynomial control systems analysis and design have been attacked successfully, due to the fact, that semidefinite programs can be solved very efficiently on a computer. In [12], for example, a semidefinite programming approach based on state dependent Riccati techniques was proposed from which one obtains global stability only in case of a quadratic Lyapunov function. In [13], a semidefinite programming approach for feedback passivation was proposed. Feedback passivation is a well-established design strategy with a long history in control. However, one obstacle of this design tool is that a (fictitious) passivating output of the control system must be known, which may be hard to find, since no systematic procedures are available for finding passivating outputs.

In the present paper, a new computer-aided state feedback design approach is proposed which overcomes this limitation by considering polynomial control systems as linear parameter-varying (LPV) control systems. In particular, it is shown that the state feedback design can be formulated as semidefinite program by embedding polynomial control systems in parameterized linear control systems and by using duality theory. As a result, a state feedback and a control Lyapunov function (CLF) respectively, is obtained, if a certain semidefinite program is feasible. This feedback can be used to stabilize the original polynomial control system. The main advantages of the proposed design procedure are that no special requirements on the system structure are imposed, that the results are of global nature, and the linear case is included. The proposed state feedback design strategy is illustrated on a small example.

The remainder of the paper is organized as follows: In Section II the basic idea as well as the main state feedback design approach in terms of a semidefinite program is presented. A small example is given in Section III to illustrate the proposed controller design approach. Concluding remarks are given in Section IV.

Notations: A function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is called positive definite, if $V(0) = 0, V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and positive semidefinite if $V(x) \geq 0, \forall x \in \mathbb{R}^n$. The row vector $\nabla V(x) = (\partial V/\partial x)(x)$ denotes the derivative of $V(x)$ with respect to x. A control Lyapunov function (CLF) V of the control system $\dot{x} = f(x) + G(x)u$ is a radially unbounded positive definite function such that for every nonzero $x \in \mathbb{R}^n$ there exists a $u \in \mathbb{R}^p$ such that $\dot{V}(x) = \nabla V(x)f(x) + \nabla V(x)G(x)u < 0$. A polynomial $p(x)$ is a finite linear combination of monomials, i.e., $p(x) = \sum_\alpha c_\alpha x^\alpha = \sum_\alpha c_\alpha x_1^{\alpha_1}...x_n^{\alpha_n}$, where $c_\alpha \in \mathbb{R}$ and $\alpha = [\alpha_1, ..., \alpha_n], \alpha_i \in \mathbb{N}_0$. The degree of a polynomial is defined, $d = \sum_{i=1}^n \alpha_i$. The set of all polynomials with real coefficients is written as $\mathbb{R}[x]$. A polynomial vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n,\ f(x) = [f_1(x), ..., f_n(x)]^T$ is a vector field with $f_i \in \mathbb{R}[x]$, i.e., the entries of the vector field are polynomial functions in x.

II. State Feedback Design

The following problem is considered in this paper:

Stabilization Problem 1 Given a polynomial control system of the form[1]

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^p$ is the input. f is a polynomial vector field with $f(0) = 0$ and G is a polynomial map, i.e., $f_i, G_{ij} \in \mathbb{R}[x]$. Find a state feedback $u = k_1(x)$, such that the closed-loop is globally asymptotically stable w.r.t. the origin $x = 0$.

To put the polynomial control system (1) in relation with linear control systems, one can write the control system (1) as follows:

$$\dot{x} = A(x)x + G(x)u. \tag{2}$$

Note that the matrix $A(x)$ is not unique, i.e., there are different matrices $A(x)$ to write $f(x) = A(x)x$ (cf. also Remark 5). However, the basic idea is now to turn the state dependency of $A(x), G(x)$ into an parameter dependency, let say θ. Then, one arrives at the following linear control system $\dot{x} = A(\theta)x + G(\theta)u$, parameterized by θ. Instead of solving the original stabilization problem described above, one may solve the following problem:

Stabilization Problem 2 Given the parameterized linear control systems

$$\dot{x} = A(\theta)x + G(\theta)u. \tag{3}$$

Find a parameterized polynomial state feedback $u = k_2(x, \theta)$, and a common, parameter–independent, polynomial control Lyapunov function (CLF)[2] $V = V(x) > 0$ such that

$$\nabla V(x)[A(\theta)x + G(\theta)k_2(x, \theta)] < 0,$$

is satisfied in certain domain $\mathcal{D}_{(x,\theta)} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ which contains the points $(x, x) \in \mathcal{D}_{(x,\theta)}$ for all $x \in \mathbb{R}^n$.

Note that Problem 2 is more difficult than Problem 1, since the feedback has to stabilize for certain θ's with a common CLF V and not just for $\theta = x$. The motivation of introducing the domain $\mathcal{D}_{(x,\theta)} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is to reduce the conservatism of the state feedback design. Note that the smaller the distance between the domain $\mathcal{D}_{(x,\theta)}$ and the set $\{(x, x) \mid x \in \mathbb{R}^n\}$ becomes, the less conservative the state feedback design will be. However, Problem 2 has one significant advantage, namely, the sum of squares decomposition can be applied to search for a solution, which cannot be applied to the original problem. That the latter stabilization problem implies to original stabilization problem, is due to the fact, that a common parameter independent CLF has to be found in a certain domain $\mathcal{D}_{(x,\theta)}$. This is summarized in the next lemma:

Lemma 1 A solution of the Stabilization Problem 2 implies a solution the Stabilization Problem 1.

Proof Assume the Stabilization Problem 2 has a solution, i.e., there exists a parameterized feedback law $u = k_2(x, \theta)$ and a common CLF $V = V(x) > 0$ such that

$$\nabla V(x)[A(\theta)x + G(\theta)k_2(x, \theta)] < 0, \tag{4}$$

holds for all $(x, \theta) \in \mathcal{D}_{(x,\theta)}$ and for all nonzero $x \in \mathbb{R}^n$. Then, the Lyapunov inequality (4) holds in particular for $x = \theta$, which proofs stability of the original Stabilization Problem 1, with $u = k_1(x) = k_2(x, x)$. $\square$

To arrive at a semidefinite programming solution via the sum of squares decomposition, several steps are necessary since the unknown polynomial functions $\nabla V, k_2$ do not appear linearly in the Lyapunov inequality

$$\nabla V(x)A(\theta)x + \nabla V(x)G(\theta)k_2(x, \theta) < 0, \tag{5}$$

due to bilinear term $\nabla V(x)G(\theta)k_2(x, \theta)$. In the linear case without parameterizations, i.e., A, G are constant matrices, it is well-known [1] that the change of variable $z = Px$ with a quadratic CLF $V(x) = x^T Px$, $P > 0$ leads to a convex solution. Note that this change of coordinates can be perfectly interpreted in terms of duality theory via conjugate Lyapunov functions [5]. The same can be also done for the inequality (5). But instead of assuming that the CLF V is a quadratic form, it is enough to assume that V is a convex function. Furthermore, the domain $\mathcal{D}_{(x,\theta)}$, respectively, the domain $\mathcal{D}_{(z,\theta)}$ after the change of coordinates has to be specified, such that a convex formulation is possible and such that $\mathcal{D}_{(x,x)} \subseteq \mathcal{D}_{(x,\theta)}$ holds. Notice that the domain $\mathcal{D}_{(x,\theta)}$ specified below play a crucial role in the state feedback design, since it is introduced to restrict the parameter variation θ to the domain $\mathcal{D}_{(x,\theta)}$, i.e., to certain states in the state space. Therefore the conservatism in the design inequalities can be reduced and the design inequalities be satisfied more easily. This is worked out in the next Theorem.

Theorem 1 If the inequality

$$z^T A(\theta)\nabla V^\star(z)^T + z^T G(\theta)m(z, \theta) < 0, \tag{6}$$

has a solution for all (z, θ) in the domain

$$\mathcal{D}_{(z,\theta)} = \{(z, \theta) \ : \ z^T z - \theta^T \theta \geq 0, z^T \theta - \theta^T \theta \geq 0\},$$

where $V^\star : \mathbb{R}^n \to \mathbb{R}$ is a (strictly) convex, positive definite, polynomial CLF of the form $V^\star(z) = \frac{1}{2} z^T Pz + W^\star(z)$. $P > 0$, W is a higher degree (non–quadratic) polynomial,

[1]By putting an integrator in front of the control system, i.e., $u = \xi$, $\dot{\xi} = v$, it is always possible to write a control system $\dot{x} = f(x, u)$ in the form $\dot{x} = f(x) + G(x)u$.

[2]Note: A positive definite polynomial function is radially unbounded [4].

and $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is a polynomial function. Then the inequality

$$\nabla V(x)A(\theta)x + \nabla V(x)G(\theta)k_2(x,\theta) < 0, \qquad (7)$$

has a solution for $\theta = x$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a convex positive definite CLF given by $V(x) = V^\star((\nabla V^\star)^{-1}(x))$.

Proof Assume that inequality (6) has a solution. Since $V^\star$ is convex, polynomial, and positive definite, the mapping $x^T = \nabla V^\star(z)$ is bijective and, of course, continuously differentiable. This follows from the fact that $\frac{\partial^2 V^\star}{\partial z^2}(z)$ is positive definite, due to (strictly) convexity of the polynomial function [4]. Hence, the inverse mapping $z^T = (\nabla V^\star)^{-1}(x)$ is defined uniquely. Now, by the change of variables, one arrives at

$$(\nabla V^\star)^{-1}(x)\left[A(\theta)x + G(\theta)m((\nabla V^\star)^{-1}(x),\theta)\right] < 0, \quad (8)$$

which is equivalent to inequality (7) with $\nabla V(x) := (\nabla V^\star)^{-1}(x)$, $u = k_2(x,\theta) = m((\nabla V^\star)^{-1}(x),\theta)$. The function V is well-defined, differentiable, convex, and positive definite. This follows from $I = \partial x^T/\partial x = \partial \nabla V^\star(z)/\partial x = (\partial^2 V^\star(z)/\partial z^2)(\partial z^T/\partial x)$, hence $\partial z^T/\partial x = \partial(\nabla V^\star)^{-1}(x)/\partial x = \partial \nabla V(x)/\partial x = \left((\partial^2 V^\star/\partial z^2)(z)\right)^{-1} > 0$, i.e., the Hessian of V is positive definite, and $V(0) = 0$, because of $x^T = \nabla V^\star(z)$. Existence follows from Schwarz's integrability condition, i.e., from the symmetry of the Hessian. Finally, because of the transformation $x^T = \nabla V^\star(z)$, inequality (7) is satisfied in the domain

$$\mathcal{D}_{(x,\theta)} = \{(x,\theta) \ : \quad \nabla V(x)\nabla V(x)^T - \theta^T\theta \geq 0, \quad \nabla V(x)\theta - \theta^T\theta \geq 0\}.$$

Next, it has to be shown that $\mathcal{D}_{(x,x)} \subseteq \mathcal{D}_{(x,\theta)}$. Note that this is in general not true. However, consider the CLF $\rho(V(x))$ instead of $V(x)$, where ρ is a strictly monotonic function with $\rho(\infty) = \infty$. Then $\rho(V(x))$ is a CLF if and only if $V(x)$ is a CLF. But for an appropriate rescaling function ρ, the first inequality and the second inequality are satisfied, as shown below. For the first inequality $\rho_1'(V(x))^2\nabla V(x)\nabla V(x)^T - \theta^T\theta \geq 0$, it must be shown that there exists a strictly monotonic function ρ_1 such that

$$\rho_1'(V(x))^2\nabla V(x)\nabla V(x)^T - x^Tx \geq 0$$

holds. But this can be satisfied for ρ_1' sufficiently large, since $\nabla V(x)\nabla V(x)^T$ is positive definite. In the same way, the second inequality $\rho_2'(V(x))\nabla V(x)^T\theta - \theta^T\theta \geq 0$ can be satisfied for $\theta = x$, because

$$\rho_2'(V(x))\nabla V(x)x - x^Tx \geq 0$$

holds for a function ρ_2 with sufficient large derivative ρ_2'. This is because a function V is strictly convex if and only if $V(y) > V(x) + \nabla V(x)(y-x)$ for all x, y and hence for $y = 0$, $\nabla V(x)x$ is positive definite. Finally, it has to be shown that ρ is well-defined. In particular, it has to be shown that ρ' takes finite values for finite arguments. Which can be easily seen except for $\rho'(0)$. This is shown now. Notice that V is of the following form:

$$V(x) = \frac{1}{2}x^TP^{-1}x + W(x). \qquad (9)$$

This follows from $(\partial^2 V/\partial x^2)(0) = P^{-1}$ and from $(\partial^2 V/\partial x^2)(x) = \left((\partial^2 V^\star/\partial z^2)(z)\right)^{-1}$. Hence, ρ_i is well-defined because of $\nabla V(x) = (P^{-1}x + \nabla W(x))$, more precisely, because of ∇V is linear for small[3] x and hence $\nabla V(x)x$, $(\nabla V(x))(\nabla V(x))^T$ are quadratic for small x. Since, $\nabla V(x)x$, $(\nabla V(x))(\nabla V(x))^T$ are quadratic for small x a finite value $\rho'(0)$ is enough to dominate x^Tx. Finally, to get the function ρ, one takes a function such that $\rho'(V(x)) > sup\{\rho_1'(V(x)), \rho_2'(V(x))\}$. □

Remark 1 An explicit expression of V can be also obtained via convex analysis. More precisely, V is the conjugate function of $V^\star$, i.e., $V(x)=\max_{z\in\mathbb{R}^n}\{x^Tz - V^\star(z)\}$, from which the convexity and differentiability of V follows, since $V^\star$ is strictly convex. From the perspective of convex analysis, the proposed feedback design can be interpreted as a design based on duality since the inequality (7) is the dual to the inequality (6) and V and $V^\star$ are dual pair in terms of conjugate function duality (see also [5]).

Note that the unknowns $V^\star, m$ in the inequality (6) appear linearly. In addition, a solution, i.e., a CLF for the bilinear inequality (7) is obtained from the inequality (6). Finally, to arrive at a semidefinite programming solution, a sum of squares relaxation of the design inequalities $V^\star(z) > 0$ and (6) has to be done. Furthermore, the convex domain $\mathcal{D}_{(z,\theta)}$ has to be incorporated in the design inequalities. The choice of the domain $\mathcal{D}_{(z,\theta)}$ is basically motivated by the fact that the stabilization problem can be posed as a semidefinite program. However, notice that the inequality $z^Tz - \theta^T\theta \geq 0$ restricts the "magnitude" of θ w.r.t. z and $z^T\theta - \theta^T\theta \geq 0$ restricts the "phase" w.r.t. z.

Semidefinite Program The following polynomial design inequalities for the state feedback design can be solved by a semidefinite program (cf. Appendix, Lemma 1):

$$\begin{aligned} V^\star(z) - \mu_1 z^Tz &= SOS \\ -(z^TA(\theta)\nabla V^\star(z)^T + z^TG(\theta)m(z,\theta) + \mu_2 z^Tz) & \quad (10) \\ -(z^Tz - \theta^T\theta)s_1(x,\theta) - (z^T\theta - \theta^T\theta)s_2(x,\theta) &= SOS \end{aligned}$$

where SOS stands for a sum of squares polynomial and μ_1, μ_2 are a small positive constants, to ensure a strict inequality. s_1, s_2 are also SOS and $V^\star(z) = \frac{1}{2}z^TPz + W^\star(z)$, where $P > 0, W^\star(z) \geq 0$, convex. Convexity of $W^\star$ can be enforced by the (affine) first order convexity condition $W^\star(y) > W^\star(x) + \nabla W^\star(x)(y-x)$ [2]. If this semidefinite program is feasible, $V(x) = V^\star((\nabla V^\star)^{-1}(x))$ is a CLF for the Stabilization Problem 2.

[3] $(\partial^2 V/\partial x^2)(0) = P^{-1}$.

Stabilizing Feedback To recover a stabilizing feedback for the Stabilization Problem 1/2, different approaches are possible. One approach is to compute the CLF V from $V^\star$ and use Sontag's formula to obtain a stabilizing feedback [16], [14]. Another approach is to compute the CLF V from $V^\star$, and, in a second step, solve the inequality (7) for $\theta = x$, via semidefinite programming and the sum of squares decomposition, to obtain a stabilizing feedback $u = k_3(x)$. Note that the recovery of V from $V^\star$ is not an easy task.[4] However, one can bypass this problem as follows. Notice that the problem is to find a stabilizing state feedback $u = k_1(x)$ such that

$$(\nabla V^\star)^{-1}(x)\,[f(x) + G(x)k_1(x)] < 0,$$

holds. Although $(\nabla V^\star)^{-1}$ is not explicitly known, it is known implicitly, since $x^T = \nabla V^\star(z)$ holds. This can be utilized to solve

$$z^T\,(f(x) + G(x)k_1(x)) + s(x,z)^T(x - \nabla V^\star(z)^T) < 0,$$

where k_1 and s are vector-valued polynomial function. Notice that first whenever $x - \nabla V^\star(z) = 0$, $z = (\nabla V^\star)^{-1}(x)$ holds (cf. also Appendix, Lemma 1) and second that the rescaling function ρ in the proof of Theorem 1 is not needed in the design.

Remark 2 Note that the proposed semidefinite program is a sufficient condition for the Stabilization Problem 1, due to the fact that the Stabilization Problem 2 is solved. This fact makes the design procedure conservative.

Remark 3 In the "classical" case of a quadratic Lyapunov function $V^\star(z) = \frac{1}{2}z^TPz$, $P > 0$, V is given by $V(x) = \frac{1}{2}x^TP^{-1}x$. Hence, the stabilizing feedback can be easily computed via $u = k_2(x,\theta) = m(P^{-1}x,\theta)$, if $P < I$. Notice also that for a quadratic Lyapunov function the function $\rho(V(x))$ in the proof of Theorem 1 can be replaced by $\rho \cdot V(x)$, where ρ is a sufficient large constant. Finally, noticed that the form $V^\star(z) = \frac{1}{2}z^TPz + W^\star(z)$ implies that the linearization of the control system (1) must be stabilizable.

Remark 4 Up to now, only the global stabilization problem was investigated. If no global solution exists, a stabilizing feedback in a region of the state space may be feasible. Furthermore, instead of just designing a stabilizing feedback law, also performance considerations can be included, for example, by imposing a certain decay rate, i.e., $\dot{V}^\star < -\epsilon V^\star$.

Remark 5 Helpful can be also the following extension: The matrix function A in (2) is not unique, i.e., there are different matrices $A(x)$ to write $f(x) = A(x)x$. However, one can introduce a certain matrix function H (see also [15]). In particular, one can replace in the inequalities (6) and (7), $A(\theta)$ by $A(\theta) - H_\rho(\theta)$, with

$$H_\rho(\theta) \quad = \quad \rho(\theta)(\theta^T\theta I - \theta\theta^T).$$

$\rho(\theta) \geq 1$ is a given to specified function which takes preferable large values for θ. Notice that H is symmetric and positive semidefinite and $H_\rho(\theta)x = 0$ if and only if $x = c \cdot \theta$, $c \in \mathbb{R}$ and hence Lemma 1 and Theorem 1 are still valid, but the inequality (7) may (but must not) be easier to satisfy. Hence, beside the domain $D_{(z,\theta)}$ a second element is introduced which may reduces conservatism of the feedback design.

[4] This is typical situation if one solves a (primal) problem via the dual problem: One gets the optimal value or the feasibility certificate but not the actual optimal or feasible point.

III. Example

In this section the state feedback design is applied to stabilize a part of a jet engine compressor (Figure 1).

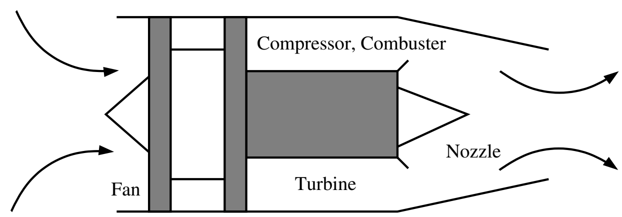

Fig. 1. Jet Engine Compressor.

A model that describes the dynamics of a jet engine compressor is the Moore–Greitzer model [8]. After a coordinate transformation, which normalizes the variables and shifts the desired equilibrium point to the origin, the dynamics of the Moore–Greitzer model can be written as

$$\begin{aligned}\dot{\phi} &= -\phi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \\ \dot{\psi} &= u \\ \dot{R} &= \sigma R\left(-2\phi - \phi^2 - R\right),\end{aligned} \tag{11}$$

where ϕ represents the overall mass flow, ψ the pressure rise, R the squared amplitude of circumferential flow asymmetry, and σ a positive constant. However, here, only the "no–stall model" is considered, that is, the Moore–Greitzer model with $R(t) = 0$. The "no–stall model" is obtained when the initial condition $R(0)$ is zero, i.e., $R(0) = 0$. Hence, the "no–stall model" is

$$\begin{aligned}\dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 &= u,\end{aligned} \tag{12}$$

where $x = [x_1\ x_2]^T = [\phi\ \psi]^T$. Using the state feedback design introduced in the previous section, one has to write the system (12) as parameterized linear control system form (3). By introducing the independent parameter θ, we obtain

$$\dot{x} = A(\theta)x + G(\theta)u, \tag{13}$$

with

$$A(\theta) = \begin{bmatrix} -\frac{3}{2}\theta - \frac{1}{2}\theta^2 & -1 \\ 0 & 0 \end{bmatrix},$$

$$G(\theta) = G = [0, 1]^T$$

Using the state feedback design inequalities (10) and making an Ansatz for the degrees of the polynomials V, m, s_1 and s_2 with undetermined coefficients, where $V^\star, s_1$ and s_2 must have even degree. Thus, the the following polynomial optimization problem is obtained:

$$\begin{aligned} V^\star(z) &\geq \mu_1 z^T z \\ -z^T A(\theta)\nabla V^\star(z)^T &+ z^T G(\theta) m(z,\theta) - \mu_2 z^T z \qquad (14) \\ -(z^T z - \theta^T\theta)s_1(x,\theta) &- (z^T\theta - \theta^T\theta)s_2(x,\theta) \geq 0 \end{aligned}$$

where μ_1 and μ_2 are positive parameters. Note, that the polynomial optimization problem (14) can be solved by replacing positive definiteness of polynomials with the sum of squares relaxation (cf. (10)), which can be solved using semidefinite programming. The software package SOSTOOLS [11] was used to solve the semidefinite program that results from (14). If the semidefinite program (14) finds no solution, no conclusion can be made. One has to use other μ_1 and μ_2 parameters or other degrees for the polynomials $V^\star, m, s_1$ and s_2. The parameters μ_1 and μ_2 and the degrees of the polynomials $V^\star, m, s_1$ and s_2 were chosen in a recursive way in the state feedback design. The semidefinite program (14) was solved with fixed parameters and fixed degrees of the polynomials. If the simulation results were not satisfactory, new parameters and new degrees of the polynomials were chosen and the semidefinite program (14) was solved again. Finally, the semidefinite program (14) was solved with $\mu_1 = 1$, $\mu_2 = 10$, a quadratic storage function $V^\star$, a polynomial state feedback m consisting of all monomials of degree one and three, and polynomials s_1 and s_2 consisting of all monomials of degree two. Finally, the CLF and the state feedback is given by

$$V(x) = 0.85x_1^2 - 0.61x_1x_2 + 0.22x_2^2 \qquad (15)$$

$$k_1(x) = 6.19x_1 - 2.68x_2 + 1.72x_1^3 - 1.56x_1^2x_2 + 0.43x_1x_2^2 - 0.10x_2^3. \qquad (16)$$

A phase plot of the closed-loop system is shown in Figure 2. Figure 3 shows the initial response of the closed-loop state trajectories of the system (12) with the initial condition $x_0 = [1\ 1]^T$. Simulations show that the the state feedback achieves good performance and demonstrates the utility of the proposed state feedback design. Notice, that the origin of the closed-loop system is globally asymptotically stable. Furthermore, the Lyapunov function V can be used to design other stabilizing feedbacks, i.e., Sontag's formula [16], [14], as shown in Figure 3.

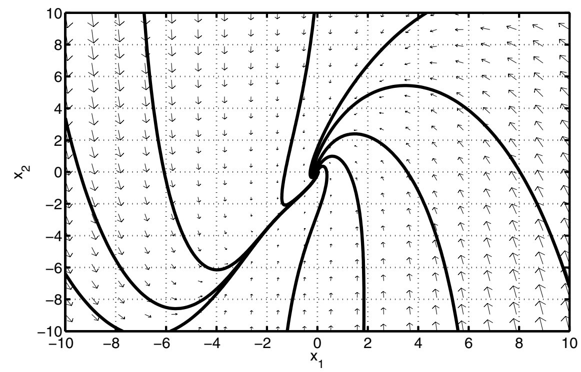

Fig. 2. Phase plot of the closed-loop state trajectories x_1 and x_2.

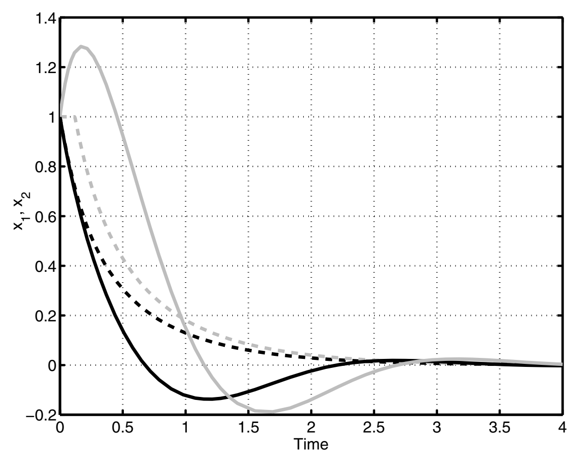

Fig. 3. Closed-loop state trajectories x_1 (black) and x_2 (gray). The solid curves correspond to state feedback (16) and the dashed curves correspond to Sontag's formula.

IV. CONCLUSIONS

In this paper, a new state feedback design for polynomial control systems has been proposed. The design is based on LPV systems and duality. The advantage of the proposed design is that no special requirements on the system structure are imposed and that the results are of nonlocal (global) nature. A disadvantage of the proposed design is that it is conservative and the CLF has to be convex. However, the design allows in contrast to many other nonlinear state feedback design a computer-aided design with nonquadratic CLFs. In particular, the stabilizing feedback law can be efficiently computed via semidefinite programming, i.e., by convex optimization. Finally, the applicability of this approach was illustrated on a small model of a jet engine compressor.

REFERENCES

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM, 1994.

[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[3] M.D. Choi, T.Y. Lam, and B. Reznick. Sums of squares of real polynomials. In *Proceedings of Symposia in Pure Mathematics 58(2)*, pages 103–126, 1995.

[4] A. Van Den Essen. *Polynomial Automorphisms*. Birkhäuser, 2000.

[5] R. Goebel, A. Teel, T. Hu, and Z. Lin. Dissipativity for dual linear differential inclusions through conjugate storage functions. In *Conference on Decision and Control (CDC), Paradise Island, Bahamas*, pages 2700–2705, 2004.

[6] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard. Some controls applications of sum of squares. In *Conference on Decision and Control (CDC), Maui, Hawaii*, pages 4676 – 4681, 2003.

[7] P. Kokotović and M. Arcak. Constructive nonlinear control: a historical perspective. *Automatica*, 37:637–662, May 2001.

[8] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. Wiley, 1995.

[9] A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum of squares decomposition. In *IEEE Conference on Decision and Control, Las Vegas, Nevada*, pages 3482–3487, 2002.

[10] P. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.

[11] S. Prajna, A. Papachristodoulou, and P. A. Parrilo. *SOSTOOLS for MATLAB*, 2004. Available from http://www.cds.caltech.edu/sostools.

[12] S. Prajna, A. Papachristodoulou, and F. Wu. Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach. In *Proc. of the Asian Control Conf. (ASCC), Melboourne, Australia*, 2004.

[13] T. Raff, C. Ebenbauer, and F. Allgöwer. Feedback passivation of an electrostatic microactuator: A semidefinite programming approach. In *Symposium on Nonlinear Control Systems (NOLCOS), Stuttgart, Germany*, pages 1181–1186, 2004. Preprint.

[14] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control*. Springer Verlag, 1997.

[15] J. S. Shanna and J. R. Cloutier. Existence of SDRE stabilization feedback. *IEEE Transaction of Automatic Control*, 48(3):513–517, 2003.

[16] E.D. Sontag. *Mathematical Control Theory*. Springer Verlag, 1998.

APPENDIX

In the following, a short summary of the sum of squares decomposition is given. More details can be found for example in [10]. Basically, the sum of squares decomposition is used to check if a polynomial in several variables in positive semidefinite. Recently, a lot of attention has been paid to the application of the sum of squares decomposition to polynomial control systems [12], [13]. The sum of squares decomposition is used in the present paper to solve the polynomial (in)equalities given by (6).

A polynomial $p \in \mathbb{R}[x]$ of degree d is a sum of squares if there exists a finite number of polynomials $p_i \in \mathbb{R}[x]$ such that p can be written as

$$p(x) = \sum_i p_i^2(x).$$

It is well known that not every positive definite polynomial can be represented as a sum of squares. However, with the help of the so called the "Gram Matrix" method, an answer to the question "When is a polynomial a sum of squares?" can be given:

Theorem 1: [3]. A polynomial $p \in \mathbb{R}[x]$ of degree $2d$ has a sum of squares decomposition if and only if there exists a positive semidefinite matrix Q such that

$$p(x) = m^T Q m,$$

where m is the vector of all monomials in $x_1, ..., x_n$ of degree less or equal to d, i.e., $m = [1, x_1, x_2, ..., x_n, x_1x_2, ..., x_n^d]$. There exists $\binom{n+d}{n}$ such monomials.

This representation theorem tells us that all sum of squares polynomials can be parameterized by the cone (convex set) of positive semidefinite matrices. Often it is useful and sufficient to check positive semidefiniteness of a polynomial p on a subset in $\mathbb{R}^n$ constrained by $c(x) \geq 0$, $c \in \mathbb{R}[x]$, i.e., the set $\{x \in \mathbb{R}^n \ : \ c(x) \geq 0\}$. Then, the following lemma is helpful (cf. e.g. [9]).

Lemma 1: Let $c \in \mathbb{R}[x]$. A polynomial $p \in \mathbb{R}[x]$ is positive semidefinite on the set $\{x \in \mathbb{R}^n \ : \ c(x) \geq 0\}$ if there exists a positive semidefinite polynomial $s \in \mathbb{R}[x]$ such that $p(x) - c(x)s(x)$ is positive semidefinite.

What makes these results especially interesting from an engineering point of view is the fact that there exists efficient numerical algorithms, namely semidefinite programming algorithms, which allow to test this condition on a computer. In [10], the gap between the "Gram Matrix" method and semidefinite programming has been bridged by showing that the existence of a sum of squares decomposition can be solved by semidefinite programming.