A New Method for Stabilization of Networked Control Systems with Random Delays [4]

LIQIAN ZHANG [1] YANG SHI [2] TONGWEN CHEN [2] [3] BIAO HUANG [1]

Abstract— **We consider the stabilization problem for a kind of networked control systems in discrete-time domain with random delays. The sensor-to-controller and controller-to-sensor delays are modeled as two Markov chains, and the resulting closed-loop systems are jump linear systems with two modes. The necessary and sufficient conditions on the existence of the stabilizing controllers are established. It is shown that the state-feedback gains are different with different modes. An iterative linear matrix inequality (LMI) approach is employed to calculate the state-feedback gains.**

I. INTRODUCTION

JNetworked control systems are the feedback control loops closed through a real time network. That is, in networked control systems, communication networks are employed to exchange the information and control signals (reference input, plant output, control input, etc.) between control system components (sensors, controllers, actuators, etc.) [9]. The main advantages of networked control systems are low cost, reduced weight, simple installation and maintenance, and high reliability. As a result, networked control systems have been widely applied to many complicated control systems, such as, manufacturing plants, vehicles, aircraft, and spacecraft [6].

Despite of the great advantages and wide applications, communication networks in the control loops make the analysis and design of a networked control system complicated. One main issue is the network-induced delays (sensor-to-controller and controller-to-actuator), which occur when sensors, actuators, and controllers exchange data across the network. The delays may be constant, time-varying, and in most cases, random. It is known that the occurrence of delay degrades the stability and control performance of the control loop system. Many researchers have paid attention on the study of the stability, controller design for stabilization and performance achievement purposes for networked control systems under the existence of network-induced delay. In [5], the stability analysis and control design of networked control systems were studied when the network-induced delay at each sampling time (sensor-to-controller delay τ_k+controller-to-actuator delay d_k) is random and less than one sampling time. The controller given there depends on sensor-to-controller delay τ_k. The results in [5] have recently been extended to the case with longer delay by Hu *et al.* [2]. In [9], the stability of networked control systems was analyzed by a hybrid system approach when the induced delay is deterministic (constant or time-varying) and the controller gain is constant, and in [4], a switched system approach was used to study the stability of networked control systems with constant controller gain. In [2],the maximum of the network-induced delay preserving the closed-loop stability for given plant and controller was considered. It is noticed that in all of the aforementioned papers, the plant is in continuous-time domain. For the discrete-time domain case, Krtolica *et al.* [3] and Xiao *et al.* [7] modelled the random delays as Markov chains such that the closed-loop system is a jump linear system with one mode. It is noticed that in [3], the state-feedback gain is mode-independent, and in [7], the state-feedback gain only depends on the delay from sensor to controller.

In this paper, we consider the stabilization problem of networked control systems with a discrete-time plant. The two random delays (sensor-to-controller and controller-to-sensor) are modelled as two different Markov chains, and the resulting closed-loop systems are jump linear systems with two modes characterized by two Markov chains. Since at each sampling time, when the controller is designed, the current states, the current sensor-to-controller delay (τ_k), and last time sensor-to-actuator delay (d_{k-1}) can be known, our goal is to design a state-feedback controller whose gain depends on both τ_k and d_{k-1}. In this way, the conservativeness of the stabilization conditions should be reduced. The necessary and sufficient conditions on the existence of stabilizing controllers are given, and an iterative linear matrix inequality (LMI) approach is used to calculate the controllers. An inverted pendulum example is considered to illustrate the proposed method.

II. PROBLEM STATEMENT

Consider the NCS setup in Figure 1, where the plant is a linear time-invariant discrete-time system, $\tau_k > 0$ is the random time delay from sensor to controller, $d_k > 0$ is

[1]Department of Chemical and Materials Engineering, University of Alberta, Edmonton, Alberta, Canada, T6G 2G6.
[2]Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta, Canada, T6G 2V4.
[3]Author for Correspondence: E-mail: tchen@ece.ualberta.ca; Telephone: 1-780-492-3940; Fax: 1-780-492-1811.
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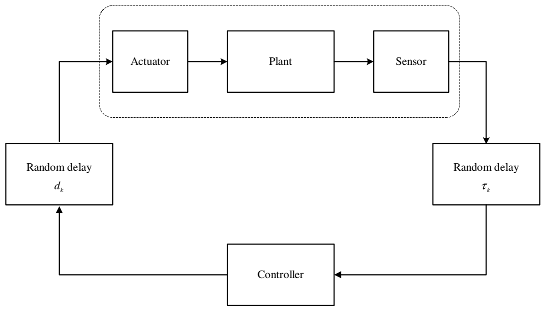

Fig. 1. Networked control system with random delays

the random time delay from controller to actuator, and the controller is to be designed.

Here, it is assumed that both τ_k and d_k are bounded, that is,

$$0 \le \tau_k \le \tau, \; 0 \le d_k \le d.$$

In real communication systems, time delays are usually correlated with the last time delays. It is reasonable to model two random delays τ_k and d_k as two homogeneous Markov chains that take values in $\mathcal{M} = \{0, 1, \ldots, \tau\}$ and $\mathcal{N} = \{0, 1, \ldots, d\}$, and their transition probability matrices are $\Lambda = [\lambda_{ij}]$ and $\Pi = [\pi_{rs}]$ respectively. That is, τ_k and d_k jump from mode i to j and from mode r to s respectively with probabilities λ_{ij} and π_{rs}, which are defined by

$$\begin{aligned} \lambda_{ij} &= \Pr(\tau_{k+1} = j \,|\tau_k = i) \\ \pi_{rs} &= \Pr(d_{k+1} = s \,|d_k = r) \end{aligned}$$

where $\lambda_{ij}, \pi_{rs} \ge 0$ and

$$\sum_{j=0,}^{\tau} \lambda_{ij} = 1, \;\; \sum_{s=0}^{d} \pi_{rs} = 1, \tag{1}$$

for all $i, j \in \mathcal{M}$ and $r, s \in \mathcal{N}$.

Remark 1: It is noted in [7] that modelling of τ_k and d_k as two Markov chains are quite general, and the package loss can be included naturally. It is usually assumed that the controller always uses the most recent data. Thus, if at sampling time k, $x(k - \tau_k)$ is available, then at sampling time $k+1$, if there are delays longer than 1 or package loss, we still have $x(k - \tau_k)$ to use. This means the delay τ_k can increase at most 1 at each step, or

$$\Pr(\tau_{k+1} > \tau_k + 1) = 0.$$

For d_k, we have similar comments.

Assume that the model of the plant is a linear time-invariant discrete-time model as follows

$$x(k+1) = Ax(k) + Bu(k). \tag{2}$$

It is noticed that when the controller is designed, we cannot know the exact d_k, but τ_k and d_{k-1}. Consequently, the controller gain can be designed depending on τ_k and d_{k-1}, that is,

$$\begin{aligned} u(k) &= F(\tau_k, d_{k-1}) x(k - \tau_k - d_k), \\ x(t) &= \phi(t), \; t \in \{-\tau - d, \ldots, 0\}. \end{aligned} \tag{3}$$

Hence, the closed-loop system from (2) and (3) can be expressed as

$$x(k+1) = Ax(k) + BF(\tau_k, d_{k-1}) x(k - \tau_k - d_k). \tag{4}$$

At sampling time k, if we augment the state-variable as

$$X(k) = \begin{bmatrix} x(k)^T & x(k-1)^T & \cdots & x(k-\tau-d)^T \end{bmatrix}^T, \tag{5}$$

then the closed-loop system (4) can be written as

$$X(k+1) = \left(\tilde{A} + \tilde{B} F(\tau_k, d_{k-1}) \tilde{E}(\tau_k, d_k)\right) X(k), \tag{6}$$

$$X(0) = \begin{bmatrix} \phi(0)^T & \phi(-1)^T & \cdots & \phi(-\tau-d)^T \end{bmatrix}^T,$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \in \mathbb{R}^{n(1+\tau+d)\times n(1+\tau+d)},$$

$$\tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n(1+\tau+d)\times p},$$

$$\tilde{E}(\tau_k, d_k) = \begin{bmatrix} 0 & \cdots & 0 & I & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n(1+\tau+d)},$$

and $\tilde{E}(\tau_k, d_k)$ has all elements being zeros except for the $(1 + \tau_k + d_k)$th block being identity. It can be seen that the closed-loop system (6) is a jump linear system with two modes modelled by different homogeneous Markov chains. Throughout this paper, we use the following definition:

Definition 1: The system in (6) is stochastically stable if for every finite $X_0 = X(0)$ and initial mode $\tau_0 = \tau(0) \in \mathcal{M}$ and $d_{-1} = d(-1) \in \mathcal{N}$, there exists a finite $W > 0$ such that the following holds

$$\mathcal{E}\left\{ \sum_{k=0}^{\infty} \|X(k)\|^2 \,|X_0, \tau_0, d_{-1} \right\} < X_0^T W X_0. \tag{7}$$

Our objective for this paper is to find the state-feedback gain $F(\tau_k, d_{k-1})$ such that the closed-loop system (6) is stochastically stable. In the following, if we know that $\tau_k = i, d_{k-1} = r$, $F(\tau_k, d_{k-1})$ is denoted as $F(i, r)$.

III. Main Results

With Definition 1, the necessary and sufficient conditions on the stochastic stability of closed-loop system (6) can be obtained.

Theorem 1: The closed-loop system (6) is stochastically stable if and only if there exists $P(i, r) > 0$ such that the following matrix inequality series:

$$\begin{aligned} &L(i, r) \\ &= \left\{ \textstyle\sum_{s=0}^{d} \pi_{rs} \left(\tilde{A} + \tilde{B} F(i, r) \tilde{E}(i, s)\right)^T \right. \\ &\left. \bar{P}(i, s) \left(\tilde{A} + \tilde{B} F(i, r) \tilde{E}(i, s)\right) \right\} - P(i, r) < 0 \end{aligned} \tag{8}$$

holds for all $i \in \mathcal{M}$ and $r \in \mathcal{N}$, where

$$\bar{P}(i,s) = \sum_{j=0}^{\tau} p_{ij} P(j,s). \quad (9)$$

Proof: Sufficiency: For the closed-loop system (6), consider the quadratic function which is given by

$$V(X(k),k) = X(k)^T P(\tau_k, d_{k-1}) X(k).$$

Noticing (9) and (6), we have

$$\begin{aligned}
&\mathcal{E}\left\{\Delta V(X(k),k)X(k+1)\right\} \\
= \; &\mathcal{E}\left\{X(k+1)^T P(\tau_{k+1}, d_k) X(k+1) \,|X_k, \right. \\
&\left.\tau_k = i, d_{k-1} = r\right\} - X(k)^T P(i,r) X(k)\} \\
= \; &\left[\sum_{s=0}^{d}\sum_{j=0}^{\tau} \pi_{rs} \left(X(k)^T \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right)^T\right)\right. \\
&\left. p_{ij} P(j,s) \left(\left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right) X(k)\right)\right] \\
&- X(k)^T P(i,r) X(k) \\
= \; &X(k)^T \left[\sum_{s=0}^{d} \pi_{rs} \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right)^T \bar{P}(i,s)\right. \\
&\left.\left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right) - P(i,r)\right] X(k).
\end{aligned}$$

Thus, if $L(i,r) < 0$, then

$$\begin{aligned}
&\mathcal{E}\left\{\Delta V(X(k))\right\} \\
&= \mathcal{E}\left\{V(X(k+1), k+1) \,|X_k, \tau_k = i, d_{k-1} = r\right\} \\
&- V(X(k), k) \\
&\leq -\lambda_{\min}(-L(i,r)) X(k)^T X(k) \\
&\leq -\beta X(k)^T X(k) = -\beta \left\|X(k)\right\|^2,
\end{aligned}$$

where $\beta = \inf\{\lambda_{\min}(-L(i,r)), i \in \mathcal{M}, r \in \mathcal{N}\} > 0$. From the above inequality, we can see that for any $T \geq 1$,

$$\begin{aligned}
&\mathcal{E}\{V(X(T+1), T+1)\} - \mathcal{E}\{V(X_0, 0)\} \\
&\leq -\beta \mathcal{E}\{\textstyle\sum_{t=0}^{T} \left\|X(t)\right\|^2\}
\end{aligned}$$

or

$$\begin{aligned}
&\mathcal{E}\{\textstyle\sum_{t=0}^{T} \left\|X(t)\right\|^2\} \\
&\leq \tfrac{1}{\beta}\left(\mathcal{E}\{V(X_0, 0) - \mathcal{E}\{V(X(T+1), T+1)\}\right) \\
&\leq \tfrac{1}{\beta}\mathcal{E}\{V(X_0, 0),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathcal{E}\{\textstyle\sum_{t=0}^{\infty} \left\|X(t)\right\|^2\} \quad &\leq \tfrac{1}{\beta}\mathcal{E}\{V(X_0, 0) \\
&= \tfrac{1}{\beta} X(0)^T P(\tau_0, d_{-1}) X(k).
\end{aligned}$$

From Definition 1, the stochastic stability is obtained.

Necessity: Assume that the closed-loop system (6) is stochastically stable. That is, we have

$$\mathcal{E}\left\{\sum_{k=0}^{\infty} \left\|X(k)\right\|^2 |X_0, \tau_0, d_0\right\} < X_0^T W X_0. \quad (10)$$

Consider the following function

$$\begin{aligned}
&X(t)^T \tilde{P}(T-t, \tau_t, d_{t-1}) X(t) \\
&\triangleq \mathcal{E}\left\{\textstyle\sum_{k=t}^{T} X(k)^T Q(\tau_k, d_{k-1}) X(k) \,|X_t, \tau_t, d_{t-1}\right\}.
\end{aligned} \quad (11)$$

with $Q(\tau_k, d_{k-1}) > 0$. Assume that $X(k) \neq 0$. Since $Q(\tau_k, d_{k-1}) > 0$, as T increases, either $X(t)^T \tilde{P}(T - t, \tau_t, d_{t-1}) X(t)$ is monotonically increasing or it increases monotonically until

$$\mathcal{E}\left\{X(k)^T Q(\tau_k, d_{k-1}) X(k) \,|X_t, \tau_t, d_{t-1}\right\} = 0$$

for all $k \geq k_1 \geq t$. From (10), it can be seen that $X(t)^T \tilde{P}(T-t, \tau_t, d_{t-1}) X(t)$ is bounded above, and thus the following limit exists:

$$\begin{aligned}
&X(t)^T P(i,r) X(t) \\
\triangleq \; &\lim_{T\to\infty} X(t)^T \tilde{P}(T-t, \tau_t = i, d_{t-1} = r) X(t) \\
= \; &\lim_{T\to\infty} \mathcal{E}\left\{\sum_{k=t}^{T} X(k)^T Q(\tau_k, d_{k-1}) X(k) \,|X_t\right. \\
&\left.\tau_t = i, d_{t-1} = r\right\}
\end{aligned} \quad (12)$$

Since this is valid for any $X(t)$, we have

$$P(i,r) = \lim_{T\to\infty} \tilde{P}(T-t, \tau_t = i, d_{t-1} = r). \quad (13)$$

From (12), it can be seen that $P(i,r) > 0$ since $Q(\tau_k, d_{k-1}) > 0$. Let us consider

$$\begin{aligned}
&\mathcal{E}\left\{X(t)^T \tilde{P}(T-t, \tau_t, d_{t-1}) X(t) - \right. \\
&X(t+1)^T \tilde{P}(T-t-1, \tau_{t+1}, d_t) X(t+1) \,|X_t, \\
&\left.\tau_t = i, d_{t-1} = r\right\} \\
&= X(t)^T Q(i,r) X(t).
\end{aligned} \quad (14)$$

Notice that

$$\begin{aligned}
&\mathcal{E}\left\{X(t+1)^T \tilde{P}(T-t-1, \tau_{t+1}, d_t) X(t+1) \,|X_t, \right. \\
&\left.\tau_t = i, d_{t-1} = r\right\} \\
&= X(t)^T \textstyle\sum_{j=0}^{\tau}\sum_{s=0}^{d} \pi_{rs}\lambda_{ij} \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right)^T \\
&\tilde{P}(T-t-1, j, s) \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right) X(t).
\end{aligned}$$

This, together with (14), implies that

$$\begin{aligned}
&X(t)^T \left[\tilde{P}(T-t, \tau_t, d_{t-1}) - \sum_{s=0}^{d} \pi_{rs}\right. \\
&(\tilde{A} + \tilde{B}F(i,r))\tilde{E}(i,s)^T \sum_{j=0}^{\tau} \lambda_{ij} \tilde{P}(T-t-1, j, s) \\
&\left.(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s))\right] X(t) \\
= \; &X(t)^T Q(i,r) X(t).
\end{aligned}$$

Letting $T \to \infty$ and noticing that (13) and $Q(i,r) > 0$, we have

$$\begin{aligned}
&P(i,r) - \textstyle\sum_{s=0}^{d} \pi_{rs} \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right)^T \\
&\quad \textstyle\sum_{j=0}^{\tau} \lambda_{ij} P(j,s) \left(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,s)\right) \\
&> 0.
\end{aligned}$$

□

Theorem 1 gives necessary and sufficient conditions on the existence of the state-feedback stabilizating gain.

However, since the given conditions in (8) are nonlinear, we need to find a method to solve them. To this end, in the following theorem, the equivalent conditions to (8) are given.

Theorem 2: There exists a controller in (3) such that the closed-loop system (4) is stochastically stable if and only if there exist $\bar{X}(i,s) > 0$, $P(i,r) > 0$ and $F(i,r)$ such that the following LMI:

$$\begin{bmatrix} -P(i,r) & V(i,r) \\ V(i,r)^T & -D(i) \end{bmatrix} < 0 \tag{15}$$

with

$$V(i,r) = \Big[(\pi_{r0})^{\frac{1}{2}}(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,0) \cdots \\ (\pi_{rd})^{\frac{1}{2}}(\tilde{A} + \tilde{B}F(i,r)\tilde{E}(i,d)\Big]$$
$$D(i) = \mathrm{diag}\{\bar{X}(i,0), \ldots, \bar{X}(i,d)\}$$

holds for all $i \in \mathcal{M}$ and $r,s \in \mathcal{N}$ with the constraint of

$$\bar{X}(i,s) = \bar{P}(i,s)^{-1}, \tag{16}$$

where $\bar{P}(i,s) = \sum_{j=0}^{\tau} \lambda_{ij} P(j,s)$.

Proof: The proof is obtained by the Schur complement and letting $\bar{X}(i,s) = \bar{P}(i,s)^{-1}$. □

The conditions stated in Theorem 2 are in fact a set of linear matrix inequalities (LMIs) with some inversion constraints. Though they are nonconvex, there are some existing methods to solve them. In this paper, it is suggested to use an iterative LMI approach, called as the cone complementarity linearization (CCL) algorithm to calculate $F(i,r)$ from Theorem 2 (see [1] and [8] for details).

IV. Numerical Example

Consider the cart and inverted pendulum problem in Figure 2 [7]. The state variables are $x, \dot{x}, \theta$, and $\dot{\theta}$. Assume that $m_1 = 1kg, m_2 = 0.5kg, L = 1m$, and there are no friction surfaces. The sampling time is $T_s = 0.1$ second, and the random delays exist in $\tau_k \in \{0,1\}$ and $d_k \in \{0,1\}$, and their transition probability matrices are given by

$$\Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}.$$

The controllers are designed using the discretized linearized model, which has the state-space model as

$$x(k+1) = A_d x(k) + B_d u(k),$$

where

$$A_d = \begin{bmatrix} 1.0000 & 0.1000 & -0.0166 & -0.0005 \\ 0 & 1.0000 & -0.3374 & -0.0166 \\ 0 & 0 & 1.0996 & 0.1033 \\ 0 & 0 & 2.0247 & 1.0996 \end{bmatrix},$$
$$B_d = \begin{bmatrix} 0.0045 \\ 0.0896 \\ -0.0068 \\ -0.1377 \end{bmatrix}.$$

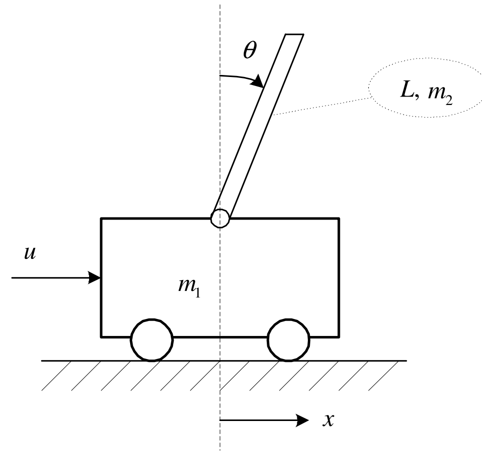

Fig. 2. Cart and inverted pendulum

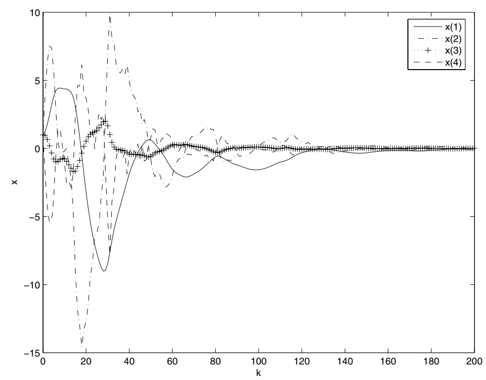

Fig. 3. States of the closed-loop system

Since A_d has eigenvalues at $1, 1, 1.5569, 0.6423$, the discretized system is unstable. To stabilize it, by Theorem 2, we can obtain the controllers as follows:

$$F(0,0) = \begin{bmatrix} 0.2331 & 1.1476 & 21.4372 & 4.8034 \end{bmatrix},$$
$$F(0,1) = \begin{bmatrix} 0.8370 & 0.8133 & 28.5973 & 6.3474 \end{bmatrix},$$
$$F(1,0) = \begin{bmatrix} -0.6518 & 1.2327 & 8.0582 & 6.0030 \end{bmatrix},$$
$$F(1,1) = \begin{bmatrix} -0.1640 & 0.7335 & 12.4178 & 7.6657 \end{bmatrix}.$$

The states of the closed-loop system caused by the discretized model and the obtained controller are shown in Figure 1 when $x(-2) = x(-1) = x(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$. It can be seen that the closed-loop system is stochastically stable.

V. Conclusion

This paper has presented a new method to the stabilization of a kind of networked control systems with random communication delays. By modelling the delays as Markov chains, the closed-loop systems can be expressed as jump linear systems with two modes. Necessary and sufficient conditions of stochastic stability for the jump linear systems are obtained in terms of a set LMIs with matrix inversion constraints, from which the state-feedback gain can be solved by an existing iterative LMI algorithm. It is shown that the state-feedback gain depends on the two modes. A numerical example has been considered to illustrate the main results.

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