# State-Feedback Controller Synthesis for Parameter-Dependent LTI Systems

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*Abstract*— **In this paper we show that the class of polynomially parameter-dependent quadratic (PPDQ) Lyapunov functions of specified degree characterizes the stability of single-parameter dependent linear, time-invariant, (s-PDLTI) systems. Using PPDQ Lyapunov functions we provide necessary and sufficient conditions for the stability of s-PDLTI systems. Checking the feasibility of these conditions can be cast as a convex, finite-dimensional programming problem. Finally, we extend this analysis result to state-feedback parameter-dependent controller synthesis for s-PDLTI systems.**

## I. Introduction

In this paper we consider single-parameter dependent linear, time-invariant (s-PDLTI) systems

$$\dot{x} = A_\rho x + B_\rho u, \quad \rho \in \Omega, \tag{1}$$

where $A_\rho \in \mathbb{R}^{n\times n}$ and $B_\rho \in \mathbb{R}^{n\times m}$ are *polynomially* parameter-dependent matrices of the form

$$A_\rho := \sum_{i=0}^{n_a} \rho^i A_i, \quad B_\rho := \sum_{i=0}^{n_b} \rho^i B_i, \tag{2}$$

and $\Omega$ is a compact subset of $\mathbb{R}$. Stability analysis criteria for the corresponding uncontrolled system

$$\dot{x} = A_\rho x, \quad \rho \in \Omega, \tag{3}$$

over a compact interval have been proposed in [1], [2]. Saydy et al. [3], [4] have given necessary and sufficient conditions for the stability of (3) using guardian maps. Lyapunov-based stability criteria have also been proposed, but these are only sufficient, hence conservative [5], [6], [7], [8], [9], [10]. Recently, it was shown that the stability of (3) is *equivalent* to the stability of an auxiliary system which depends affinely on the parameter $\rho$ [11]. Well-known results from the theory of affinely-dependent PDLTI systems can then be used to analyze (3). Despite the previous references (with the exception of [11]), Lyapunov-based sufficient and *necessary* conditions for (3) are not known. Neither nonconservative synthesis methods are currently available (notable exception is the recent paper by Bliman [12]). It is the purpose of this paper to contribute to the current state of knowledge on Lyapunov-based analysis and (most importantly) synthesis criteria for (3) and (1). Specifically, two new results are shown in this paper. The first result (Theorem 2.1) shows that necessary and sufficient conditions for the stability of (3) can be derived via the use of polynomially parameter-dependent quadratic (PPDQ) Lyapunov functions *of a known degree*. The second result of the paper (Theorem 3.1) proposes a nonconservative approach for constructing parameter-dependent state-feedback controllers of the form $u(x) = K_\rho x$ for (1)-(2). Specifically, it is shown that a polynomially, parameter-dependent gain matrix $K_\rho$ that stabilizes the system over a compact interval exists, if and only if a certain pair of parameterized matrix inequalities is feasible. The existence proof is constructive, thus yielding a method to compute the gain matrix $K_\rho$. It is also shown that both the proposed analysis and synthesis results can be cast as *finite-dimensional* convex optimization problems of bounded complexity in terms of LMIs.

It should be noted that for systems affected by polynomial time-invariant uncertainty Chesi [13] has also provided computationally attractive LMI conditions. However, these conditions are not Lyapunov based. Moreover, although PPDQ Lyapunov functions for the analysis of (3) have been proposed recently [14], [12], the degree of the polynomial dependence is not known a priori in these references. On the contrary, Theorem 2.1 below provides an explicit upper bound on the degree of the PPDQ Lyapunov function.

The synthesis result developed in this paper can be useful for designing controllers in two different cases. First, it can be used for designing gain-scheduled controllers for LPV systems with a slowly varying parameter. This is because any stability conditions for (3) with the parameter $\rho$ frozen are necessary and sufficient for robust stability of (3) with respect to an arbitrarily slowly varying parameter $\rho(t) \in \Omega$; see, for instance, [15]. Furthermore, our method can be used for on-site parameter tuning for plants containing a constant parameter whose actual value is not known a priori, but its range $\Omega$ is known at the time of control design. Moreover, the actual value of the parameter is known during controller implementation. One may then design a class of controllers parameterized by $\rho \in \Omega$ and then tune the parameter $\rho$ when the controller is actually implemented ("in situ" controller tuning).

The following notation will be used throughout this paper.

Given a matrix $A \in \mathbb{R}^{n\times n}$, $\mathcal{N}(A)$ will denote the null space of the matrix $A$. The notation $\widehat{A}$ will be used to denote the matrix of dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ defined by $\widehat{A} := D_n^+(A \oplus A)D_n$, where $A \oplus A = I_n \otimes A + A \otimes I_n$ is the Kronecker sum of matrix $A$ with itself, and $D_n$ is the *duplication matrix* [16], [17] of dimension $n^2 \times \frac{1}{2}n(n+1)$. Moreover, the following notation will be adapted from [14], [18]. Given an integer $q \geq 0$, the symbol $\rho^{[q]} \in \mathbb{R}^q$ will be used to denote the vector $\rho^{[q]} := (1\ \rho\ \rho^2\ \cdots\ \rho^{q-1})^\intercal$, and $\hat{J}_k$ and $\check{J}_k$ will be used to denote the matrices $\hat{J}_k := \begin{bmatrix} I_k & 0_{k\times 1}\end{bmatrix}$ and $\check{J}_k := \begin{bmatrix} 0_{k\times 1} & I_k\end{bmatrix}$, respectively.

## II. An Analysis Result

Consider the following uncontrolled s-PDLTI system with polynomial dependence on a single parameter $\rho$

$$\dot{x} = \mathcal{A}_\rho x, \quad \mathcal{A}_\rho := \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i, \quad \rho \in \Omega, \tag{4}$$

where $\mathcal{A}_i \in \mathbb{R}^{n\times n}$ for $i = 0, 1, \ldots, \nu_a$ and $\Omega$ any compact subset of $\mathbb{R}$. The following theorem is an extension of a similar result in [18], and states that the stability of (4) is *characterized* via the use of PPDQ Lyapunov functions of known polynomial degree. A related result can also be found in [11].

*Theorem 2.1:* Consider the polynomially parameter-dependent matrix in (4), and assume that $\dim[\bigcap_{i=1}^{\nu_a} \mathcal{N}(\widehat{\mathcal{A}}_i)] = \ell$. Then the following two statements are equivalent:

(i) $\mathcal{A}_\rho := \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i$ is Hurwitz for all $\rho \in \Omega$.

(ii) There exist real symmetric matrices $P_i$ ($i = 0, 1, \ldots, m_p$), such that

$$\mathcal{A}_\rho P_\rho + P_\rho \mathcal{A}_\rho^\intercal < 0, \quad \forall \rho \in \Omega, \tag{5}$$

$$P_\rho = \sigma_\rho \left( \sum_{i=0}^{m_p} \rho^i P_i \right) > 0, \quad \forall \rho \in \Omega, \tag{6}$$

where,

$$m_p := \nu_a \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \ell\}, \tag{7}$$

and where $\sigma_\rho = -\text{sign}(\det \widehat{\mathcal{A}}_\rho)$ with $\det \widehat{\mathcal{A}}_\rho \neq 0$ for all $\rho \in \Omega$. Moreover, if $\Omega$ is connected, without loss of generality we can take $\sigma_\rho = +1$ for all $\rho \in \Omega$.

*Proof:* It easy to show that $\widehat{\mathcal{A}}_\rho = \sum_{i=0}^{\nu_a} \rho^i \widehat{\mathcal{A}}_i$. Then using Lemma A.1 in the Appendix one obtains that

$$\text{Adj}\left( \sum_{i=0}^{\nu_a} \rho^i \widehat{\mathcal{A}}_i \right) = \sum_{i=0}^{m_p} \rho^i N_i,$$

for some constant matrices $N_i$ ($i = 0, 1, \ldots, m_p$) where $m_p$ as in (7). The rest of the proof now follows as in Theorem 3.1 of [18]. ∎

The parameterized matrix inequalities (5)-(6) can be transformed *exactly* (that is, without conservatism) to a set of LMIs using the following lemma. In the sequel we assume that $\Omega$ is compact and connected. Without loss of generality we may take $\Omega = [-1, +1]$.

*Lemma 2.1 ([18]):* Let $\Theta \in \mathbb{R}^{nk\times nk}$. Then the matrix inequality

$$\left(\rho^{[k]} \otimes I_n\right)^\intercal \Theta \left(\rho^{[k]} \otimes I_n\right) < 0 \tag{8}$$

holds for all $\rho \in [-1, 1]$ if and only if there exist matrices $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ such that

$$D = D^\intercal > 0, \quad G + G^\intercal = 0, \tag{9}$$

$$\Theta < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^\intercal \begin{bmatrix} -D & G \\ G^\intercal & D \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}. \tag{10}$$

Using now the fact that the polynomial matrix $P_\rho$ in (6) can be written as $P_\rho = \left(\rho^{[k]} \otimes I_n\right)^\intercal P_\Sigma \left(\rho^{[k]} \otimes I_n\right)$ where $k = \lceil \frac{m_p}{2} \rceil + 1$ for some $P_\Sigma \in \mathbb{R}^{nk\times nk}$, one can easily show that the matrix in (5) can be written compactly as follows

$$R_\rho := \mathcal{A}_\rho P_\rho + P_\rho \mathcal{A}_\rho^\intercal = \left(\rho^{[k+\nu_a]} \otimes I_n\right)^\intercal R_\Sigma \left(\rho^{[k+\nu_a]} \otimes I_n\right) \tag{11}$$

where,

$$\begin{aligned} R_\Sigma &:= H^\intercal P_\Sigma F + F^\intercal P_\Sigma H, && (12)\\ H &:= (\hat{J}_k \hat{J}_{k+1} \cdots \hat{J}_{k+\nu_a-1}) \otimes I_n, && (13)\\ F &:= (\hat{J}_k \hat{J}_{k+1} \cdots \hat{J}_{k+\nu_a-1}) \otimes \mathcal{A}_0^\intercal && (14)\\ &\quad + \sum_{i=1}^{\nu_a} (\check{J}_k \check{J}_k \cdots \check{J}_{k+i-1} \hat{J}_{k+i} \cdots \hat{J}_{k+\nu_a-1}) \otimes \mathcal{A}_i^\intercal. \end{aligned}$$

Note, in particular, that $R_\Sigma$ is linear in $P_\Sigma$.

In light of Lemma 2.1 we are now ready to provide the following necessary and sufficient condition for the stability of (4) for $|\rho| \leq 1$ in terms of LMIs.

*Theorem 2.2:* Let the parameter-dependent matrix $\mathcal{A}_\rho = \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i$, where $\mathcal{A}_i \in \mathbb{R}^{n\times n}$ with $\dim[\bigcap_{i=1}^{\nu_a} \mathcal{N}(\widehat{\mathcal{A}}_i)] = \ell$ and let $\kappa := \lceil \frac{m_p}{2} \rceil + \nu_a$ where,

$$m_p := \nu_a \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \ell\}. \tag{15}$$

Then, $\mathcal{A}_\rho$ is Hurwitz for all $|\rho| \leq 1$ if and only if there exist symmetric matrices $P_\Sigma \in \mathbb{R}^{n(\kappa-\nu_a+1)\times n(\kappa-\nu_a+1)}$, $D \in \mathbb{R}^{n\kappa\times n\kappa}$ and a skew-symmetric matrix $G \in \mathbb{R}^{n\kappa\times n\kappa}$, such that

$$P_0 > 0, \quad D = D^\intercal > 0, \quad G + G^\intercal = 0, \tag{16}$$

$$R_\Sigma < \begin{bmatrix} \hat{J}_\kappa \otimes I_n \\ \check{J}_\kappa \otimes I_n \end{bmatrix}^\intercal \begin{bmatrix} -D & G \\ G^\intercal & D \end{bmatrix} \begin{bmatrix} \hat{J}_\kappa \otimes I_n \\ \check{J}_\kappa \otimes I_n \end{bmatrix}, \tag{17}$$

where $R_\Sigma = R_\Sigma(P_\Sigma)$ as in (12)-(14).

*Proof:* From Theorem 2.1 the matrix $\mathcal{A}_\rho$ is Hurwitz for all $|\rho| \leq 1$ if and only if there exist a PPDQ Lyapunov matrix of degree $m_p$, such that

$$P_\rho = \left(\rho^{[k]} \otimes I_n\right)^\intercal P_\Sigma \left(\rho^{[k]} \otimes I_n\right) > 0, \tag{18}$$

and

$$R_\rho = \left(\rho^{[k+\nu_a]} \otimes I_n\right)^\intercal R_\Sigma \left(\rho^{[k+\nu_a]} \otimes I_n\right) < 0, \tag{19}$$

for all $\rho \in [-1, +1]$. Using Lemma 2.1, inequality (19) is equivalent to (17) for some symmetric matrix $D$ and skew-symmetric matrix $G$. Next, notice that if $\mathcal{A}_\rho$ is nominally stable, then (19) implies that $P_0 > 0$. On the contrary, the condition $P_0 > 0$ along with (19) imply the nominal stability of $\mathcal{A}_\rho$. The latter two conditions imply that $P_\rho > 0$ for all $|\rho| \leq 1$, since inequality (19) implies the nonsingularity of $P_\rho$ for all $|\rho| \leq 1$; see [15] and [18]. ■

## III. State-Feedback Synthesis

The results of the previous section can be used to design *parameter-dependent* state-feedback controllers for the controlled s-PDLTI system (1)-(2). This is shown in the next theorem.

*Theorem 3.1:* Let $\Omega \subset \mathbb{R}$ be a compact interval. Then the following statements are equivalent:

(i) There exists a polynomially parameter-dependent state feedback controller $u = K_\rho x$ that stabilizes the system (1)-(2) for all $\rho \in \Omega$.

(ii) There exist an integer $m_p$ and symmetric matrices $P_i$ $(i = 0, 1, \ldots, m_p)$ such that

$$\begin{aligned} &P_\rho := \sum_{i=0}^{m_p} \rho^i P_i, \quad P_\rho > 0, \quad \forall \rho \in \Omega, \\ &A_\rho P_\rho + P_\rho A_\rho^\intercal < B_\rho B_\rho^\intercal, \quad \forall \rho \in \Omega. \end{aligned} \tag{20}$$

If statement (i) holds for a gain matrix $K_\rho \in \mathbb{R}^{m \times n}$ of degree $n_k$, then statement (ii) holds with $m_p \leq \nu_a\big(\frac{1}{2}n(n+1) - 1\big)$ where $\nu_a := \max\{n_a, n_b + n_k\}$. Conversely, if (ii) holds, then a stabilizing state feedback gain in (i) is given by

$$K_\rho = -\mu_\rho B_\rho^\intercal P_\rho^{-1}, \tag{21}$$

where

$$\mu_\rho := \det(P_\rho)/\epsilon, \quad \epsilon := \min_{\rho \in \Omega} \det(P_\rho). \tag{22}$$

Moreover, $K_\rho$ is a polynomial matrix of degree $n_k \leq n_b + m_p(n_a - 1)$.

*Proof:* Suppose (i) holds for a matrix $K_\rho$ of degree $n_k$. Then the closed-loop system is described by (4) with $\mathcal{A}_\rho := A_\rho + B_\rho K_\rho$. The matrix $\mathcal{A}_\rho$ has degree $\nu_a := \max\{n_a, n_b + n_k\}$. Then Theorem 2.1 implies satisfaction of (5)-(6) for some $P_\rho$ of degree $m_p \leq \nu_a\big(\frac{1}{2}n(n+1) - 1\big)$. Let now $N_\rho$ be the null space of $B_\rho^\intercal$. Then multiplying the Lyapunov inequality in (5) by $N_\rho$ from the right and by $N_\rho^\intercal$ from the left, one obtains

$$N_\rho^\intercal (A_\rho P_\rho + P_\rho A_\rho^\intercal) N_\rho < 0.$$

By Finsler's theorem [19], there exists $\tau_\rho > 0$ such that

$$A_\rho P_\rho + P_\rho A_\rho^\intercal < \tau_\rho B_\rho B_\rho^\intercal$$

for each $\rho \in \Omega$. Since $\Omega$ is compact and $\tau_\rho$ is a continuous function of $\rho$ (see Lemma A.2 in the Appendix), $\tau_{\max} := \max_{\rho \in \Omega} \tau_\rho > 0$ is well defined. Then, redefining $P_\rho$ to be $P_\rho / \tau_{\max}$, we have (20). Thus we have (i) $\Rightarrow$ (ii). The converse can be proved by direct substitution. Specifically, with the control gain matrix (21) one obtains

$$\begin{aligned} &(A_\rho + B_\rho K_\rho) P_\rho + P_\rho (A_\rho + B_\rho K_\rho)^\intercal \\ &\qquad = A_\rho P_\rho + P_\rho A_\rho^\intercal - 2\mu_\rho B_\rho B_\rho^\intercal < 0, \quad \forall\ \rho \in \Omega, \end{aligned}$$

where it is noted that $\mu_\rho \geq 1$ by definition. This proves that $K_\rho$ is stabilizing. Since $\deg \mu_\rho \leq n m_p$ (see (A.4)) from the expression $K_\rho P_\rho = -\mu_\rho B_\rho^\intercal$ it follows that $n_k + m_p \leq n_a m_p + n_b$. Therefore, $n_k \leq n_b + m_p(n - 1)$. ■

The synthesis condition (20) is a pair of polynomially-parameterized LMIs and hence it can be converted exactly to finite dimensional LMIs by eliminating the parameter $\rho$ using Lemma 2.1. The details are left to the reader.

**Example 1** Consider the following controlled polynomial s-PDLTI system

$$A_\rho = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \rho \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_\rho = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The uncontrolled system is stable for $\rho < -2$ and unstable otherwise. Using Theorem 3.1 we can design the following parameter-dependent feedback gain that will ensure that the closed-loop system is stable for all $\rho \in [-1, +1]$

$$\begin{aligned} K_\rho = &\begin{bmatrix} -89.579 & -69.882 \end{bmatrix} + \rho \begin{bmatrix} 4.4698 & -35.807 \end{bmatrix} \\ &+ \rho^2 \begin{bmatrix} 70.575 & 18.009 \end{bmatrix}. \end{aligned}$$

The polynomial Lyapunov matrix that ensures the stability of the closed-loop system is computed by the solution of the LMI problem (20) and is given by

$$P = \begin{bmatrix} 0.49120 & -0.29686 \\ -0.29686 & 0.38053 \end{bmatrix} + \rho \begin{bmatrix} 0.088923 & 0.24193 \\ 0.24193 & -0.35769 \end{bmatrix}.$$

For this example $\epsilon = 0.008496$. The closed-loop system matrix is third order and is given in equation (23) on the next page. It can be easily verified that this matrix is Hurwitz for all $\rho \in [-1, +1]$.

## IV. Conclusions

Non-conservative (exact) analysis and state-feedback synthesis results are proposed for s-PDLTI systems. Both analysis and synthesis conditions can be implemented and tested efficiently in terms of finite-dimensional linear matrix inequalities. The results are useful for control design of slowly parameter-varying systems and for *in situ* controller implementation and tuning of controllers for s-PDLTI systems.

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$$A_{\rm cl} = \begin{bmatrix} -177.16 & -138.76 \\ 2 & 1 \end{bmatrix} + \rho \begin{bmatrix} -78.640 & -141.50 \\ -89.579 & -68.882 \end{bmatrix} + \rho^2 \begin{bmatrix} 145.62 & 0.21074 \\ 4.4698 & -35.807 \end{bmatrix} + \rho^3 \begin{bmatrix} 70.575 & 18.009 \\ 70.575 & 18.009 \end{bmatrix} \tag{23}$$

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## APPENDIX

The following lemma deals with the degree of the adjoint of the polynomial matrix $A_\rho = \sum_{i=0}^{p} \rho^i A_i$ of degree $p$.

*Lemma A.1:* Consider matrices $A_i \in \mathbb{R}^{n\times n}$ $(i = 0, 1, \ldots, p)$ with $\dim[\bigcap_{i=1}^{p} \mathcal{N}(A_i)] = q$. Then

$$\mathrm{Adj}\,(A_\rho) := \mathrm{Adj}\left(\sum_{i=0}^{p} \rho^i A_i\right) = \sum_{i=0}^{\mu} \rho^i N_i \tag{A.1}$$

for some constant matrices $N_i$, $(i = 0, 1, \ldots, \mu)$, where

$$\mu \leq p \, \min\{n-1, n-q\}. \tag{A.2}$$

*Proof:* Recall that the determinant of a matrix $F \in \mathbb{R}^{n\times n}$ can be computed from [20]

$$\det F = \sum_{a\in \mathbf{A}} \mathrm{sign}(a) \prod_{i=1}^{n} F_{i,a_i}, \tag{A.3}$$

where $a := (a_1, a_2, \ldots, a_n)$, $\mathbf{A}$ is the set of permutations of $\{1, 2, \ldots, n\}$, and $\mathrm{sign}(a)$ is the signature of the permutation $a$ taking the values of either $+1$ or $-1$. The determinant of $A_\rho = \sum_{i=0}^{p} \rho^i A_i$ is thus a sum of $n!$ terms, each term being the product of $n$ elements. Moreover, each of these elements is chosen from a different row and column of the matrix $A_\rho$. Therefore, for every possible permutation $(a_1, a_2, \ldots, a_n)$,

$$\deg \prod_{i=1}^{n} F_{i,a_i} \leq np,$$

which together with (A.3), yields that

$$\deg\left(\det \sum_{i=0}^{p} \rho^i A_i\right) \leq np. \tag{A.4}$$

Assume now that $\dim[\bigcap_{i=1}^{p} \mathcal{N}(A_i)] = q$. Then there exist $q$ of linearly independent constant vectors $v_1, v_2, \ldots, v_q \in \mathbb{R}^n$ such that

$$A_i v_j = 0, \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q.$$

Choose $n-q$ linearly independent constant vectors $u_1, u_2, \ldots, u_{n-q} \in \mathbb{R}^n$ such that the matrix

$$T = \Big[u_1, u_2, \ldots, u_{n-q}, v_1, v_2, \ldots, v_q\Big] \tag{A.5}$$

is invertible. Furthermore,

$$\begin{aligned}\det \sum_{i=0}^{p} \rho^i A_i &= \det\left(T^{-1}\big(\sum_{i=0}^{p} \rho^i A_i\big)T\right) \\ &= \det T^{-1} \det\Big(\sum_{i=0}^{p} \rho^i A_i T\Big) \\ &= \det T^{-1} \det\,\Big[\bar{u}_1, \cdots, \bar{u}_{n-q}, \bar{v}_1, \cdots, \bar{v}_q\Big],\end{aligned}$$

where $\bar{u}_i = \sum_{j=0}^{p} \rho^j A_j u_i,\ i = 1, 2, \ldots, n-q$ and $\bar{v}_i = A_0 v_i,\ i = 1, 2, \ldots, q$. Since $\bar{v}_i$ are constant vectors, together with the determinant formula (A.3), one has

$$\det \left[\bar{u}_1, \ldots, \bar{u}_{n-q}, \bar{v}_1, \ldots, \bar{v}_q\right]$$
$$= \sum_{a_1 \neq a_2 \neq \ldots \neq a_n} \pm(\bar{u}_{1,a_1} \bar{u}_{2,a_2} \ldots \bar{u}_{(n-q),a_{(n-q)}} \bar{v}_{1,a_{(n-q+1)}} \ldots \bar{v}_{q,a_n}).$$

For every possible permutation $(a_1, a_2, \ldots, a_n)$, we have that $\deg \bar{u}_{1,a_1} \cdots \bar{u}_{(n-q)a_{(n-q)}} \bar{v}_{1,a_{(n-q+1)}} \cdots \bar{v}_{q,a_n} = \deg \bar{u}_{1,a_1} \cdots \bar{u}_{(n-q)a_{(n-q)}} \leq p(n-q)$. It follows that $\deg\left(\det\left[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-q}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_q\right]\right) \leq p(n-q)$ and hence

$$\deg\left(\det \sum_{i=0}^{p} \rho^i A_i\right) \leq p\,(n-q).$$

The result now follows from the fact that

$$\left[\mathrm{Adj}\,(A_\rho)\right]_{ij} = (-1)^{i+j} \det(A_\rho)_{[ji]}, \qquad 1 \leq j, i \leq n$$

■

The following lemma is an extension of the well-known Finsler's Lemma to the case of continuously parameter-varying matrices.

*Lemma A.2 (Parameter-dependent Finsler's Lemma):* Let the parameter-dependent matrices $B_\rho \in \mathbb{R}^{n\times m}$ and $P_\rho \in \mathbb{R}^{n\times n}$. Suppose that $\mathrm{rank}\, B_\rho < n$ and $P_\rho = P_\rho^{\mathsf{T}}$ for all $\rho \in \Omega$ and that

$$N_\rho^{\mathsf{T}} P_\rho N_\rho < 0, \qquad \forall \rho \in \Omega, \tag{A.6}$$

where $N_\rho$ is the null space of $B_\rho$. Then there exists a $\tau_\rho > 0$ such that

$$P_\rho < \tau_\rho B_\rho B_\rho^{\mathsf{T}}, \qquad \forall \rho \in \Omega. \tag{A.7}$$

Moreover, $\tau_\rho$ can be chosen to depend continuously on $\rho$.

*Proof:* The existence of a $\tau_\rho > 0$ satisfying (A.7) for each $\rho \in \Omega$ follows from the standard Finsler's Theorem [19]. It remains to show that $\tau_\rho$ can be selected to depend continuously on $\rho$. To this end, let $\phi(\rho, \tau)$ denote the minimum eigenvalue of $\tau B_\rho^{\mathsf{T}} B_\rho - P_\rho$. Clearly, $\phi$ is continuous in both its arguments and defines a continuous surface $\Phi$ in the $(\rho, \tau, \phi)$-space. Moreover, $\phi(\rho, \tau_2) \geq \phi(\rho, \tau_1)$ whenever $\tau_2 > \tau_1$. From the proof of Finsler's Theorem [19] it follows that for each $\rho \in \Omega$ there exists a unique $\hat{\tau}_\rho$ such that $\phi(\rho, \hat{\tau}_\rho) = 0$. Furthermore, $\phi(\rho, \tau) > 0$ for all $\tau > \hat{\tau}_\rho$ and $\phi(\rho, \tau) < 0$ for all $\tau < \hat{\tau}_\rho$. This means that the surface $\Phi$ intersects transversally the level set $\phi = 0$. Moreover, this intersection defines a continuous curve $\tau = \hat{\tau}_\rho$ on the $(\rho, \tau)$-plane. Then $\tau_\rho := \hat{\tau}_\rho + 1$ is continuous in $\rho$ and satisfies (A.7). ■