2005 American Control Conference
June 8-10, 2005. Portland, OR, USA



# Results on solution sets to hybrid systems with applications to stability theory

R. Goebel and A.R. Teel

***Abstract*— Motivated by questions in stability theory for hybrid dynamical systems, we establish some fundamental properties of the set of solutions to such systems. Using the notion of a hybrid time domain and general results on set and graphical convergence, we establish under weak regularity and local boundedness assumptions that the set of solutions is sequentially compact and "upper semicontinuous" with respect to initial conditions and system perturbations. The latter means that each solution to the system under perturbations is close to some solution of the unperturbed system on a compact hybrid time domain. The general facts are then used to establish several results for the behavior of hybrid systems that have asymptotically stable compact sets. For example, it is shown that the basin of attraction for a compact attractor is (relatively) open, the attractivity is uniform from compact subsets of the basin of attraction, and asymptotic stability is robust with respect to small perturbations.**

## I. INTRODUCTION

The development of effective nonlinear controllers requires a clear understanding of stability and its robustness in nonlinear systems. For differential equations, this theory is well established and nicely summarized in the book by Khalil [16], for example. For discontinuous and/or switching systems, the theory is more recent and not yet complete.

In the last decade, important elements of stability theory for differential inclusions have been established. Clarke et al. [9] showed the existence of smooth Lyapunov functions for asymptotically stable differential inclusions, and in the process, that the asymptotic stability is a robust property. (Related results are in [20], [6], and [31]. For discrete-time "difference inclusions" see [15].) Ryan [27] established a general invariance principle, extending the seminal work of Krasovskii [17] and LaSalle [19], [18]. Artstein and coworkers [2], [1] provided novel singular perturbation and averaging results for differential inclusions. See also [30].

The goal of this paper is to provide some of the tools that will allow the mentioned results to be extended to hybrid inclusions: systems where the state flows according to a differential inclusion and also jumps according to a difference inclusion. In differential inclusions, the main facts from which (robust) stability results follow are that sets of solutions are sequentially compact under mild growth conditions (in particular, the limit of solutions is a solution) and "upper semicontinuous" with respect to initial conditions and system perturbations (every perturbed solution is close, in an appropriate sense, to some unperturbed one).

One of the first obstacles to considering such results for hybrid systems is the fact that ordinary time may be insufficient to describe the evolution of a hybrid system. Several characterizations of potentially suitable hybrid time domains are given in the literature. We point to Tavernini [29], Michel [23], Lygeros et al. [21], and van der Schaft and Schumacher [32] for particular examples. Most recently, the concurrent conference papers by Collins [10] and the authors et al. [11] have proposed treating the number of jumps as an independent variable and parameterizing the state of a hybrid system by $(t, j)$ – that is, $x(t, j)$ is the state at time $t$ and after $j$ jumps. In [11], the motivation for such parameterization was that it naturally allows for the use of graphical convergence to solutions of hybrid systems. Such convergence, and other tools of set-valued analysis, are well-developed and used in nonsmooth analysis, see Rockafellar and Wets [26], and Aubin and Cellina [4] for applications to differential inclusions. The need to rely on nonclassical analysis is quite strong in hybrid systems, as, for example, the standard concepts like uniform convergence are not well-suited to handle discontinuous solutions.

Earlier results on the continuity of hybrid solutions with respect to initial conditions include those by Tavernini [29], Broucke and Arapostathis [7], and Lygeros et al. [22]. These give, respectively: continuity near "regular states" under strong continuity properties of the data; existence of continuous selections from sets of solutions when Zeno behaviors are excluded; and continuity of solutions under a uniqueness assumption. The work of Collins [10], for systems with a compact state space, contains a statement about the upper semicontinuity of a map from initial conditions to jump values that are possible after a given number of jumps. In our work, we have no uniqueness assumptions, permit Zeno behaviors, and allow a noncompact state space. The regularity assumptions in our work, here and in [11], extend those in [10] beyond compact state spaces, and appear to be the weakest possible for the results reported here. For example, from the differential inclusions describing the continuous evolution of the hybrid system we do not require more than what is needed for upper semicontinuity of solutions when no discrete behaviors are present; similarly for the difference inclusions describing the jumps.

R. Goebel's address is 3518 NE 42 Street, Seattle, WA 98105; *rafal@ece.ucsb.edu*.

A.R. Teel is with Center for Control Engineering and Computation, Department of Electrical and Computer Engineering, University of California, Santa Barbara 93106-9560; *teel@ece.ucsb.edu*.

Research by R. Goebel was carried out at Center for Control Engineering and Computation.

Research by A.R. Teel was partially supported by the Army Research Office under Grant no. DAAD19-03-1-0144, the National Science Foundation under Grant no. CCR-0311084 and ECS-0324679, and by the Air Force Office of Scientific Research under Grant no. F49620-03-1-0203.

0-7803-9098-9/05/$25.00 ©2005 AACC

Related results on the behavior of solutions under regularization or perturbation of the system include Johansson et al. [13], Hiskens and Pai [12], Prieur [24], and Prieur and Astolfi [25]. The last work addresses robust stabilization of a certain nonholonomic chained system. We note though that, to our knowledge, a systematic study of robust stability of hybrid systems has not been previously carried out.

In what follows, we establish that the solution set for hybrid inclusions satisfying basic conditions is sequentially compact (Theorem 4.2) and upper semicontinuous (Corollaries 4.4 and 5.4). As applications, we show basic properties of compact attractors for hybrid systems, in particular that their asymptotic stability is semiglobally practically robust with respect to perturbations. Results on a general LaSalle-like invariance principle and the construction of smooth Lyapunov functions for asymptotically stable hybrid systems, based on the foundations established here, can be found in Sanfelice et al. [28] and Cai et al. [8].

## II. Hybrid Inclusions

We write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$ and $\mathbb{N}$ for $\{0, 1, \ldots\}$. We call a subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ \ldots \leq t_J$. We say $S$ is a *hybrid time domain* if for all $(T, J) \in S$,

$$S \ \cap \ ([0, T] \times \{0, 1, \ldots, J\})$$

is a compact hybrid domain; equivalently, if $S$ is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with $T$ finite or $T = +\infty$. Hybrid time domains were proposed in [10] and [11]. They are essentially equivalent to "hybrid time trajectories" of [21], [5], and [22], but give a more prominent role to the "discrete" variable $j$. On each hybrid domain there is a natural ordering of points: $(t, j) \preceq (t', j')$ if $t+j \leq t'+j'$ (equivalently, $t \leq t'$ and $j \leq j'$). Points from two different hybrid time domains need not be comparable.

By a *hybrid arc* we will understand a function $x$ defined on a hybrid time domain, and such that $x(t, j)$ is locally absolutely continuous in $t$ for a fixed $j$. Alternatively, one could think of a hybrid arc as a set-valued mapping from $\mathbb{R}_{\geq 0} \times \mathbb{N}$ whose domain is a hybrid time domain (for a set-valued mapping $M$, the *domain* $\operatorname{dom} M$ is the set of arguments for which the value is nonempty). A sample solution of a hybrid system (corresponding to the height in the Bouncing Ball example, see for example [22]) in the hybrid coordinates is shown in Figure 1.

The state of a hybrid system is often given by a "continuous" variable and "discrete" one. We will not explicitly distinguish between the two. The set of potential values of the discrete variable – often consisting of descriptive elements like "off" or "on" – can be identified with a subset of integers. This leads to more compact notation.

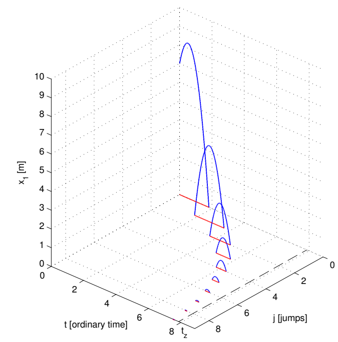


Fig. 1. *Solution of a hybrid system in hybrid coordinates*

A hybrid system $\mathcal{H}$ will be given on a state space $O$ by set-valued mappings $F$ and $G$ describing, respectively, the continuous and the discrete evolutions, and sets $C$ and $D$ where these evolutions may occur. A hybrid arc $x : \operatorname{dom} x \mapsto O$ is a *solution to the hybrid system* $\mathcal{H}$ if

(S1) For all $j \in \mathbb{N}$ and almost all $t$ such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j)). \qquad (1)$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j+1) \in G(x(t, j)). \qquad (2)$$

The fundamental conditions on $\mathcal{H}$ that will enable us to show, among other things, that an appropriately understood limit of solutions to $\mathcal{H}$ is itself a solution, are:

*(A0)* $O \subset \mathbb{R}^n$ *is an open set.*

*(A1)* $C$ *and* $D$ *are relatively closed sets in* $O$*.*

*(A2)* $F : O \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and locally bounded, and* $F(x)$ *is nonempty and convex for all* $x \in C$*.*

*(A3)* $G : O \rightrightarrows O$ *is outer semicontinuous and* $G(x)$ *is nonempty for all* $x \in D$*.*

The set $C$ is relatively closed in $O$ if $C = O \cap \overline{C}$, where $\overline{C}$ is the closure of $C$; similarly for $D$. The mapping $F$ is *locally bounded* if for any compact $K \subset O$ there exists $m > 0$ such that $F(K) \subset m\mathbb{B}$, where $\mathbb{B}$ denotes the closed unit ball. Outer semicontinuity is defined in Section III.

A solution to $\mathcal{H}$ is called *maximal* if it cannot be extended, and *complete* if its domain is unbounded. Clearly, complete solutions are maximal. In this framework, a solution is *Zeno* if it is complete and $\operatorname{dom} x$ is bounded in the $t$-direction. Basic existence and extension of solutions properties of $\mathcal{H}$ are summarized below. In a slightly different setting, most of them were stated in [5].

Below, $T_C(x)$ is the *tangent cone* to $C$ at $x \in C$. It is the set of all $v \in \mathbb{R}^n$ for which there exists a sequence of real numbers $\alpha_i \searrow 0$ and a sequence $v_i \to v$ such that for $i = 1, 2, \ldots$, $x + \alpha_i v_i \in C$. For details, see [3] or [26].

*Proposition 2.1:* *Assume (A0)-(A2). If $x^0 \in D$ or the following condition holds:*

*(VC) $x^0 \in C$ and for some neighborhood $U$ of $x^0$, for all $x' \in U \cap C$, $T_C(x') \cap F(x') \neq \emptyset$,*

*then there exists a solution $x$ to $\mathcal{H}$ with $x(0,0) = x^0$ and $\operatorname{dom} x \neq (0,0)$. If, for all $x^0 \in C \cup D$, either*

*(VD) $x^0 \in D$ and $G(x^0) \cap (C \cup D) \neq \emptyset$,*

*or (VC) holds, then every solution to the hybrid system can be extended to a maximal solution, and a maximal solution $x$ is either complete or eventually leaves every compact subset of $O$.*

The viability condition (VC) is automatically satisfied at each point $x^0$ in the interior of $C$. Thus, when $C \cup D = O$, either (VC) or (VD) holds for all $x^0 \in C \cup D$.

## III. Preliminaries – Set Convergence

Consider a sequence $\{S_i\}_{i=1}^{\infty}$ of sets in $\mathbb{R}^n$. Its *outer limit*, denoted $\limsup_{i\to\infty} S_i$, is the set of all $x \in \mathbb{R}^n$ for which there exists a subsequence $\{S_{i_k}\}_{k=1}^{\infty}$ and points $x_{i_k} \in S_{i_k}$, such that $x_{i_k} \to x$. The *inner limit*, denoted $\liminf_{i\to\infty} S_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist $x_i \in S_i$ such that $x_i \to x$. The *limit* of the sequence exists if the outer and inner limits agree, and then $\lim_{i\to\infty} S_i = \limsup_{i\to\infty} S_i = \liminf_{i\to\infty} S_i$. The inner and outer limits always exist, are closed (Proposition 4.4, [26]) but may be empty. If the outer limit is empty (and then so are the inner limit and the limit), the sequence *escapes to the horizon*; equivalently: for all $\rho > 0$ there exists $i_0$ such that for all $i > i_0$, $S_i \cap \rho\mathbb{B} = \emptyset$.

*Example 3.1:* Let $S_i = [a_i, b_i] \subset \mathbb{R}$. The limit of $S_i$'s exists if and only if $a_i$'s and $b_i$'s converge (to finite or infinite limits). Then, $\lim_{i\to\infty} S_i = [\lim_{i\to\infty} a_i, \lim_{i\to\infty} b_i]$ if the latter two limits are finite (otherwise the infinite "endpoints" are not in the limit). In general, the inner limit is the interval with endpoints $a = \limsup_{i\to\infty} a_i$, $b = \liminf_{i\to\infty} b_i$ if $a \le b$; otherwise it is empty. The outer limit need not be an interval; for example if $S_{2i-1} = [1,2]$, $S_{2i} = [3,4]$, then $\limsup_{i\to\infty} S_i = [1,2] \cup [3,4]$.

*Lemma 3.2: Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of hybrid time domains, and suppose $\lim_{i\to\infty} S_i = S$. Then $S$ is a hybrid time domain. If each $S_i$ is unbounded, then so is $S$.*

Set convergence has a certain uniformity property, which applies to unbounded sets if truncations are considered.

*Theorem 3.3: ([26], Thm 4.10) For a sequence $\{S_i\}_{i=1}^{\infty}$ and a closed set $S$, $\lim_{i\to\infty} S_i = S$ if and only if for all $\varepsilon > 0$, $\rho > 0$, there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$,*

$$S \cap \rho\mathbb{B} \subset S_i + \varepsilon\mathbb{B}, \qquad S_i \cap \rho\mathbb{B} \subset S + \varepsilon\mathbb{B}.$$

As an immediate consequence of this fact, one can show that arcs eventually get close to their omega limits.

*Example 3.4:* Let $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ be a complete hybrid arc. The omega limit of $x$, denoted $\Omega(x)$, is the set of all accumulation points of $x(t,j)$ as $t+j \to +\infty$. Equivalently,

$$\Omega(x) = \limsup_{i\to\infty} S_i \quad \text{where} \quad S_i = \{x(t,j) \mid t+j \ge i\}.$$

A general property of set limits implies $\Omega(x)$ is closed. If $x$ is uniformly bounded, then $\Omega(x) \neq \emptyset$, and Theorem 3.3 yields: for all $\varepsilon > 0$ there exists $m_\varepsilon$ such that for all $(t,j) \in \operatorname{dom} x$ with $t+j \ge m_\varepsilon$, $x(t,j) \in \Omega(x) + \varepsilon\mathbb{B}$.

An important property of set convergence is that, much like for real numbers, a sequence of sets either diverges or has a convergent subsequence. We will rely on this in selecting subsequences of solutions to hybrid systems.

*Theorem 3.5: ([26], Thm 4.18) Every sequence $\{S_i\}_{i=1}^{\infty}$ of nonempty subsets of $\mathbb{R}^n$ either escapes to the horizon or has a subsequence converging to a nonempty set $S$.*

Set convergence can be used to give sequential definitions of continuity of set-valued mappings. A mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at $x$ if for all sequences $x_i \to x$, $y_i \in M(x_i)$ with $y_i \to y$, we have $y \in M(x)$. Equivalently, $M$ is outer semicontinuous at $x$ if for all sequences $x_i \to x$ we have $\limsup_{i\to\infty} M(x_i) \subset M(x)$. Inner semicontinuity and continuity can also be defined, see Chapter 5 in [26]. The mapping $M$ is outer semicontinuous on $\mathbb{R}^n$ if and only if the *graph* of $M$:

$$\operatorname{gph} M := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in M(x)\}$$

is a closed set (Theorem 5.7 in [26]). Our assumptions can also be phrased in terms of graphs: $F : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous if $\operatorname{gph} F$, equal to $\{(x,y) \mid x \in O, y \in F(x)\}$, is relatively closed in $O \times \mathbb{R}^n$.

Outer semicontinuous mappings have closed values. If the mapping is also locally bounded, the values are compact. For locally bounded set-valued mappings with closed values, outer semicontinuity agrees with what is often referred to as *upper semicontinuity*: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x'$ with $\|x'-x\| < \delta$, $M(x') \subset M(x) + \varepsilon\mathbb{B}$. This need not be true in general.

## IV. Graphical convergence of solutions

Set convergence gives rise to a nonclassical concept of convergence of functions and set-valued mappings. A sequence $\{M_i\}_{i=1}^{\infty}$ of set-valued mappings *converges graphically* to a set-valued mapping $M$ if $\lim_{i\to\infty} \operatorname{gph} M_i = \operatorname{gph} M$. See Chapter 5 in [26] for more details.

Even if each $M_i$ is a function, the graphical limit may be set-valued. For example, take $M_i : [0,1] \mapsto [0,1]$ given by $M_i(x) = x^i$. The graphical limit is $M(x) = 0$ for $x \in [0,1)$, $M(1) = [0,1]$. Graphical convergence can easily treat mappings with different domains.

We will usually be interested in graphical convergence of hybrid arcs subject to some boundedness assumptions. We will say that a sequence of hybrid arcs $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ is *locally eventually bounded with respect to $O$* if

> for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset O$ such that for all $i > i_0$, all $(t,j) \in \operatorname{dom} x_i$ with $t+j < m$, $x_i(t,j) \in K$.

If a locally eventually bounded with respect to $O$ sequence converges graphically to $x$, then $x(t,j) \in O$ for all $(t,j) \in \operatorname{dom} x$. More importantly, we have the following result.

*Lemma 4.1: Assume (A0)-(A3). Let $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$, $i = 1, 2, \ldots$ be solutions to $\mathcal{H}$. Suppose that $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded with respect to O and converges graphically to a set-valued mapping $x : \mathbb{R}_{\geq 0} \times \mathbb{N} \mapsto \mathbb{R}^n$. Then $\operatorname{dom} x$ is a hybrid domain and $x$ is a solution to $\mathcal{H}$.*

Application of Theorem 3.5 to graphs of solutions to $\mathcal{H}$ (see also Theorem 5.36, [26]), together with Lemma 4.1 and Lemma 3.2, yields the following important result.

*Theorem 4.2: Assume (A0)-(A3). Let $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ be a locally eventually bounded with respect to O sequence of solutions to $\mathcal{H}$. Then there exists a subsequence of $x_i$'s graphically converging to a solution of $\mathcal{H}$. Such a limiting solution is complete if each $x_i$ is complete, or more generally, if no subsequence of $x_i$'s has uniformly bounded domains (i.e. for any $m > 0$, there exists $i_m \in \mathbb{N}$ such that for all $i > i_m$, there exists $(t, j) \in \operatorname{dom} x_i$ with $t + j > m$).*

We write $\mathcal{S}(x^0)$ for the set of all maximal solutions to $\mathcal{H}$ from $x^0$, $\mathcal{S}(K)$ for all those from a set $K$, and call $\mathcal{H}$ *forward complete* at $x^0$ if each $x \in \mathcal{S}(x^0)$ is complete.

Results given so far did not require any growth conditions on $G$; local boundedness was needed for $F$. In what follows, we need a related condition on $G$:

*(A4) $G : O \rightrightarrows O$ is locally bounded.*

*Theorem 4.3: Assume (A0)-(A4). Suppose $\mathcal{H}$ is forward complete at $x^0$. Then for any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$ and a compact subset $K \subset O$ such that any $x_\delta \in \mathcal{S}(x^0 + \delta\mathbb{B})$ satisfies $x_\delta(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x_\delta$, $(t, j) \preceq (T, J)$.*

Graphical convergence gives rise to a notion of closeness of solutions. For locally eventually bounded sequences, this closeness takes the form as in the Corollary below.

*Corollary 4.4: Assume (A0)-(A4). Suppose that $\mathcal{H}$ is forward complete at every $x^0 \in K$ for some compact set $K$. For any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$ with the following property: for any solution $x_\delta \in \mathcal{S}(K + \delta\mathbb{B})$ there exists a solution $x$ to $\mathcal{H}$ with $x(0, 0) \in K$ such that*

*(a) for all $(t, j) \in \operatorname{dom} x$ with $t \leq T$, $j \leq J$ there exists $s$ such that $(s, j) \in \operatorname{dom} x_\delta$, $|t - s| < \varepsilon$, and*

$$\|x(t, j) - x_\delta(s, j)\| \leq \varepsilon.$$

*(b) for all $(t, j) \in \operatorname{dom} x_\delta$ with $t \leq T$, $j \leq J$ there exists $s$ such that $(s, j) \in \operatorname{dom} x$, $|t - s| \leq \varepsilon$, and*

$$\|x_\delta(t, j) - x(s, j)\| \leq \varepsilon;$$

To conclude this section, we mention a result by Collins [10]. Here, it can be derived from Theorem 4.2. Following [10], we say that a given set of solutions to $\mathcal{H}$ is uniformly non-Zeno if there exist $T > 0$ and $J \in \mathbb{N}$ so that for any solution in that set, in any time period of length $T$, at most $J$ jumps can occur (more specificaly, if $(t, j), (t', j') \in \operatorname{dom} x$, then $|t - t'| \leq T$ implies $|j - j'| \leq J$). Suppose that a compact set $K \subset O$ is forward invariant, that is any $x \in \mathcal{S}(K)$ is such that $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. Then either the set $\mathcal{S}(K)$ of all solutions to $\mathcal{H}$ with $x(0, 0) \in K$ is uniformly non-Zeno or there exists an instantaneous Zeno solution (a complete solution $x$ with $\operatorname{dom} x = \{0\} \times \mathbb{N}$) starting in $K$.

## V. Perturbations of hybrid systems

Below, we consider a sequence of hybrid systems $\mathcal{H}_i$ given by sets $C_i$, $D_i$ and mappings $F_i : O \rightrightarrows \mathbb{R}^n$, $G_i : O \rightrightarrows O$ on the open set $O$. Since we do not require existence or outer semicontinuity from the solutions to $\mathcal{H}_i$ as studied for $\mathcal{H}$, in Sections II, III, we do not need (A1)-(A4) to hold for $\mathcal{H}_i$. We do assume the following:

(C1) Sequences of sets $\{C_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$ are such that $O \cap \limsup_{i\to\infty} C_i \subset C$, $O \cap \limsup_{i\to\infty} D_i \subset D$.

(C2) Sequences of set-valued mappings $\{F_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$ are such that $F_\infty(x) \subset F(x)$, $G_\infty(x) \subset G(x)$ for all $x \in O$, where $F_\infty$, $G_\infty$ denote the outer graphical limits of $F_i$'s, $G_i$'s.

(C3) The sequence $\{F_i\}_{i=1}^{\infty}$ is locally uniformly bounded: for any compact set $K \subset O$ there exists $m > 0$ such that for any $i = 1, 2, \ldots$ $F_i(K) \subset m\mathbb{B}$.

(C4) The sequence $\{G_i\}_{i=1}^{\infty}$ is locally uniformly bounded with respect to $O$: for any compact set $K \subset O$ there exists a compact $K' \subset O$ such that for any $i = 1, 2, \ldots$, $G_i(K) \subset K'$.

The outer graphical limit of $\{F_i\}_{i=1}^{\infty}$ is the mapping $F_\infty$ such that $\operatorname{gph} F_\infty = \limsup_{i\to\infty} \operatorname{gph} F_i$. Assumption (C1) holds, in particular, when the sequences of $C_i$'s and $D_i$'s converge, and $C = O \cap \lim_{i\to\infty} C_i$, and similarly for $D$. An analogous statement can be made about (C2).

*Theorem 5.1: Let $x_i : \operatorname{dom} x_i \mapsto \mathbb{R}^n$ be a solution to a hybrid system $\mathcal{H}_i$, $i = 1, 2, \ldots$. Suppose that the sequence $\{x_i\}_{i=1}^{\infty}$ is locally uniformly bounded with respect to O and its graphical limit $x$ exists. Then $x$ is a solution to $\mathcal{H}$.*

Below, $\mathcal{S}_i(x^0)$ is the set of maximal solutions to $\mathcal{H}_i$ from $x^0$; later in this section $\mathcal{S}_\delta(x^0)$ has a similar meaning.

*Corollary 5.2: Suppose $\mathcal{H}$ is forward complete at $x^0$. Then, for any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$, $i_0 > 0$, and a compact subset $K \subset O$ such that for all $i > i_0$, any $x \in \mathcal{S}_i(x^0 + \delta\mathbb{B})$ satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$, $(t, j) \preceq (T, J)$.*

Robustness analysis of $\mathcal{H}$ calls for consideration of perturbations $\mathcal{H}_\delta$, given on $O$ by sets $C_\delta$, $D_\delta$, and mappings $F_\delta$, $G_\delta$, for a continuously varying parameter $\delta > 0$. We will be interested in $\mathcal{H}_\delta$'s that have the *convergence property*: for any sequence $1 > \delta_1 > \delta_2 > \ldots > 0$ converging to 0, sequences $\{C_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$ and $\{F_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$ satisfy assumptions (C1), (C2), (C3), and (C4), where for each $i =$, $C_i = C_{\delta_i}$, similarly for $D_i$, $F_i$, $G_i$.

*Example 5.3:* Let $\alpha : O \mapsto \mathbb{R}_{\geq 0}$ be continuous and such that, for all $x \in O$, $x + \alpha(x)\mathbb{B} \subset O$. Then, one considers systems $\mathcal{H}_\delta$ on $O$, given for $\delta \in (0, 1)$ by the sets

$$\begin{aligned} C_\delta &= \{x \in O \mid x + \delta\alpha(x)\mathbb{B} \cap C \neq \emptyset\}, \\ D_\delta &= \{x \in O \mid x + \delta\alpha(x)\mathbb{B} \cap D \neq \emptyset\}, \end{aligned}$$

and the mappings $F_\delta : O \rightrightarrows \mathbb{R}^n$, $G_\delta : O \rightrightarrows O$ given by

$$F_\delta(x) = \operatorname{con} F\left(x + \delta\alpha(x)\mathbb{B}\right) + \delta\alpha(x)\mathbb{B},$$
$$G_\delta(x) = \{y \mid y \in \eta + \delta\alpha(\eta)\mathbb{B},\ \eta \in G\left(x + \delta\alpha(x)\mathbb{B}\right)\}.$$

Above, $\operatorname{con} S$ isthe convex hull of a set $S$. Obviously, $F_\delta$ is convex-valued. It can be checked that $F_\delta$ is nonempty-valued on $C_\delta$, same for $G_\delta$ on $D_\delta$. Finally, it can be verified that $\mathcal{H}_\delta$'s have the convergence property. We add that such perturbations combine those used for differential or difference inclusions by [9], [31], [15].

*Corollary 5.4: Suppose that $\mathcal{H}$ is forward complete at every $x^0 \subset K$ for some compact set $K$. Suppose that the perturbations $\mathcal{H}_\delta$ have the convergence property. Then, for any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta^* > 0$ with the following property: for any $\delta \in (0, \delta^*]$ and any $x_\delta \in \mathcal{S}_\delta(K + \delta\mathbb{B})$ there exists a solution $x$ to $\mathcal{H}$ with $x(0, 0) \in K$ such that conditions (a) and (b) of Corollary 4.4 hold.*

*Example 5.5:* Given a hybrid system $\mathcal{H}$ on $O$ with $F$ nonempty and convex valued on $O$ (not just on $C$), consider an augmented system $\widetilde{\mathcal{H}}_\delta$ with state space $\widetilde{O} = O \times \mathbb{R}$ and

$$\widetilde{C}_\delta = (C \times \mathbb{R}_{\geq 0}) \cup (O \times [0, \delta]), \qquad \widetilde{D}_\delta = D \times \mathbb{R}_{\geq \delta},$$
$$\widetilde{F}(\tilde{x}) = F(x) \times \{1\}, \qquad \widetilde{G}(\tilde{x}) = G(x) \times \{0\},$$

with the augmented state $\tilde{x} = (x, \tau)$ for $\tau \in \mathbb{R}$. Similar augmented systems with $\delta > 0$ were considered in [13] to eliminate Zeno behavior. Indeed, when $\delta > 0$ jumps are separated by at least $\delta$ amount of time. On the other hand, when $\delta = 0$, the behavior of the $x$ component of the solution is exactly that of $\mathcal{H}$. Such a temporal regularization has the convergence property. Thus, the conclusions of Corollary 5.4 are valid. In turn, the $x$ component of each solution to the regularized system is close to some solution of $\mathcal{H}$ on compact hybrid time domains.

## VI. Applications to Stability: Compact Attractors

For a hybrid system $\mathcal{H}$ with state space $O$, the compact set $\mathcal{A} \subset O$ is said to be *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each $x \in \mathcal{S}(\mathcal{A} + \delta\mathbb{B})$ is complete and satisfies $\|x(t, j)\|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$ (i.e. the distance from $\mathcal{A}$ to $x$ is less than $\varepsilon$). It is said to be *attractive* if there exists $\mu > 0$ such that each $x \in \mathcal{S}(\mathcal{A} + \mu\mathbb{B})$ is complete and satisfies $\lim_{t+j\to\infty} |x(t, j)|_{\mathcal{A}} = 0$. The set of points from which solutions are complete and converge to $\mathcal{A}$ is called the *basin of attraction* for $\mathcal{A}$ and is denoted $\mathcal{B}_{\mathcal{A}}$. The set $\mathcal{A}$ is said to be *locally asymptotically stable* if it is both stable and attractive. The set $\mathcal{A}$ is said to be *uniformly attractive* from the compact set $\mathcal{K} \subset O$ if each $x \in \mathcal{S}(\mathcal{K})$ is complete and for each $\varepsilon > 0$ there exists $m$ such that $x \in \mathcal{S}(\mathcal{S})$ and $t + j \geq m$ imply $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$.

*Proposition 6.1:*

(i) *Suppose that for each $x_0 \in C \cup D$, either (VC) or (VD) of Proposition 2.1 holds. Then for any locally asymptotically stable compact set, the basin of attraction is open relative to $C \cup D$.*

(ii) *If a compact set $\mathcal{A}$ is forward invariant and uniformly attractive from a compact set containing a neighborhood of $\mathcal{A}$ in $C \cup D$ then it is stable, and hence locally asymptotically stable.*

(iii) *A locally asymptotically stable compact set is uniformly attractive from each compact subset of its basin of attraction.*

(iv) *For each compact subset $K$ of the basin of attraction,*

$$\operatorname{reach}_{T,J}(K) := \{x(t, j) \mid x \in \mathcal{S}(K), (t, j) \preceq (T, J)\}.$$

*is compact for any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$. Furthermore, $\cup_{(T,J)\in\mathbb{R}_{\geq 0}\times\mathbb{N}} \mathit{reach}_{T,J}(K)$ is compact.*

The results above, when specialized to differential inclusions and difference inclusions, have appeared recently in [9], [31], [15], and [14].

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KLL}$ if it is continuous, $\beta(\cdot, t, j)$ is zero at zero and nondecreasing, $\beta(s, \cdot, j)$ and $\beta(s, t, \cdot)$ are nonincreasing and converge to zero as the argument becomes unbounded. A function $\omega : X \to \mathbb{R}_{\geq 0}$ is a proper indicator of a compact $A \subset X$ with respect to an open $X$ if $\omega(x) = 0$ if and only if $x \in A$, and $\omega(x_i) \to \infty$ when $\lim x_i \notin X$ or $\lim |x_i| = \infty$.

*Theorem 6.2: Suppose the compact set $\mathcal{A}$ is locally asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}} = (C \cup D) \cap X$, where $X$ is an open subset of $O$. Let $\omega$ be a proper indicator of $A$ with respect to $X$. Then, there exists $\beta \in \mathcal{KLL}$ such that, for all solutions starting in $\mathcal{B}_{\mathcal{A}}$,*

$$\omega(x(t, j)) \leq \beta(\omega(x(0, 0)), t, j) \qquad \forall (t, j) \in \operatorname{dom} x\ . \quad (3)$$

*Furthermore, if $\mathcal{H}_\delta$, $\delta \in (0, 1)$, is any family of perturbations of $\mathcal{H}$ with the convergence property, then for each compact set $\mathcal{K} \subset \mathcal{B}_{\mathcal{A}}$ and each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, the solutions $x_\delta$ of $\mathcal{H}_\delta$ from $\mathcal{K}$ satisfy, for all $(t, j) \in \operatorname{dom} x_\delta$,*

$$\omega(x_\delta(t, j)) \leq \beta(\omega(x_\delta(0, 0)), t, j) + \varepsilon\ . \quad (4)$$

The result above does not addresses the question of existence of solutions to perturbed systems $\mathcal{H}_\delta$. Under the current assumptions, not much can be said about existence, as the convergence property does not assume any regularity of $C_\delta$, $D_\delta$, $F_\delta$, and $G_\delta$. When $\mathcal{H}_\delta$ is obtained by an "outer perturbation" of $\mathcal{H}$ as in Example 5.3, then the data for $\mathcal{H}_\delta$ has the properties (A1), (A2), (A3), and if furthermore $C \cup D = O$ for the original system, and consequently $C_\delta \cup D_\delta = O$, then solutions to $\mathcal{H}_\delta$ do exist for any initial point in $O$. However, even then the question of existence of solutions to a system with exogeneous inputs $\dot{x} \in F(x + e)$ if $x + e \in C$ and $x^+ \in G(x + e)$ if $x + e \in D$, for example to $\mathcal{H}$ under measurement error, remains unanswered. We do not pursue this topic in this paper.

Finally, the following result is inspired by and relies on the result of Collins [10] for unperturbed systems that have compact state space (mentioned at the end of Section IV).

*Theorem 6.3: Under the assumptions of Theorem 6.2, if $\mathcal{H}$ has no instantaneous Zeno solutions in $\mathcal{B}_{\mathcal{A}}$, then for each* *compact $\mathcal{K} \subset \mathcal{B}_{\mathcal{A}}$, there exists $\delta^* > 0$ such that all solutions to $\mathcal{H}_\delta$, $\delta \in (0, \delta^*]$, starting in $\mathcal{K}$ are uniformly non-Zeno.*

*Example 6.4:* Consider controlling the state $x_1$ of an integrator using an actuator, with state denoted $x_2$, that can move in the negative direction at a constant rate (normalized to minus one) but that can be reset in the positive direction instantaneously. The flow equation is given by $\dot{x}_1 = x_2$, $\dot{x}_2 = -1$. We consider the flow set $C := \left\{x : x_1 \geq 0 \text{ or } x_2 \geq \sqrt{-k_c x_1}\right\}$ where $k_c \geq 2$, the jump set $D := \overline{\mathbb{R}^2 \backslash C}$ and the reset rule

$$x_2^+ = \max\{0, -\eta x_2\} + \sqrt{-k_r x_1}$$

where $k_r > k_c$ and $\eta \in [0, 1)$. The set $\{x : x_1 \geq 0\}$ is forward invariant, trajectories converge to this set in finite time, and within this set the solutions are the same as those of the bouncing ball example (see, for example, [22]) with $g = 1$. In particular, the origin is forward invariant (the only solution from the origin is the instantaneous Zeno solution that remains at the origin) and uniformly attractive. Thus, by Proposition 6.1 (ii), the origin is globally asymptotically stable. To eliminate the instantaneous Zeno solution at the origin, we can use a temporal regularization with $\widetilde{C}_\delta$ and $\widetilde{D}_\delta$ as defined in Example 5.5 for $\delta \geq 0$, and additional dynamics $\dot{\tau} = 1 - \tau$, $\tau^+ = 0$. We note that the set $\mathbb{R}_{\geq 0}$ is forward invariant for $\tau$, independent of $\delta \geq 0$. Also, when $\delta = 0$, the $x$ component of the solution is exactly the solution without the temporal regularization and $\tau$ converges uniformly to the interval $[0, 1]$. It follows that the set compact set $\mathcal{A} := \{(x, \tau) : \|x\| = 0\ ,\ \tau \in [0, 1]\}$ is asymptotically stable when $\delta = 0$ with basin of attraction $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In particular, according to Theorem 6.2, with $z := (x, \tau)$ and $\omega(z) := \|x\| + \max\{0, \delta - 1\}$, there exists $\beta \in \mathcal{KLL}$ such that, for each solution,

$$\omega(z(t, j)) \leq \beta(\omega(z(0, 0)), t, j).$$

Furthermore, for each $\varepsilon > 0$ and compact set $K \subset \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ there exists $\delta^* > 0$ such that the solutions of the hybrid system $\widetilde{H}_\delta$, $\delta \in (0, \delta^*]$, satisfy

$$\omega(z_\delta(t, j)) \leq \beta(\omega(z_\delta(0, 0)), t, j) + \varepsilon.$$

In particular, for $\tau(0, 0) \in [0, 1]$,

$$\|x_\delta(t, j)\| \leq \beta(\|x_\delta(0, 0)\|, t, j) + \varepsilon.$$

## REFERENCES


[1] Z. Artstein. Invariant measures of differential inclusions applied to singular perturbations. *J. Diff. Eq.*, 152:289–307, 1999.

[2] Z. Artstein and A. Vigodner. Singularly perturbed ordinary differential equations with dynamic limits. *Proc Royal Society of Edinburgh*, 126A:541–569, 1996.

[3] J.-P. Aubin. *Viability theory*. Birkhauser, 1991.

[4] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, 1984.

[5] J.-P. Aubin, J. Lygeros, M. Quincampoix, S. Sastry, and N. Seube. Impulse differential inclusions: a viability approach to hybrid systems. *IEEE Trans. Automat. Control*, 47(1):2–20, 2002.

[6] A. Bacciotti and L. Rosier. *Liapunov Functions and Stability in Control Theory*, volume 267 of *Lecture Notes in Control and Information Sciences*. Springer Verlag, 2001.

[7] M. Broucke and A. Arapostathis. Continuous selections of trajectories of hybrid systems. *Systems & Control Lett.*, 47:149–157, 2002.

[8] C. Cai, A.R. Teel, and R. Goebel. Converse Lyapunov theorems and robust asymptotic stability for hybrid systems. In *Proc 24th American Control Conference*, 2005.

[9] F.H. Clarke, Y.S. Ledyaev, and R.J. Stern. Asymptotic stability and smooth Lyapunov functions. *J. Diff. Eq.*, 149(1):69–114, 1998.

[10] P. Collins. A trajectory-space approach to hybrid systems. In *16th International Symposium on Mathematical Theory of Networks and Systems*, 2004.

[11] R. Goebel, J. Hespanha, A. Teel, C. Cai, and R. Sanfelice. Hybrid systems: generalized solutions and robust stability. In *Stuttgart Symposium on Nonliear Control Systems*, 2004.

[12] I.A. Hiskens and M.A. Pai. Trajectory sensitivity analysis of hybrid systems. *IEEE Trans. Circuits Syst - Part I: Fund. Theory and Apps.*, 47(2):204–220, 2000.

[13] K.H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. On the regularization of Zeno hybrid automata. *Systems & Control Letters*, 38:141–150, 1999.

[14] C.M. Kellett. *Advances in converse and control Lyapunov functions*. PhD thesis, University of California, Santa Barbara, 2002.

[15] C.M. Kellett and A.R. Teel. Smooth Lyapunov functions and robustness of stability for difference inclusions. *Systems & Control Letters*, 52:395–405, 2004.

[16] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd edition, 2002.

[17] N. N. Krasovskii. *Problems of the theory of stability of motion*. Stanford Univ. Press, 1963. Translation of Russian edition, Moscow 1959.

[18] J. P. LaSalle. An invariance principle in the theory of stability. In *Differential equations and dynamical systems*. New York: Academic Press, 1967.

[19] J. P. LaSalle and S. Lefschetz. *Stability by Lyapunov s direct method*. New York: Academic Press, 1961.

[20] Y.S. Ledyaev and E.D. Sontag. A Lyapunov characterization of robust stabilization. *Nonlinear Analysis*, 37:813–840, 1999.

[21] J. Lygeros, K.H. Johansson, S. Sastry, and M. Egerstedt. On the existence of executions of hybrid automata. In *Proc 38th IEEE Conference on Decision and Control, Phoenix, 1999*, pages 2249–2254, 1999.

[22] J. Lygeros, K.H. Johansson, S.N. Simić, J. Zhang, and S.S. Sastry. Dynamical properties of hybrid automata. *IEEE Trans. Automat. Control*, 48(1):2–17, 2003.

[23] A.N. Michel and B. Hu. Towards a stability theory of general hybrid dynamical systems. *Automatica*, 35(3):371–384, 1999.

[24] C. Prieur. Perturbed hybrid systems, applications in control theory. In A. Zinober and D. Owens, editors, *Nonlinear and adaptive control*, volume 281 of *Lecture Notes in Control and Inform. Sci.*, pages 285–294. Springer, Berlin, 2003.

[25] C. Prieur and A. Astolfi. Robust stabilization of chained systems via hybrid control. *IEEE Trans. Automat. Control*, 48(10):1768–1772, 2003.

[26] R.T. Rockafellar and R. J-B Wets. *Variational Analysis*. Springer, 1998.

[27] E.P. Ryan. An integral invariance principle for differential inclusions with applications in adaptive control. *SIAM J. Control Optim.*, 36(3):960–980, 1998.

[28] R.G. Sanfelice, R. Goebel, and A.R. Teel. Results on convergence in hybrid systems via detectability and an invariance principle. In *Proc. 24th American Control Conference*, 2005.

[29] L. Tavernini. Differential automata and their discrete simulators. *Nonlinear Anal.*, 11(6):665–683, 1987.

[30] A.R. Teel, L. Moreau, and D. Nesic. A unified framework for input-to-state stability in systems with two time scales. *IEEE Transactions on Automatic Control*, 48(9):1526–1544, 2003.

[31] A.R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM Control Optim. Calc. Var.*, 5:313–367, 2000.

[32] A. van der Schaft and H. Schumacher. *An Intruduction to Hybrid Dynamical Systems*, volume 251 of *Lect. Notes in Contr. and Inform. Sci.* Springer, 2000.