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Optimal Output Transitions for Dual-Stage Systems: Disk Drive Example

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***Abstract*— The minimum-energy output-transition problem for dual-stage systems is solved in this article. The objective is to find optimal feedforward inputs that change the system output from an initial value $\underline{y}$ at time t_i to a final value $\overline{y}$ at time t_f during a specified output-transition time-interval $[t_i, t_f]$. The main contribution of this article is the solution of the optimal output-transition problem for systems with actuator redundancy (dual-stage systems). Additionally, we show that the use of pre- and post-actuation inputs, outside the output-transition time-interval, can lead to substantial reduction of the output-transition cost. In particular, the method is applied to a dual-stage disk drive model and simulation results are presented. The results show that the output-transition cost (input-energy) can be reduced by 97% with the use of pre- and post-actuation inputs when compared to standard methods that do not use such pre- and post-actuation.**

I. Introduction

The minimum-energy output-transition problem for dual-stage systems is solved in this article. The objective is to find optimal feedforward inputs that change the system output from an initial value $\underline{y}$ at time t_i to a final value $\overline{y}$ at time t_f. In such output-transition problems, the output should be maintained at a constant value outside the output-transition time-interval — maintaining the output constant outside of the transition time-interval is critical to reduce the time lost to useful operations. For example, in the disk-drive application ([1], [2]), the goal is to move the position of a read-write head from one data track (initial value) to the next desired track (final value); read and write operations cannot be performed (before and after the output-transition) if the output position is not precisely maintained at the desired track. Similar output-transition problems also arise in other flexible structure applications such as: (a) positioning the endpoint of large-scale manipulators ([3], [4], [5]) and (b) nano-scale positioning and manipulation using relatively small piezoactuators ([6], [7]). The main contribution of this article is the solution of the optimal output-transition problem for systems with actuator redundancy (dual-stage systems). Additionally, we show that the use of pre- and post-actuation inputs (outside the output-transition time-interval) can lead to substantial reduction of the output-transition cost. In particular, the method is applied to a dual-stage disk drive model described in [8] and simulation results are presented. The results show that the output-transition cost (input-energy) can be reduced by 97% with the use of pre- and post-actuation inputs when compared to standard methods that do not use such pre- and post-actuation.

As opposed to the problem of changing the output-point on a flexible structure, the problem of changing the complete configuration (i.e., the state) of a flexible structure has been well studied in literature, e.g., [1]-[5], [9], [10], [11]. These techniques, which solve the state-to-state transition problem (referred to as the state-transition problem), can also be used to find a solution to the output-transition problem. In particular, output-transitions without residual vibrations can be obtained by requiring that the flexible system maneuver between equilibrium configurations (rigid-body rest configurations). (*These* rest states, $\underline{x}$ *and* $\overline{x}$*, are chosen to result in the initial and final output values,* $\underline{y}$ *and* $\overline{y}$*, respectively.*) Once the boundary states, at the beginning and end of the output-transition, are chosen to be the rest states (i.e., $x(t_i) = \underline{x}$, and $x(t_f) = \overline{x}$), a solution to the output-transition problem can be found by solving the standard, minimum-energy, state-transition problem from the initial state ($x(t_i)$) to the final state ($x(t_f)$); this is referred to as the rest-to-rest state-transition approach ([5], [9], [11]). However, the solution found with this choice of the rest-to-rest boundary states $\{x(t_i) = \underline{x},\ x(t_f) = \overline{x}\}$ may not lead to optimal (minimum input-energy) output-transition. On the other hand, arbitrary choices of the boundary states $\{x(t_i),\ x(t_f)\}$ are also not acceptable; they may not allow the output to be maintained at a constant value after the completion of the output-transition (i.e., without residual vibrations) for any choice of bounded inputs. Therefore, the standard optimal state-to-state transition approach cannot be used to directly solve the optimal output-transition problem.

The output-transition problem was previously considered in [12], in which an inversion-based technique was used to plan an output-transition along a prescribed output trajectory. However, the approach in [12] requires the user to specify (a priory) the set of acceptable output trajectories during the output-transition (using polynomials); it is unclear how the set of output trajectories is chosen such that it includes the optimal output trajectory. Similar pre-specification of a desired output trajectory is also needed in [13], which uses optimal filtering to achieve smooth transitions between output-trajectory segments in industrial positioning systems. In contrast, our previous work [14], [15] directly solved the optimal output-transition problem. It does not require the pre-specification of the output trajectory; rather, the best output trajectory is obtained as the result of the optimization procedure.

Previous solutions to the optimal output-transition problem were restricted to square systems (same number of outputs as inputs). In contrast, the current article solves the optimal output-transition problem for linear systems with actuator redundancy. The method is applied to a dual-stage disk drive model [8]. While the addition of the second actuator leads to the 17% reduction of the output-transition cost, we show that the significant factor in the cost reduction is the use of the pre- and post-actuation inputs. In general, the proposed optimal output-transition approach uses both pre- and post-actuation inputs to reduce the output-transition cost. (*It is noted that the standard state-to-state transition approach uses neither pre- nor post-actuation.*) Pre-actuation input has to be applied to the system before the output-transition is initiated; therefore, it requires preview information of an impending output-transition (the amount of required preview time is quantified in [16]). However, pre-actuation will not be applicable if such preview information of the desired output-transition is not available. For such cases, a modified output-transition approach is proposed that only uses post-actuation (without pre-actuation). The implications of using (or not using) pre-actuation are investigated for the disk-drive seek control problem. The simulation results show significant reduction of the output-transition cost with the use

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of the proposed optimal output-transition method (with or without pre-actuation) when compared to the use of the standard approach that is based on state-to-state transition (which uses neither pre-actuation nor post-actuation). For the dual-stage example studied in this article, we show that for the same amount of input-energy, the use of pre- and post-actuation reduces the required transition time by 67%. While this article considers the minimization of input-energy, the concept of using pre- and post-actuation is general and can be used to optimize other criteria such as output-transition time and input magnitudes; for example the combined minimization of transition time and energy was studied in [17].

This paper is organized in the following format. The dual-stage optimal output-transition problem is formulated in Section II. The optimal output-transition (OOT) solution is derived in Section III, and the OOT solution without using the pre-actuation is presented in Section IV. Application to the dual-stage disk drive is in Section V. Our conclusion is in Section VI.

II. Problem Formulation

We consider a linear, time-invariant, dynamical system in the state-space form, described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_1 u_1(t) + B_2 u_2(t) \\ y(t) &= Cx(t) \end{cases}, \forall\, t \in (-\infty, \infty) \quad (1)$$

where the system state is $x(t) \in \Re^n$. Furthermore, it is assumed that the system is dual-input $\{u_1(t), u_2(t)\}$ and single-output $y(t)$ (DISO) and *controllable*. Next, we consider the problem of changing the output position from one value to another within a finite time-interval, called the output-transition time-interval. It is noted that the output should be maintained constant (at the desired value) outside the output-transition time-interval. Formally, the output-transition problem [14], [15] is defined as follows.

Definition 1: **The output-transition problem** is to find bounded input-state trajectories $\{u_1(\cdot), u_2(\cdot), x_{ref}(\cdot)\}$ that satisfy the system equations in (Eq. 1) and the following two conditions:

1. The output-transition condition: The output is transferred from an initial value $\underline{y}$ to a final value $\overline{y}$ within the output-transition time-interval $[t_i, t_f]$, and is maintained constant at the desired value before and after the output-transition, i.e.,

$$\begin{array}{lllll} y_{ref}(t) &=& \underline{y} &=& C\underline{x} \quad \forall\, t \le t_i \\ y_{ref}(t) &=& \overline{y} &=& C\overline{x} \quad \forall\, t \ge t_f \end{array} \quad (2)$$

where $y_{ref}(t) := Cx_{ref}(t)$; t_i and t_f denote the times when the output transition starts and completes, respectively. Furthermore, the states $\underline{x}$ and $\overline{x}$ denote the initial and final equilibrium states (i.e., $A\underline{x} = A\overline{x} = 0$), which are chosen to result in the desired initial and final output values ($\underline{y}$ and $\overline{y}$).

2. The delimiting-state condition: The state approaches the equilibrium configuration as time goes to (plus or minus) infinity, i.e.,

$$x_{ref}(t) \to \underline{x} \text{ as } t \to -\infty; \text{ and } x_{ref}(t) \to \overline{x} \text{ as } t \to \infty. \quad (3)$$

For controllable systems, there exists at least one solution such that the desired output transition is achieved, e.g. by setting the state $x(t) = \underline{x}$ during the pre-transition time ($t \le t_i$) and the state $x(t) = \overline{x}$ during the post-transition time ($t \ge t_f$) the output-transition problem becomes a state-transition problem which can be solved by existing techniques (see, for example, [18]). In this article, we want to choose the input that minimizes the input-energy performance. The optimal output-transition (OOT) problem is stated as follows.

Definition 2: **The optimal (minimum energy) output-transition problem (OOT)** is to find bounded input-state trajectories $\{u_1(\cdot), u_2(\cdot), x_{ref}(\cdot)\}$ that solve the output-transition problem (see Definition 1), and minimize the following input-energy cost function,

$$J = \int_{-\infty}^{\infty} U(t)\, dt := \int_{-\infty}^{\infty} \left[R_1\{u_1(t)\}^2 + R_2\{u_2(t)\}^2\right] dt \quad (4)$$

$$= \int_{-\infty}^{t_i} U(t)\, dt + \int_{t_i}^{t_f} U(t)\, dt + \int_{t_f}^{\infty} U(t)\, dt$$

$$:= J_{pre} + J_{tran} + J_{post} \quad (5)$$

where the positive constants R_1 and R_2 represent the weighting factors between the two inputs.

III. The Optimal Output-transition (OOT) Solution

A. Inverse input law

First, we derive an input law that allows the output to be maintained at the desired value. This input is unique and is obtained by using the dynamic inversion technique (see [19]) described as follows.

Assumption 1 (Relative degree): Throughout the rest of the article, we assume that the system (Eq. 1) has a relative degree r, see [19].

From the definition of relative degree, the r^{th}-derivative output equation can be written explicitly in terms of the input as

$$y^{(r)}(t) = CA^r x(t) + CA^{r-1}B_1 u_1(t) + CA^{r-1}B_2 u_2(t) \quad (6)$$

where the term $CA^{r-1}B_1 \neq 0$ and/or the term $CA^{r-1}B_2 \neq 0$. Without loss of generality, we assume that the term $CA^{r-1}B_1 \neq 0$, i.e., the inverse input is described in terms of the first input $u_1(t)$. Then the first input $u_1(t)$ given by the following inverse input law,

$$u_1(t) := u_{1,inv}(t)$$

$$= \frac{1}{CA^{r-1}B_1}\left[y_d^{(r)}(t) - CA^r x(t)\right] - \frac{CA^{r-1}B_2}{CA^{r-1}B_1} u_2(t), \quad (7)$$

maintains exact output tracking where $y_d(t)$ is the desired output trajectory. We observe that if the term $CA^{r-1}B_2 \neq 0$, the inverse input $u_{1,inv}$ will depend on the choice of the second input u_2. Assumption 1 also implies that there exists a coordinate transformation matrix, $T_{\xi\eta}$, whose first r rows are chosen as C, CA, $CA^2, \ldots, CA^{r-1}$, and the remaining $n-r$ rows are chosen such that the matrix $T_{\xi\eta}$ is invertible, i.e.,

$$T_{\xi\eta} := \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{r-1} \\ \hline T_\eta \end{array}\right]. \quad (8)$$

It is noted that the set $\{C,\ CA, \ldots,\ CA^{r-1}\}$ is linearly independent (see [19]), therefore we can always find T_η (in Eq. 8) such that $T_{\xi\eta}$ is invertible. The transformation $T_{\xi\eta}$ partitions the state $x(t)$ into two components: (i) the first r terms are the output and its derivatives up to order $r-1$ (denoted by ξ), and (ii) the remaining components are called the internal state of the system (denoted by η), i.e.

$$T_{\xi\eta} \cdot x(t) = \left[\begin{array}{c} \xi(t) \\ \eta(t) \end{array}\right], \quad (9)$$

where $\xi(t) := \left[y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)\right]^T$. Conversely, let the inverse of the transformation $T_{\xi\eta}$ be defined by

$$x(t) = T_{\xi\eta}^{-1} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} := [\Phi_\xi \mid \Phi_\eta] \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}. \tag{10}$$

B. Post-actuation ($t \geq t_f$)

During post-transition interval, the output needs to be maintained at a constant value (i.e., $y(t) = \overline{y}, \ \forall \ t \geq t_f$), see (Eq. 2). In order to achieve this output-tracking requirement, the first input $u_1(t)$, during the post-transition interval, must be chosen as the inverse input given by (Eq. 7). However, the second input $u_2(t)$ can be chosen arbitrarily, therefore, it can be optimally chosen to minimize the cost. We derive the optimal output-transition solution during the post-transition interval by the following steps.

First, to simplify the derivation, the system state is shifted as $\hat{x}(t) := x(t) - \overline{x}$ where the state $\overline{x}$ is the final equilibrium state; consequently, the transformed components (ξ, η) in the shifted coordinates become

$$\begin{aligned} \begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} &= T_{\xi\eta}\hat{x}(t) = T_{\xi\eta}\left\{x(t) - \overline{x}\right\} = \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} - T_{\xi\eta}\overline{x} \\ &:= \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} - \begin{bmatrix} \overline{\xi} \\ \overline{\eta} \end{bmatrix}. \end{aligned} \tag{11}$$

Since the output along the post-transition interval is constant, i.e. $y(t) = \overline{y}, \ \forall t \in [t_f, \ \infty)$, the component $\xi(t) = \overline{\xi}, \ \forall t \in [t_f, \infty)$. In other word, the shifted component $\hat{\xi}(t) = 0$ during the post-transition interval. Furthermore, the state in the shifted coordinates becomes (from Eq. 10)

$$\hat{x}(t) = [\Phi_\xi \mid \Phi_\eta] \begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} = \Phi_\eta \hat{\eta}(t). \tag{12}$$

Second, we rewrite the inverse input (Eq. 7) in terms of the shifted state $\hat{x}(t)$ as

$$\begin{aligned} \hat{u}_{1,inv}(t) &= \frac{y_d^{(r)}(t) - CA^r x(t)}{CA^{r-1}B_1} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_2(t) \\ &= \frac{y_d^{(r)}(t) - CA^r\left\{\hat{x}(t) + \overline{x}\right\}}{CA^{r-1}B_1} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_2(t). \end{aligned}$$

Note that (i) the r^{th}-derivative of the output $y_d^{(r)}(t) = 0$ since the output is constant during the post-transition interval, (ii) the term $CA^r\overline{x} = 0$ since the state $\overline{x}$ is an equilibrium state, and (iii) the shifted state $\hat{x}(t) = \Phi_\eta\hat{\eta}(t)$ as derived in (Eq. 12). So the above inverse input becomes

$$\hat{u}_{1,\ inv}(t) = -\frac{CA^r\Phi_\eta}{CA^{r-1}B_1}\hat{\eta}(t) - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_2(t). \tag{13}$$

Third, we rewrite the system equation (Eq. 1) in terms of the shifted state $\hat{x} = x(t) - \overline{x}$ as

$$\begin{aligned} \dot{\hat{x}}(t) = \dot{x}(t) &= A\left\{\hat{x}(t) + \overline{x}\right\} + B_1u_1(t) + B_2u_2(t) \\ &= A\hat{x}(t) + B_1u_1(t) + B_2u_2(t) \end{aligned}$$

where the term $A\overline{x} = 0$ since the state $\overline{x}$ is an equilibrium state. Then, apply the coordinate transformation $T_{\xi\eta}$ (Eq. 9) and apply the inverse input law given by (Eq. 13) as the first input, to the above system equation, we have

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\xi}}(t) \\ \dot{\hat{\eta}}(t) \end{bmatrix} &= T_{\xi\eta}AT_{\xi\eta}^{-1}\begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} + T_{\xi\eta}B_1u_1(t) + T_{\xi\eta}B_2u_2(t) \\ &= T_{\xi\eta}AT_{\xi\eta}^{-1}\begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} + T_{\xi\eta}B_2u_2(t) \\ &\quad + T_{\xi\eta}B_1\left(-\frac{CA^r\Phi_\eta}{CA^{r-1}B_1}\hat{\eta}(t) - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_2(t)\right) \\ &= \left(T_{\xi\eta}AT_{\xi\eta}^{-1} - \frac{T_{\xi\eta}B_1CA^rT_{\xi\eta}^{-1}}{CA^{r-1}B_1}\right)\begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} \\ &\quad + T_{\xi\eta}\left(B_2 - \frac{B_1CA^{r-1}B_2}{CA^{r-1}B_1}\right)u_2(t) \\ &:= \begin{bmatrix} A_\xi & 0 \\ A_{\eta\xi} & A_\eta \end{bmatrix}\begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2\eta} \end{bmatrix}u_2(t). \end{aligned} \tag{14}$$

It is observed that the above system equation has a lower-triangular structure as a result of the coordinated transformation $T_{\xi\eta}$ and the inverse input law. Since the component $\hat{\xi}(t) = 0$ during the post-transition interval, the system equation (Eq. 14) is then reduced to the *internal dynamics* given by

$$\dot{\hat{\eta}}(t) = A_\eta\hat{\eta}(t) + B_{2\eta}u_2(t). \tag{15}$$

Fourth, rewrite the post-transition cost from (Eq. 5) in terms of the shifted coordinate, by substituting the first input $u_1(t)$ from (Eq. 13), i.e.

$$\begin{aligned} J_{post} &= \int_{t_f}^{\infty} U(t)\ dt = \int_{t_f}^{\infty}\left[R_1\{u_1(t)\}^2 + R_2\{u_2(t)\}^2\right]dt \\ &= \int_{t_f}^{\infty}\left[\hat{\eta}^T\Phi_\eta^T\left\{\frac{(A^r)^TC^TR_1CA^r}{(CA^{r-1}B_1)^2}\right\}\Phi_\eta\hat{\eta}\right. \\ &\quad + u_2\left\{R_2 + \frac{B_2^T(A^{r-1})^TC^TR_1CA^{r-1}B_2}{(CA^{r-1}B_1)^2}\right\}u_2 \\ &\quad \left. + 2u_2\left\{\frac{B_2^T(A^{r-1})^TC^TR_1CA^r}{(CA^{r-1}B_1)^2}\right\}\Phi_\eta\hat{\eta}\right] dt \\ &:= \int_{t_f}^{\infty}\left[\hat{\eta}^TQ\hat{\eta} + u_2Nu_2 + 2u_2S\hat{\eta}\right]\ dt. \end{aligned} \tag{16}$$

Now the optimal output-transition problem during the post-transition interval becomes a standard linear-quadratic (LQ) optimal control problem in terms of the internal state. In particular, the problem is reduced to find the input $u_2(t)$ that minimizes the post-transition cost given by (Eq. 16) subject to the internal dynamics (Eq. 15). The following assumption is to ensure the existence of a bounded solution to the LQ optimal control problem.

Assumption 2 (Necessary conditions for LQ optimal control): Throughout the rest of the article, we assume that the internal dynamics (Eq. 15) is controllable under the input $u_2(t)$, i.e., the pair $(A_\eta, B_{2\eta})$, as described in (Eq. 15), is controllable. Furthermore, we assume that the the pair $(A_\eta, \sqrt{Q})$ is observable.

Remark 1 (On the necessary condition assumption): The above controllability and observability of the internal dynamics are assumed in order to simplify the derivation of the solution. In general, the pair $(A_\eta, B_{2\eta})$ may not be controllable or the pair $(A_\eta, \sqrt{Q})$ may not be observable. In such systems, additional procedures are required to take the account of the uncontrollable and the unobservable subspace of the internal state. This treatment of the subspaces of the internal state is similar to procedures found in Refs. [14], [15].

Lemma 1 (Post-actuation): Given an internal state $\eta(t_f)$ at the final transition time t_f, the post-actuation inputs

$\{u_{1,post}(t), u_{2,post}(t)\}$ that satisfy the output-transition conditions (Eqs. 2 and 3) and minimize the post-transition cost (Eq. 16) are given by

$$u_{1,post}(t) = -\frac{CA^r\Phi_\eta}{CA^{r-1}B_1}\Big\{\eta(t)-\overline{\eta}\Big\} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_{2,post}(t) \quad (17)$$

$$u_{2,post}(t) = -N^{-1}\left(B_{2\eta}^T W_{post} + S^T\right)\Big\{\eta(t)-\overline{\eta}\Big\} \quad (18)$$

where the symmetric matrix W_{post} is the solution to the algebraic Riccati equation (ARE) given by

$$A_\eta^T W_{post} + W_{post}A_\eta + Q - (W_{post}B_{2\eta} + S^T)N^{-1}(B_{2\eta}^T W_{post} + S) = 0. \quad (19)$$

Furthermore, the cost associated with these post-actuation inputs is uniquely specified in terms of the choice of the internal state $\eta(t_f)$ at the completion of the output-transition and is given by

$$J_{post}\{\eta(t_f)\} = \Big\{\eta(t_f)-\overline{\eta}\Big\}^T W_{post}\Big\{\eta(t_f)-\overline{\eta}\Big\}. \quad (20)$$

Proof: By applying the first input $u_1(t)$ to the system as the inverse input (Eq. 13), for any choice of the second input $u_2(t)$ that is bounded, the output is maintained constant at the final value $\overline{y}$ during the post-transition interval, thus the output-transition condition (Eq. 2) is satisfied. It is also observed that the system equation which is reduced to the internal dynamics (Eq. 15) and the post-transition cost (Eq. 16) becomes solely dependent on the choice of the second input $u_2(t)$.

The derivation of the optimal second input $u_{2,post}(t)$ in (Eq. 18) that minimizes the post-transition cost J_{post} (Eq. 16) subject to the internal dynamics (Eq. 15) and the derivation of the associated cost (Eq. 20) can be found in many standard optimal control textbooks, e.g. in Ref. [18], and is omitted in this proof for brevity. We also note that in the expressions for the post-actuation inputs (Eqs. 17 and 18), the shifted internal state $\hat{\eta}(t)$ is replaced by the component in the original coordinate using (Eq. 11), i.e. replace $\hat{\eta}(t)$ by $\eta(t)-\overline{\eta}$.

Assumption 2 implies that the optimal LQ input will result in the strictly-stable closed-loop system. In other words, by applying the optimal second input $u_{2,post}(t)$ into the internal dynamics (Eq. 15), the shifted internal state $\hat{\eta}(t)$ will asymptotically approach zero, i.e. $\lim_{t\to\infty}\hat{\eta}(t) = 0$ or $\lim_{t\to\infty}\eta(t) = \overline{\eta}$. In addition, the first input (Eq. 17) enforces the output to be at the final value $\overline{y}$, so the component $\xi(t) = \overline{\xi}$ during the post-transition interval. Thus, the state approaches the final equilibrium state as time goes to infinity since the component $\{\xi(t), \eta(t)\}$ asymptotically approach the equilibrium configuration $\{\overline{\xi}, \overline{\eta}\}$. Therefore, the post-actuation inputs (Eqs. 17 and 18) satisfy the delimiting state condition (Eq. 3). ■

Remark 2 (Post-transition internal state): The internal state $\eta(t)$ that appears in the expressions for the post-actuation inputs (Eqs. 17 and 18) is uniquely specified in terms of the choice of the internal state $\eta(t_f)$ at the completion of the output-transition, and can be obtained from off-line simulation of the closed-loop internal dynamics:

$$\begin{aligned}\dot{\eta}(t) &= \dot{\hat{\eta}}(t) = A_\eta\hat{\eta}(t) + B_{2\eta}u_{2,post}(t) \\ &= A_\eta\eta(t) - A_\eta\overline{\eta} + B_{2\eta}u_{2,post}(t) \\ &= \Big\{A_\eta - B_{2\eta}N^{-1}\left(B_{2\eta}^T W_{post} + S^T\right)\Big\}\eta(t) + \\ &\quad \Big\{B_{2\eta}N^{-1}\left(B_{2\eta}^T W_{post} + S^T\right) - A_\eta\Big\}\overline{\eta} \\ &:= A_{\eta,post}^{cl}\ \eta(t) + B_{\eta,post}^{cl}\ \overline{\eta}\end{aligned} \quad (21)$$

with the initial condition $\eta(t_f)$ at the time t_f, for all time $t \geq t_f$.

C. Pre-actuation ($t \leq t_i$)

The pre-actuation input is obtained by using procedures similar to that used in the previous subsection for finding the post-actuation input. We begin by shifting the system state as $\check{x}(t) := x(t) - \underline{x}$ where the state $\underline{x}$ is the initial equilibrium state; consequently, the transformed components (ξ, η) in this shifted coordinate become

$$\begin{bmatrix}\check{\xi}(t)\\ \check{\eta}(t)\end{bmatrix} = T_{\xi\eta}\check{x}(t) = T_{\xi\eta}\{x(t)-\underline{x}\} = \begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} - T_{\xi\eta}\underline{x} := \begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} - \begin{bmatrix}\underline{\xi}\\ \underline{\eta}\end{bmatrix}. \quad (22)$$

Since the output along the pre-transition interval ($t \leq t_i$) is constant, i.e. $y(t) = \underline{y}$, the component $\xi(t) = \underline{\xi}$, $\forall t \in (\infty, t_i]$. In other words, the shifted component $\check{\xi}(t) = 0$ during the pre-transition interval. Similarly, the inverse input (Eq. 7) during the pre-transition interval becomes

$$\check{u}_{1,inv}(t) = -\frac{CA^r\Phi_\eta}{CA^{r-1}B_1}\check{\eta}(t) - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_2(t). \quad (23)$$

By applying the coordinate transformation $T_{\xi\eta}$ and the above inverse input law to the system equation (Eq. 1), the system equation during the pre-transition interval is reduced to the *internal dynamics* given by

$$\dot{\check{\eta}}(t) = A_\eta\check{\eta}(t) + B_{2\eta}u_2(t) \quad (24)$$

where the matrices A_η and $B_{2\eta}$ are the same as described in (Eq. 14) for the post-transition interval.

Next, rewrite the pre-transition cost from (Eq. 5) in terms of the shifted coordinate, by substituting the first input $u_1(t)$ from (Eq. 23), i.e.,

$$\begin{aligned}J_{pre} &= \int_{-\infty}^{t_i} U(t)\ dt = \int_{-\infty}^{t_i}\left[R_1\{u_1(t)\}^2 + R_2\{u_2(t)\}^2\right]dt \\ &= \int_{-\infty}^{t_i}\Bigg[\check{\eta}^T\Phi_\eta^T\left\{\frac{(A^r)^TC^TR_1CA^r}{(CA^{r-1}B_1)^2}\right\}\Phi_\eta\check{\eta} \\ &\quad + u_2\left\{R_2 + \frac{B_2^T(A^{r-1})^TC^TR_1CA^{r-1}B_2}{(CA^{r-1}B_1)^2}\right\}u_2 \\ &\quad + 2u_2\left\{\frac{B_2^T(A^{r-1})^TC^TR_1CA^r}{(CA^{r-1}B_1)^2}\right\}\Phi_\eta\check{\eta}\Bigg]\ dt \\ &:= \int_{t_f}^{\infty}\left[\check{\eta}^TQ\check{\eta} + u_2Nu_2 + 2u_2S\check{\eta}\right]\ dt.\end{aligned} \quad (25)$$

Assumption 2 is also assumed on the internal dynamics (Eq. 24), i.e. the pair $(A_\eta, B_{2\eta})$ is controllable and the pair $(A_\eta, \sqrt{Q})$ is observable, in order to ensure the existence of a bounded solution to the LQ optimal control problem.

Lemma 2 (Pre-actuation): Given an internal state $\eta(t_i)$ at the initial transition time t_i, then the pre-actuation inputs $\{u_{1,pre}(t), u_{2,pre}(t)\}$ that satisfy the output-transition conditions (Eqs. 2 and 3) and minimize the pre-transition cost (Eq. 25) are given by

$$u_{1,pre} = -\frac{CA^r\Phi_\eta}{CA^{r-1}B_1}\Big\{\eta(t)-\underline{\eta}\Big\} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}u_{2,pre}(t) \quad (26)$$

$$u_{2,pre} = -N^{-1}\left(B_{2\eta}^T W_{pre} + S^T\right)\Big\{\eta(t)-\underline{\eta}\Big\} \quad (27)$$

where the symmetric matrix W_{pre} is the solution to the algebraic Riccati equation (ARE) given by

$$-A_\eta^T W_{pre} - W_{pre}A_\eta + Q \\ - (-W_{pre}B_{2\eta} + S^T)N^{-1}(-B_{2\eta}^T W_{pre} + S) = 0. \quad (28)$$

Furthermore, the cost associated with these pre-actuation inputs is uniquely specified in terms of the choice of the internal state $\eta(t_i)$ at the initiation of the output-transition and is given by

$$J_{pre}\{\eta(t_i)\} = \Big\{\eta(t_i) - \underline{\eta}\Big\}^T W_{pre}\Big\{\eta(t_i) - \underline{\eta}\Big\}. \quad (29)$$

Proof: Note that the optimal input $u_{2,pre}(t)$ that minimizes the pre-transition cost J_{pre} in (Eq. 25) subject to the internal dynamics in (Eq. 24) can be derived by reversing the time direction. Specifically, it is equivalent to find the input $u_2(t)$ that minimizes the *reversed* pre-transition cost

$$J_{pre} = \int_{t_i}^{\infty} \left[\check{\eta}^T Q\check{\eta} + u_2 N u_2 + 2u_2 S\check{\eta}\right] \, dt$$

subject to the *reversed* internal dynamics

$$\dot{\check{\eta}}(t) = -A_\eta\check{\eta}(t) - B_{2\eta}u_2(t).$$

The rest of the proof follows similar arguments as described in the post-actuation case (Lemma 1), thus is omitted for brevity. ■

Remark 3 (Pre-transition internal state): The internal state $\eta(t)$ that appears in the expressions for the pre-actuation inputs (Eqs. 26 and 27) is uniquely specified in terms of the choice of the internal state $\eta(t_i)$ at the initiation of the output-transition, and can be obtained from off-line simulation of the closed-loop *reversed* internal dynamics:

$$\begin{aligned} \dot{\eta}(t) &= \dot{\check{\eta}}(t) \;=\; -A_\eta\check{\eta}(t) - B_{2\eta}u_{2,pre}(t) \\ &= -A_\eta\eta(t) + A_\eta\underline{\eta} - B_{2\eta}u_{2,pre}(t) \\ &= -\Big\{A_\eta - B_{2\eta}N^{-1}\Big(B_{2\eta}^T W_{pre} + S^T\Big)\Big\}\eta(t) - \\ &\quad \Big\{B_{2\eta}N^{-1}\Big(B_{2\eta}^T W_{pre} + S^T\Big) - A_\eta\Big\}\underline{\eta} \\ &:= -A_{\eta,pre}^{cl}\,\eta(t) - B_{\eta,pre}^{cl}\,\underline{\eta} \end{aligned} \quad (30)$$

backward in time with the initial condition $\eta(t_i)$ at the time t_i, for all time $t \le t_i$.

D. Transition interval ($t_i \le t \le t_f$)

Since the output is required to be maintained constant at the initial value $\underline{y}$ before the initiation of output-transition, it implies that the state component $\xi(t)$ must be maintained constant at the initial equilibrium configuration $\underline{\xi}$. So the only component of the state $x(t_i)$ at the initiation of the output-transition that can be varied while satisfying the output-transition conditions (Eqs. 2 and 3) is the internal state component $\eta(t_i)$. Similarly, the state component $\xi(t)$ must be maintained constant at the final equilibrium configuration $\overline{\xi}$ after the completion of the output-transition. So the only component of the state $x(t_f)$ at the completion of the output-transition that can be varied while satisfying the output-transition conditions (Eqs. 2 and 3) is the internal state component $\eta(t_f)$. Therefore, the acceptable boundary states $\{x(t_i), x(t_f)\}$ of the output-transition must be chosen as

$$x(t_i) = T_{\xi\eta}^{-1}\Big[\underline{\xi}^T \mid \eta(t_i)^T\Big]^T \text{and } x(t_f) = T_{\xi\eta}^{-1}\Big[\overline{\xi}^T \mid \eta(t_f)^T\Big]^T \quad (31)$$

Furthermore, we define the *boundary condition* Ψ which is the components of the state, at the initiation and completion of the output transition, that can be freely varied while satisfying the conditions for the output-transition problem, i.e.

$$\Psi := \Big[\eta(t_f)^T \quad \eta(t_i)^T\Big]^T. \quad (32)$$

Given a pair of acceptable boundary state $\{x(t_i), x(t_f)\}$ at the initial transition time t_i and at the final transition time t_f, respectively, the minimum-energy input that transfer the system (Eq. 1) from the initial state $x(t_i)$ to the final state $x(t_f)$ within the output-transition time $T_{tran} = t_f - t_i$ is given by (see Chapter 3 in Ref. [18])

$$u_{1,tran}(t) = R_1^{-1}B_1^T e^{A^T(t_f-t)}G^{-1}d_x \quad (33)$$

$$u_{2,tran}(t) = R_2^{-1}B_2^T e^{A^T(t_f-t)}G^{-1}d_x \quad (34)$$

where the matrix G is the *invertible* controllability grammian, defined by

$$G = \int_{t_i}^{t_f} e^{A(t_f-\tau)}\,[B_1 \;\; B_2]\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} e^{A^T(t_f-\tau)}d\tau, \quad (35)$$

and d_x denotes the transition-state difference, given by

$$d_x = x(t_f) - e^{A(t_f-t_i)}x(t_i).$$

The state difference d_x can be partitioned in terms of the boundary condition (Ψ) defined in (Eq. 32) which is the components that can be freely chosen and the fixed component ($\hat{f}$) which must be chosen as the equilibrium configuration , i.e.

$$d_x := H_1\hat{f} + H_2\Psi \quad (36)$$

where $H_1 := [\Phi_\xi \mid -\Gamma_\xi]$, $H_2 := [\Phi_\eta \mid -\Gamma_\eta]$,
$[\Gamma_\xi \mid \Gamma_\eta] := e^{A(t_f-t_i)}[\Phi_\xi \mid \Phi_\eta]$, and $\hat{f} := \Big[\overline{\xi}^T \mid \underline{\xi}^T\Big]^T$.

The transition cost when using these optimal inputs (Eqs. 33 and 34) during the transition interval is equal to

$$J_{tran} = d_x^T G^{-1}d_x = \Big(H_1\hat{f} + H_2\Psi\Big)^T G^{-1}\Big(H_1\hat{f} + H_2\Psi\Big). \quad (37)$$

E. Optimal output-transition solution

The input-energy needed for output-transition with a specified output-transition boundary condition Ψ (Eq. 32) can be obtained by substituting the pre-transition cost (Eq. 29), the post-transition cost (Eq. 20), and the transition cost (Eq. 37) into the total cost (Eq. 5) as follows.

$$J \;=\; J_{pre} + J_{tran} + J_{post} \;:=\; \Psi^T\Lambda\Psi - 2\Psi^T b + c \quad (38)$$

where $\Lambda := \begin{bmatrix} W_{post} & 0 \\ 0 & W_{pre} \end{bmatrix} + H_2^T G^{-1}H_2$,
$b := \begin{bmatrix} W_{post}\overline{\eta} \\ W_{pre}\underline{\eta} \end{bmatrix} - H_2^T G^{-1}H_1\hat{f}$, and
$c := \overline{\eta}^T W_{post}\overline{\eta} + \underline{\eta}^T W_{pre}\underline{\eta} + \hat{f}^T H_1^T G^{-1}H_1\hat{f}$.

Note that the input-energy (Eq. 38) needed for output-transition is quadratic in terms of the specified output-transition boundary condition Ψ. The optimal value for the output-transition boundary condition Ψ is found in the next Theorem, which solves the optimal output-transition (OOT) problem.

Theorem 1 (OOT): Let Assumptions 1 and 2 be true. Then, we obtain the following results.

I) The minimum-energy output-transition problem (Definition 2) always has a solution where the optimal boundary condition Ψ^* that minimizes the cost function (Eq. 38) is given by

$$\Psi^* = \left[\begin{array}{c} \eta_f^* \\ \eta_i^* \end{array} \right] = \left\{ \begin{array}{ll} \Lambda^{-1}b & \text{, if } \Lambda \text{ is invertible} \\ \Lambda^{\dagger}b & \text{, otherwise} \end{array} \right. \quad (39)$$

where $\Lambda^{\dagger}$ is the pseudo (generalized) inverse of Λ (see [20]).

II) The optimal inputs for the optimal output-transition problem is given by

$$u_{1,oot}^* \begin{cases} = \frac{CA^{r-1}B_2}{CA^{r-1}B_1} N^{-1} \left(B_{2\eta}^T W_{pre} + S^T \right) \left\{ \eta_{pre}^*(t) - \underline{\eta} \right\} \\ \quad - \frac{CA^r \Phi_\eta}{CA^{r-1}B_1} \left\{ \eta_{pre}^*(t) - \underline{\eta} \right\} \quad \text{if } t < t_i \\ = \frac{CA^{r-1}B_2}{CA^{r-1}B_1} N^{-1} \left(B_{2\eta}^T W_{post} + S^T \right) \left\{ \eta_{post}^*(t) - \overline{\eta} \right\} \\ \quad - \frac{CA^r \Phi_\eta}{CA^{r-1}B_1} \left\{ \eta_{post}^*(t) - \overline{\eta} \right\} \quad \text{if } t > t_f \\ = R_1^{-1} B_1^T e^{A^T(t_f - t)} G^{-1} \left[x_f^* - e^{A(t_f - t_i)} x_i^* \right] \\ \quad \text{if } t_i \le t \le t_f \end{cases} \quad (40)$$

and

$$u_{2,oot}^* \begin{cases} = -N^{-1} \left(B_{2\eta}^T W_{pre} + S^T \right) \left\{ \eta_{pre}^*(t) - \underline{\eta} \right\} \text{ if } t < t_i \\ = -N^{-1} \left(B_{2\eta}^T W_{post} + S^T \right) \left\{ \eta_{post}^*(t) - \overline{\eta} \right\} \text{ if } t > t_f \\ = R_2^{-1} B_2^T e^{A^T(t_f - t)} G^{-1} \left[x_f^* - e^{A(t_f - t_i)} x_i^* \right] \\ \quad \text{if } t_i \le t \le t_f \end{cases} \quad (41)$$

where the optimal boundary states are given by

$$x_i^* = T_{\xi\eta}^{-1} \left[\underline{\xi}^T \ \eta_i^{*T} \right]^T, \text{ and } x_f^* = T_{\xi\eta}^{-1} \left[\overline{\xi}^T \ \eta_f^{*T} \right]^T.$$

The optimal internal state $\eta_{post}^*(t)$ during the post-transition interval is obtained by solving the closed-loop internal dynamics (Eq. 21) where the initial condition is chosen to be η_f^* given by (Eq. 39), see Remark 2. Similarly, the optimal internal state $\eta_{pre}^*(t)$ during the post-transition interval is obtained by solving the closed-loop internal dynamics (Eq. 30) backward in time where the initial condition is chosen to be η_i^* given by (Eq. 39), see Remark 3.

III) The optimal output-transition cost using the optimal inputs (Eqs. 40 and 41) is equal to

$$J_{oot}^* = \Psi^{*T} \Lambda \Psi^* - 2\Psi^{*T} b + c. \quad (42)$$

Proof: The output-transition cost function (Eq. 38) is quadratic in terms of the boundary condition Ψ; therefore, the optimal value for the boundary condition Ψ follows from the minimization of quadratic forms, see Theorem 1 in [14] and Theorem 4.2.1 in [20]. The optimal inputs during the post-transition interval and during the pre-transition intervals follows from Lemmas 1 and 2, respectively. The optimal inputs during the transition interval are given by (Eqs. 33 and 34) where the optimal boundary states are obtained by substituting the optimal boundary condition Ψ^* in the expressions for the acceptable boundary states (Eq. 31). The optimal cost is obtained by substituting the optimal boundary condition Ψ^* into the output-transition cost function (Eq. 38). ■

IV. Optimal output-transition without pre-actuation

In general, the proposed optimal output-transition approach uses both pre- and post-actuation inputs to reduce the output-transition cost. Pre-actuation input has to be applied to the system before the output-transition is initiated; therefore, it requires preview information of the impending output-transition. However, the pre-actuation may not be applicable if such preview information of the desired output-transition is not available (e.g., when immediate output-transition is desired). For such cases, we require that the output-transition begin with the initial transition state at the equilibrium state, i.e. $x(t_i) = \underline{x}$. Then, the optimal output-transition approach can be constrained to only use post-actuation, i.e. by optimally choosing the component of the internal dynamics at the completion of the output-transition $\eta(t_f)$. Therefore, the output-transition cost function (Eq. 38) can be rewritten as

$$\tilde{J} := \eta(t_f)^T \ \tilde{\Lambda} \ \eta(t_f) - 2\eta(t_f)^T \ \tilde{b} + \tilde{c} \quad (43)$$

where

$$\tilde{\Lambda} := W_{post} + \tilde{H}_2^T G^{-1} \tilde{H}_2; \ \tilde{b} := W_{post}\overline{\eta} - \tilde{H}_2^T G^{-1} \tilde{H}_1 \tilde{f};$$
$$\tilde{c} := \overline{\eta}^T W_{post} \overline{\eta} + \tilde{f}^T \tilde{H}_1^T G^{-1} \tilde{H}_1 \tilde{f}; \ \tilde{f} := \left[\overline{\xi}^T \ \underline{\xi}^T \ \underline{\eta}^T \right]^T;$$
$$\tilde{H}_1 := \left[\Phi_\xi \ | \ -\tilde{\Gamma} \right]; \ \tilde{H}_2 := \Phi_\eta; \ \tilde{\Gamma} := e^{A(t_f - t_i)} T_{\xi\eta}^{-1}.$$

Next, the solution to optimal output-transition (OOT) problem without pre-actuation is provided in the following Lemma.

Lemma 3: The optimal internal state $\eta(t_f)$ that minimizes the output-transition cost without pre-actuation (Eq. 43) is then given by

$$\tilde{\eta}^*(t_f) = \left\{ \begin{array}{ll} \tilde{\Lambda}^{-1}\tilde{b} & \text{, if } \tilde{\Lambda} \text{ is invertible} \\ \tilde{\Lambda}^{\dagger}\tilde{b} & \text{, otherwise} \end{array} \right. \quad (44)$$

where $\tilde{\Lambda}^{\dagger}$ is the pseudo (generalized) inverse of $\tilde{\Lambda}$. The optimal control inputs, for the output-transition problem without pre-actuation, are given by

$$u_{1,nopre}^* \begin{cases} = 0 \quad \text{if } t < t_i \\ = \frac{CA^{r-1}B_2}{CA^{r-1}B_1} N^{-1} \left(B_{2\eta}^T W_{post} + S^T \right) \left\{ \tilde{\eta}^*(t_f) - \overline{\eta} \right\} \\ \quad - \frac{CA^r \Phi_\eta}{CA^{r-1}B_1} \left\{ \tilde{\eta}^*(t_f) - \overline{\eta} \right\} \quad \text{if } t > t_f \\ = R_1^{-1} B_1^T e^{A^T(t_f - t)} G^{-1} \left[\tilde{x}_f^* - e^{A(t_f - t_i)} \underline{x} \right] \\ \quad \text{if } t_i \le t \le t_f \end{cases} \quad (45)$$

and

$$u_{2,nopre}^* \begin{cases} = 0 \quad \text{if } t < t_i \\ = -N^{-1} \left(B_{2\eta}^T W_{post} + S^T \right) \left\{ \tilde{\eta}^*(t_f) - \overline{\eta} \right\} \text{ if } t > t_f \\ = R_2^{-1} B_2^T e^{A^T(t_f - t)} G^{-1} \left[\tilde{x}_f^* - e^{A(t_f - t_i)} \underline{x} \right] \\ \quad \text{if } t_i \le t \le t_f \end{cases} \quad (46)$$

where the optimal final transition states are computed from $\tilde{x}_f^* = T_{\xi\eta}^{-1} \left\{ \left[\overline{\xi} \ \ \tilde{\eta}_{post}^*(t_f) \right] \right\}^T$.

Proof: This follows from arguments similar to those in the proof of Theorem 1 because the cost without pre-actuation (Eq. 43) has the same quadratic form as in Theorem 1 (Eq. 38). ■

Remark 4 (OOT without both pre- and post-actuation): In some cases when both pre- and post-actuation are not allowed or are not applicable, e.g. in such systems that have no internal dynamics, the optimal output-transition (OOT) solution (Eqs. 40 and 41) is equivalent to the standard state-transition (SST) solution where the inputs are only applied during the transition interval. In the SST approach, the state before and after the output-transition is maintained at the equilibrium configuration, i.e. $x(t) = \underline{x}$ for $t \le t_i$ and $x(t) = \overline{x}$ for $t \ge t_f$ in order to satisfy the output-transition conditions (Eqs. 2 and 3). The optimal inputs during the transition interval can be found by finding the minimum-energy input to achieve the state transition from the initial equilibrium state $[x(t_i) = \underline{x}]$ to the final equilibrium state $[x(t_f) = \overline{x}]$. These optimal inputs has the same form as the transition inputs given in (Eqs. 33 and 34) where the transition state difference d_x is chosen as

$$d_x = x(t_f) - e^{A(t_f - t_i)} x(t_i) = \overline{x} - e^{A(t_f - t_i)} \underline{x}$$

It is noted that the optimal OOT cost that using both pre- and post-actuation is always less than or equal the SST cost because the OOT input minimizes the cost over all possible solutions to the output-transition problem (Definition 1), which includes the solution from the SST approach.

V. Dual-Stage Hard Disk Drive Example

In this section, the optimal output-transition (OOT) approach is demonstrated by using the dual-stage hard disk drive system. The tip position (the output) of the read/write head in the dual-stage disk drive is controlled by two input signals – the first input controls the voice coil motor (VCM) and the second input controls the piezo-actuated (PZT) suspension. The model and the plant parameters used in the the following simulations are given in [8]. The objective of the output-transition is to move the read/write head from an original data track at $\underline{y}$ to the next adjacent track on the disk where the head displacement is $\overline{y}$ in the prescribed time-interval. This time-interval is referred to as the track-to-track seek time. In the the following simulations, the track-to-track seek time is chosen to be 0.3 ms. We exploit the advantage of using piezoactuator which can tolerate a relatively high input (typically up to 50V) by choosing the weighting factor on the first input (VCM input) as $R_1 = 1$ and the weighting factor on the second input (PZT input) as $R_2 = 0.0001$.

TABLE I

Cost comparison between using and not using the pre- and post-actuation for the disk drive with single-stage controller (SISO) and dual-stage controller (DISO)†

Costs ($\times 10^{-1}$) (% Cost Reduction)	SISO	DISO
Not using pre- and post-actuation	4.1 (0%)	3.4 (17%)
Using only post-actuation	2.7 (34%)	0.38 (91%)
Using both pre- and post-actuation	1.6 (61%)	0.12 (97%)

†For the 0.3 ms track-to-track seek time.

A. *Effects of pre- and post-actuation inputs*

First, we consider the optimal output-transition approach that uses both pre- and post-actuation inputs as described in Section III, see Theorem 1. The input and output trajectories, when the OOT inputs (Eqs. 40 and 41) were applied, are presented in Figures 1 and 2, respectively. It is noted that, in Figures 1 and 2, the output-transition started at time t_i = 2 ms and was completed at time t_f = 2.3 ms (i.e., the length of the output-transition time-interval was 0.3 ms). Furthermore, the input-energy cost (Eq. 42) when using the OOT approach with both pre- and post-actuation is equal to $J_{oot} = 0.012$.

When the pre-actuation is not allowed, the optimal output-transition inputs are given by (Eqs. 45 and 46), and the input-energy to achieve the output-transition without using the pre-actuation input is equal to $J_{nopre} = 0.038$ which is 22% higher than the output-transition that using both the pre and post-actuation inputs.

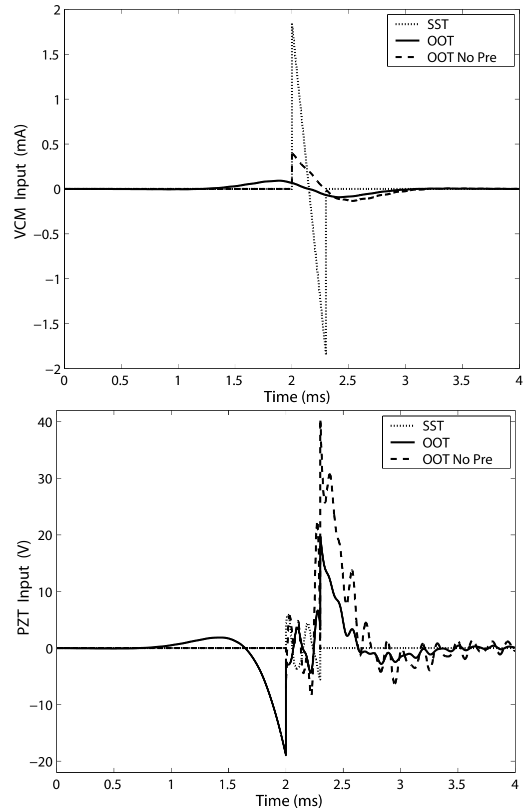

Fig. 1. Comparison of the applied inputs to the dual-stage disk drive system between using both pre- and post-actuation (OOT), using only post-actuation (OOT No Pre), and not using pre- and post-actuation (SST).

Finally, we consider the case when both pre- and post-actuation are not allowed or are not applicable. In this case, the optimal output-transition (OOT) solution becomes the standard state-transition (SST) solution, see Remark 4. The comparisons of the input and the output trajectories between using the SST approach and the OOT approach are shown in Figures 1 and 2. The input-energy cost when using the SST approach is equal to $J_{sst} = 0.34$ which is an order of magnitude higher than the input-energy required to achieve the same output-transition when using the OOT approach (with or without the pre-actuation).

It is noted that the similar results in cost reduction when using pre- and post-actuation are obtained for the case of the conventional disk drive which has a single-stage controller (i.e., the tip position of the read/write head is controlled solely by the VCM input). The cost comparisons between using and not using the pre- and post-actuation for the disk drive with single-stage controller (SISO) and dual-stage controller (DISO) are summarized in Table I, which also shows the percentage in cost reduction compared to the default case of single-stage controller without using pre- and post-actuation inputs. In Table I, we observed that the input-energy required to perform the output-transition in the dual-stage system is less than the required input-energy in the single-stage system when using the same control approach. However, the substantial cost reduction (97%) is obtained when using the proposed approach with pre- and post-actuation inputs.

B. *Effect of varying the output-transition time-interval*

The proposed approach can be used to investigate the fastest achievable output-transition time for a given limit on the input-

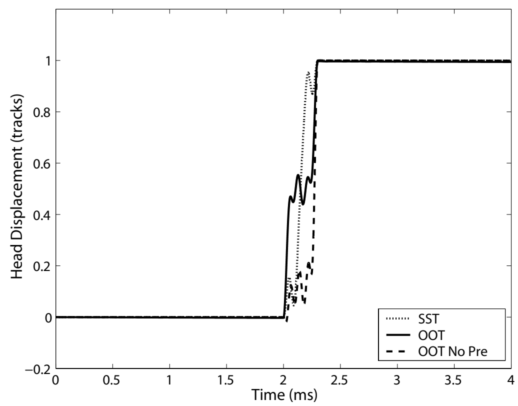

Fig. 2. Comparison of the output trajectories between using both pre- and post-actuation (OOT), using only post-actuation (OOT No Pre), and not using pre- and post-actuation (SST).

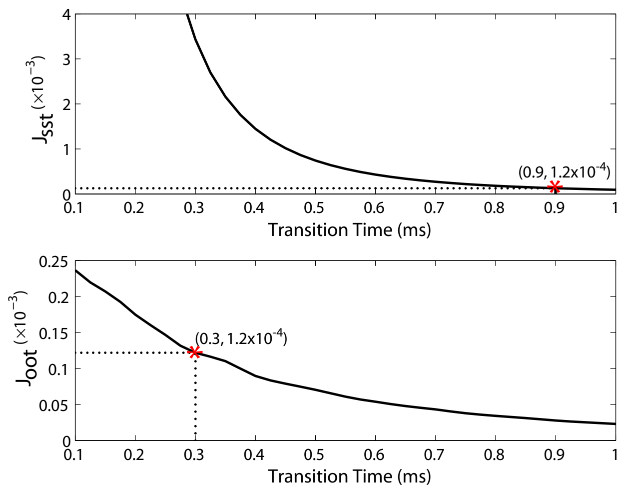

Fig. 3. Comparison of of the input-energy cost between the SST and OOT approach where the transition time is varied.

energy. For the dual-stage disk drive example, the input-energy is plotted against the output-transition time (with the OOT and SST approaches) in Figure 3. As shown in the Figure 3, the input-energy (for both SST and OOT approaches) increases as the output-transition time decreases. The simulation results show that, for a given output-transition time, the cost for OOT input is always lower than the cost for SST input as shown in Figure 3; this is expected, see Remark 4. It implies that the OOT approach can achieve faster output-transition than the SST approach with the same amount of input-energy. For example, in the dual-stage disk drive system, the cost required for 0.3-ms track-to-track seek time is $J^*_{oot} = 0.012$ when using the OOT approach, however with the same amount of input-energy the output-transition using the SST approach requires 0.9-ms to achieve the same output transition. Thus the required transition time to achieve the output transition is reduced by 67% by using the OOT approach compared to the SST approach which does not use the pre- and post-actuation.

VI. Conclusion

The direct solution to the minimum input-energy output transition for dual-stage linear systems is presented in this paper. It was shown that the proposed approach of using pre- and post-actuation inputs can substantially reduce the overall cost of the output-transition when compared to the approach based on the state-transition, that does not use pre- and post-actuation. Furthermore, when consider the same amount of input-energy, the proposed approach can achieve significantly faster output-transitions when compared to the SST approach.

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