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Min-max Feedback Model Predictive Control with State Estimation

Dong Jia and Bruce Krogh
Department of Electrical and Computer Engineering, Carnegie Mellon University
Pittsburgh, PA 15213, USA
djia@andrew.cmu.edu, krogh@ece.cmu.edu

***Abstract*— This paper proposes a state estimation scheme for output feedback min-max model predictive control (MPC). A set-membership state estimator bounds the current states of the physical system. A standard state estimator provides state feedback for the state feedback control law in MPC optimization over the prediction horizon. The paper presents a recursive method for the set-membership state estimation and conditions for bounded stability of the closed-loop system. The approach is illustrated with an example.**

I. Introduction

Model predictive control (MPC) strategies generate control inputs by solving optimal control problems at each control step using a predictive model of the system dynamics [10], [11]. We call the problem solved at each control step the MPC optimization problem. This paper concerns MPC strategies for dealing with disturbance inputs that are unknown but bounded. This leads naturally to min-max optimization problems to minimize the objective function against the worst-case value of the uncertainty. Min-max MPC problems have also been considered for robustness against parameter uncertainties [5], [9], [13]. For min-max optimization, a parameterized affine feedback law is used to obtain a solution that is less conservative than the standard open-loop MPC optimization [1], [8], [9].

When perfect state measurements are not available, controllers need to estimate the system state based on measurements of system outputs. For bounded uncertainties, set-membership state estimators have been proposed to provide bounds on system states [2], [4]. In this paper, both a standard state estimator and a set-membership state estimator are used to provide state information for min-max MPC with output feedback. The set-membership state estimator provides a bound for the current system state. A standard state estimator is incorporated in the MPC optimization to estimate the state that will be available to the feedback control law over the prediction horizon. This reflects the fact that, with output feedback, the controller takes the same control action for different states that can generate the same output.

Recursive approaches to compute set-membership state estimation have been proposed in literature [3], [6], [12], [14]. Precise computation of set-membership state estimation was proposed in [14] for systems where the initial state and disturbances are bounded by polytopes. But the precise representation and computation of the polytopes can be computationally intensive and uses excessive memory. In this paper, we use ellipsoids to approximate bounds on the current state. Ellipsoidal approximation algorithms have been given in [3], [6], [12] for recursively computing bounds on the current state of a system. Those algorithms cannot be applied directly for the min-max robust feedback MPC scheme proposed in [8], however, since they compute the optimal ellipsoidal approximation without any constraints. The min-max MPC scheme in [8] requires that the approximation should completely satisfy the prediction computed at the previous control step. We propose a method to deal with this constraint.

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II. Preliminaries and Notation

Throughout this paper, we use $\mathbf{0}$ and $\mathbf{I}$ to denote the zero matrix and the identity matrix of proper dimension, respectively. rank (A) is the rank of the matrix $A \in \Re^{m\times n}$. range (A) is the range of A. null (A) is the null space of A. C^- is the pseudo-inverse of a matrix $C \in \Re^{m\times n}$. When $m < n$ and rank $(C) = m$, $C^- = C^T\left(CC^T\right)^{-1}$. span $(\mathcal{X})$, $\mathcal{X} \subseteq \Re^m$, is the subspace of $\Re^m$ spanned from $\mathcal{X}$. $\dim(\mathcal{X})$ denotes the dimension of span $(\mathcal{X})$. $r \perp \mathcal{X}$ means that $r^T x = 0$ for all $x \in \mathcal{X}$. $\mathcal{R}^\perp$ is the orthogonal complement of the subspace $\mathcal{R} \subseteq \Re^m$.

Several operations on sets are used to simplify the presentation of our scheme. The *linear transformation* of a set $\mathcal{X} \subset \Re^m$ by a matrix A is defined as

$$A\mathcal{X} = \{y \in \Re^m | \exists x \in \mathcal{X} \ni y = Ax\}$$

The *Minkowski sum* between two set $\mathcal{X} \subset \Re^m$ and $\mathcal{Y} \subset \Re^m$ is defined as

$$\mathcal{X} \oplus \mathcal{Y} = \{z \in \Re^m | \exists x \in \mathcal{X}, y \in \mathcal{Y} \ni z = x + y\}.$$

If $\mathcal{Y}$ has only one element, i.e. $\mathcal{Y} = \{y\}$, we denote the Minkowski sum by $\mathcal{X} \oplus y$. The *Cartisan product* between two sets $\mathcal{X} \subset \Re^{m_x}$ and $\mathcal{Y} \subset \Re^{m_y}$ is defined as

$$\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \left\{ z \in \Re^{m_x+m_y} | \exists x \in \mathcal{X}, y \in \mathcal{Y} \ni z = \begin{bmatrix} x \\ y \end{bmatrix} \right\}.$$

The *over-approximation* of a set $\mathcal{X} \subset \Re^m$ is a set $\mathcal{Y} \subset \Re^m$ such that $\mathcal{X} \subset \mathcal{Y}$.

It is difficult to compute and represent sets of reachable states in general. In this work, a special class of sets, *ellipsoids*, is used to over-approximate sets of reachable states. An *ellipsoid* $\mathcal{E}(Q,c) \subset \Re^m$ is is a set of points defined by a positive semi-definite symmetric matrix $Q \in \Re^{m\times m}$ and a vector $c \in \Re^m$:

$$\mathcal{E}(Q,c) = \left\{Q^{1/2}r + c \in \Re^m | r^T r \leq 1\right\},$$

where c is the *center* of the ellipsoid. The eigenvectors of Q are the *axes* of the ellipsoid and the corresponding eigenvalues are squares of the corresponding *radii*. When Q is nonsingular, the ellipsoid can be expressed by a quadratic constraint:

$$\mathcal{E}(Q,c) = \left\{x \in \Re^m | (x-c)^T Q^{-1}(x-c) \leq 1\right\}.$$

When Q is singular, the ellipsoid is said to be *degenerate*.

In numerical computation of the set-membership estimation, we also use another special type of sets, called *cylinders*. A *cylinder* $\mathcal{X} \subseteq \Re^m$ is a set, for which there is a vector $r \in \Re^m$ such that for any point $x \in \mathcal{X}$, $x + \alpha r \in \mathcal{X}$, $\forall \alpha \in \Re$. The set of such r's composes a subspace $\mathcal{R}$, called the *direction* of the cylinder $\mathcal{X}$. An *intersection* of a cylinder $\mathcal{X}$ is a set $\mathcal{X}_{int} \subseteq \Re^m$ such that $\dim(\mathcal{X}_{int}) + \dim(\mathcal{R}) = m$ and $\mathcal{X}_{int} \oplus \mathcal{R} = \mathcal{X}$. The minimal intersection satisfies that $\exists x \in \mathcal{X}_{int}^{\min}$, $\mathcal{X}_{int}^{\min} - x \perp \mathcal{R}$. If an intersection $\mathcal{X}_{int}^{\min} = \mathcal{E}(Q', c')$ is a degenerate ellipsoid, the cylinder $\mathcal{X}$ is an *ellipsoidal cylinder*, denoted by $\mathcal{C}(P,c)$. An ellipsoidal cylinder $\mathcal{C}(P,c)$ can be defined by a quadratic constraint.

$$\mathcal{C}(P,c) = \left\{x \in \Re^m | (x-c)^T P(x-c) \leq 1\right\}.$$

III. Min-Max MPC with State Estimation

Consider an LTI system with external disturbances and measurement noise of the form

$$x(k+1) = Ax(k) + Bu(k) + Fw(t) \quad (1)$$
$$y(k) = Cx(k) + Gv(k) \quad (2)$$

subject to system constraints on states and controls presented in general as

$$g(x(k), x(k), u(k), w(k)) \leq 0, \quad (3)$$

where $x \in \Re^{m_x}$, $u \in \Re^{m_u}$ and $y \in \Re^{m_y}$ are state, control and output vectors of the system, respectively. $w \in \mathcal{W} = \mathcal{E}(Q^w, c^w) \subset \Re^{m_w}$ is a disturbance bounded by the set $\mathcal{W}$ and $v \in \mathcal{V} = \mathcal{E}(Q^v, c^v) \subset \Re^{m_v}$ is measurement noise bounded by the set $\mathcal{V}$. A, B, F, C and G are system matrices of proper dimension. We assume that the system (1)-(2) is observable and $\mathrm{rank}(C) = m_y$. For quadratic object functions, we proposed the following min-max MPC optimization in [7] for systems where the state can be measured perfectly.

Min-Max MPC

$$\min_{\mathbf{K},\bar{\mathbf{u}}} \max_{\mathbf{w}} \sum_{n=1}^{N} \|x_n^p\|_{\Gamma_n^x}^2 + \sum_{n=0}^{N-1} \|u_n^p\|_{\Gamma_n^u}^2$$

where

$$\mathbf{K} = \left\{K_1^p, K_2^p, \cdots, K_{N-1}^p\right\}$$
$$\bar{\mathbf{u}} = \left\{\bar{u}_0^p, \bar{u}_1^p, \cdots, \bar{u}_{N-1}^p\right\}$$
$$\mathbf{w} = \left\{w_0^p, w_1^p, \cdots, w_{N-1}^p\right\}$$

subject to

$$x_{n+1}^p = Ax_n^p + Bu_n^p + Fw_n^p, n = 0, 1, \cdots, N-1$$
$$u_n^p = K_n^p x_n^p + \bar{u}_n^p, n = 1, 2 \cdots, N-1$$
$$u_0^p = \bar{u}_0^p$$
$$w_n^p \in \mathcal{W} = \mathcal{E}(Q^w, c^w), n = 0, 1, \cdots, N-1$$
$$x_0^p = x(k)$$

Robustness Constraints

$$\forall \tilde{w}_n \in \mathcal{W}, n = 0, 1, \cdots, N-1$$
$$\tilde{x}_{n+1} \in \mathcal{X}_{n+1} = \mathcal{E}\left(Q_{n+1}^x, c_{n+1}^x\right)$$
$$g(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{u}_n, \tilde{w}_n) \leq 0$$

where

$$\tilde{x}_{n+1} = A\tilde{x}_n + B\tilde{u}_n + F\tilde{w}_n,$$
$$\tilde{u}_n = K_n \tilde{x}_n + \bar{u}_n^p \text{ for } n \neq 0$$
$$\tilde{u}_0 = \bar{u}_0^p$$
$$\tilde{x}_0 = x(k)$$

In [7], we developed an LMI-based scheme to compute a suboptimal solution to the above min-max MPC optimization. A time-varying affine state feedback law $u_n^p = K_n^p x_n + \bar{u}_n^p$ is introduced to reduce conservativeness.

The formulation above assumes that the state $x(k)$ is available. This work considers systems where perfect state measurements are not available. An affine state feedback control law is incorporated in MPC optimization $u_n^p = K_n^p \hat{x}_n^p + \bar{u}_n^p$, where $\hat{x}$ denotes the estimation of the system state x. The following full-order state estimator is incorporated into the MPC optimizations to provide state feedback to the affine state feedback law over the prediction horizon:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L[y(k) - C\hat{x}(k)]. \quad (4)$$

Putting system states and estimator states together, we have an augmented dynamics

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ LC & A-LC \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u(k) + \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \quad (5)$$

The following MPC optimization is formulated by incorporating the dynamics of the state estimator.

Min-Max MPC with State Estimation

$$\min_{\mathbf{K},\bar{\mathbf{u}}} \max_{x_0^p,\mathbf{w},\mathbf{v}} \sum_{n=1}^{N} \|x_n^p\|^2_{\Gamma_n^x} + \sum_{n=0}^{N-1} \|u_n^p\|^2_{\Gamma_n^u}$$

where

$$\mathbf{K} = \left\{K_1^p, K_2^p, \cdots, K_{N-1}^p\right\}$$
$$\bar{\mathbf{u}} = \left\{\bar{u}_0^p, \bar{u}_1^p, \cdots, \bar{u}_{N-1}^p\right\}$$
$$\mathbf{w} = \left\{w_0^p, w_1^p, \cdots, w_{N-1}^p\right\}$$
$$\mathbf{v} = \left\{v_1^p, v_2^p, \cdots, v_N^p\right\}$$

subject to

$$\begin{bmatrix} x_{n+1}^p \\ \hat{x}_{n+1}^p \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ LC & A-LC \end{bmatrix} \begin{bmatrix} x_n^p \\ \hat{x}_n^p \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u_n^p + \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} w_n^p \\ v_n^p \end{bmatrix}$$
$$u_n^p = K_n^p \hat{x}_n^p + \bar{u}_n^p$$
$$u_0^p = \bar{u}_0^p$$
$$w_n^p \in \mathcal{W} = \mathcal{E}\left(P^w, c^w\right)$$
$$v_n^p \in \mathcal{V} = \mathcal{E}\left(P^v, c^v\right)$$
$$x_0^p \in \mathcal{X}_0 = \mathcal{E}\left(P_0^x, c_0^x\right) = \mathcal{X}(k)$$
$$\hat{x}_0^p = \hat{x}(k)$$

Robustness Constraints

$\forall \tilde{x}_0 \in \mathcal{X}_0, \tilde{w}_n \in \mathcal{W}$ **and** $\tilde{v}_n \in \mathcal{V}, n = 0, 1, \cdots, N-1$

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{\hat{x}}_{n+1} \end{bmatrix} \in \bar{\mathcal{X}}_{n+1} = \mathcal{E}\left(Q_{n+1}^{\bar{x}}, c_{n+1}^{\bar{x}}\right)$$
$$g\left(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{u}_n, \tilde{w}_n\right) \le 0$$

where

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{\hat{x}}_{n+1} \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ LC & A-LC \end{bmatrix} \begin{bmatrix} \tilde{x}_n \\ \tilde{\hat{x}}_n \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \tilde{u}_n + \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} \tilde{w}_n \\ \tilde{v}_n \end{bmatrix}$$
$$\tilde{u}_n = K_n^p \tilde{\hat{x}}_n + \bar{u}_n^p \text{ for } n \neq 0$$
$$\tilde{u}_0 = \bar{u}_0^p$$
$$\tilde{\hat{x}}_0 = \hat{x}(k)$$

There are two estimators to estimate the current state. A standard state estimator (4) computes $\hat{x}(k)$ to provide the initial state of the estimator dynamics in the optimization. A recursive set-member state estimator that is not incorporated in the MPC optimization formulation computes $\mathcal{X}(k)$ to bound the current state. In Section IV, we formulate the set-membership estimator and present an ellipsoidal approximation scheme for numerical computation. This approximation is used as $\mathcal{X}_0$ in the min-max MPC optimization with state estimation. Let $K_{n,k}^{p*}$, $n = 1, 2, \cdots, N-1$, and $\bar{u}_{n,k}^{p*}$, $n = 0, 1, \cdots, N-1$, denote the optimal solution of the min-max MPC optimization with state estimation at control step k.

Assumption 1: There exists a set $\bar{\mathcal{X}}_{\mathcal{RI}} \subset \Re^{2m_x}$ and a control law $u(k) = K_{\mathcal{RI}}\hat{x}(k) + \bar{u}_{\mathcal{RI}}$ for system (5) such that

$$\bar{\mathcal{X}}_{\mathcal{RI}} \supseteq \begin{bmatrix} A & BK_{\mathcal{RI}} \\ LC & A + BK_{\mathcal{RI}} - LC \end{bmatrix} \bar{\mathcal{X}}_{\mathcal{RI}} \oplus \begin{bmatrix} B \\ B \end{bmatrix} \bar{u}_{\mathcal{RI}} \oplus \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} \mathcal{W} \\ \mathcal{V} \end{bmatrix}.$$

Moreover any point $\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \in \bar{\mathcal{X}}_{\mathcal{RI}}$ satisfies the constraint

$$g\left(x, x', K_{\mathcal{RI}}\hat{x} + \bar{u}_{\mathcal{RI}}, w\right) \le 0,$$

for all $w \in \mathcal{W}$, where $x' = Ax + B\left(K_{\mathcal{RI}}\hat{x} + \bar{u}_{\mathcal{RI}}\right) + Fw$. We call $\bar{\mathcal{X}}_{\mathcal{RI}}$ the robust invariant set. □

The set constraints $\mathcal{X}_n$ on states are updated based on predictions at the previous control step. After solving the min-max MPC optimization with state estimation, the controller predicts sets of reachable state as

$$\bar{\mathcal{X}}_{k+n+1|k} \supseteq \begin{bmatrix} A & BK_{n,k}^{p*} \\ LC & A + BK_{n,k}^{p*} - LC \end{bmatrix} \bar{\mathcal{X}}_{k+n|k} \oplus \begin{bmatrix} B \\ B \end{bmatrix} \bar{u}_{n,k}^{p*} \oplus \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} \mathcal{W} \\ \mathcal{V} \end{bmatrix},$$

for $n = 1, 2, \cdots, N-1$, and

$$\bar{\mathcal{X}}_{k+1|k} \supseteq \begin{bmatrix} A & \mathbf{0} \\ LC & A-LC \end{bmatrix} \begin{bmatrix} \mathcal{X}_0 \\ \{\hat{x}(k)\} \end{bmatrix} \oplus \begin{bmatrix} B \\ B \end{bmatrix} \bar{u}_{0,k}^{p*} \oplus \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & LG \end{bmatrix} \begin{bmatrix} \mathcal{W} \\ \mathcal{V} \end{bmatrix}.$$

The predictions are computed subject to the set inclusion constraint,

$$\bar{\mathcal{X}}_{k+n|k} \subseteq \bar{\mathcal{X}}_{k+n|k-1}, \text{ for } n=1,\cdots,N-1, \text{ and } \bar{\mathcal{X}}_{k+N|k} \subseteq \bar{\mathcal{X}}_{\mathcal{RI}}.$$

These constraints are feasible since $\bar{\mathcal{X}}_{k+n|k-1}$, $n = 1, \cdots, N-1$, and $\bar{\mathcal{X}}_{\mathcal{RI}}$ are used as constraints in the min-max MPC optimization with state estimation. The set constraints $\mathcal{X}_n$ are updated for step $k+1$ by

$$\mathcal{X}_n = \bar{\mathcal{X}}_{k+n+1|k}, \text{ for } n = 1, \cdots, N-1, \text{ and } \mathcal{X}_N = \bar{\mathcal{X}}_{\mathcal{RI}}.$$

When the controller has bounds on w and v, the state x of the system and the estimation $\hat{x}$ by the standard state estimator satisfy $\left[x(k)^T \, \hat{x}(k)^T\right]^T \in \bar{\mathcal{X}}_{k|k-1}$. We impose the following constraint to the set-membership state estimator

$$\begin{bmatrix} \mathcal{X}(k) \\ \hat{x}(k) \end{bmatrix} \subset \bar{\mathcal{X}}_{k|k-1}.$$

Theorem 1: If the min-max MPC with state estimation optimization is feasible at the first control step $k = 0$, it is feasible at all control steps $k \ge 0$. Furthermore, the state of the system (1)-(2) goes into the set $\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \bar{\mathcal{X}}_{\mathcal{RI}}$ within N steps and remains there for $k > N$. □

Proof: Following the proof in [8], we can show that a feasible solution for the optimization at control step k can be constructed by the optimal solution at control step k and the control parameter for the control invariant set. This implies that if the optimization is feasible at control step

$k=0$, it is feasible for all $k \geq 0$. We then can show that $\left[\begin{array}{cc} x(k)^T & \hat{x}(k)^T \end{array}\right]^T$ goes into $\bar{\mathcal{X}}_{\mathcal{RI}}$ within N step. This means that $x(k)$, the state of the system (1)-(2), goes into the set $\left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \end{array}\right] \bar{\mathcal{X}}_{\mathcal{RI}}$ within N steps. ∎

In [7], we proposed a three-step LMI-based numerical scheme to compute suboptimal solutions for the min-max MPC optimization for systems where the state can be measured perfectly and the disturbances are bounded by ellipsoids. Techniques in [7] can be easily extended for the min-max MPC optimization with state estimation. In the next section, we present a scheme to compute an ellipsoidal bound for current state.

IV. Recursive Set-Membership State Estimation

We use the following recursive set-membership state estimator to provide bound on $x(k)$:

$$\mathcal{X}(k) \supseteq \mathcal{X}_{k|k-1} \bigcap \mathcal{X}_{k|y}, \tag{6}$$

where

$$\begin{aligned} \mathcal{X}_{k|k-1} &\supseteq A\mathcal{X}(k-1) \oplus Bu(k-1) \oplus F\mathcal{W} && (7)\\ \mathcal{X}_{k|y} &= \{x | \exists v \in \mathcal{V}, \ni Cx + Gv = y(k)\} && (8) \end{aligned}$$

We use ellipsoidal approximations to represent these sets, $\mathcal{X}(k)$, $\mathcal{X}_{k|k-1}$ and $\mathcal{X}_{k|y}$. After the controller solves the optimization problem, it computes $\hat{x}(k)$ by (4). Since $\left[\begin{array}{c} x^T(k)\,\hat{x}^T(k) \end{array}\right]^T$, satisfies the bound $\bar{\mathcal{X}}_{k|k-1}$ defined as

$$\left[\begin{array}{c} x - c^{x|\bar{x}}_{k|k-1} \\ \hat{x} - c^{\hat{x}|\bar{x}}_{k|k-1} \end{array}\right]^T \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{array}\right] \left[\begin{array}{c} x - c^{x|\bar{x}}_{k|k-1} \\ \hat{x} - c^{\hat{x}|\bar{x}}_{k|k-1} \end{array}\right] \leq 1,$$

$\mathcal{X}_{k|k-1}$ is an ellipsoid $\mathcal{E}\left(Q_{k|k-1}, c_{k|k-1}\right)$, where

$$\begin{aligned} c_{k|k-1} &= c^{x|\bar{x}}_{k|k-1} - P_{11}^{-1} P_{12} \left(\hat{x}(k) - c^{\hat{x}|\bar{x}}_{k|k-1}\right) \\ Q_{k|k-1} &= \left[1 - \left(\hat{x}(k) - c^{\hat{x}|\bar{x}}_{k|k-1}\right)^T \left(P_{22} - P_{12}^T P_{11}^{-1} P_{12}\right)\right. \\ &\quad \left. \cdot \left(\hat{x}(k) - c^{\hat{x}|\bar{x}}_{k|k-1}\right)\right] P_{11}^{-1} \end{aligned}$$

Unlike $\mathcal{X}_{k|k-1}$, $\mathcal{X}_{y|k}$ is a hyper-cylinder satisfying:

$$C\mathcal{X}_{y|k} = y(k) \oplus -G\mathcal{V}.$$

Assume that $\text{rank}(G) = m_y$. Let $\mathcal{X}^{Proj}_{k|y} = \mathcal{E}\left(C^- G (Q^v)^{-1} G^T (C^-)^{-T}, C^-(y(k) - Gc^v)\right)$.

Proposition 1: $\mathcal{X}_{k|y}$ is an ellipsoidal cylinder, for which $\text{null}(C)$ is the direction and $\mathcal{X}^{Proj}_{k|y}$ is a minimal intersection. □

Proof: It is easy to verify that for any $r \in \text{null}(C)$ and $x \in \mathcal{X}_{k|y}$, $x + \alpha r \in \mathcal{X}$, $\forall \alpha \in \Re$. So, $\mathcal{X}_{k|y}$ is a cylinder. Let $\mathcal{R}$ be direction of $\mathcal{X}_{k|y}$. We have $\dim(\mathcal{R}) \geq m_x - m_y$.

Any point in $\mathcal{X}^{Proj}_{k|y}$ can be expressed as $x = C^-\left(G(Q^v)^{1/2} s + y(k) - Gc^v\right)$, where $s^T s \leq 1$. It follows $Cx = G(Q^v)^{1/2} s + y(k) - Gc^v$. Since $-(Q^v)^{1/2} s + c^v \in \mathcal{E}(Q^v, c^v)$, $x \in \mathcal{X}_{k|y}$, that is, $\mathcal{X}^{Proj}_{k|y} \subset \mathcal{X}_{k|y}$. This implies that $\mathcal{X}^{Proj}_{k|y} \oplus \mathcal{R} \subseteq \mathcal{X}_{k|y}$.

For any $x \in \mathcal{X}^{Proj}_{k|y}$ and $r \in \text{null}(C)$, $x^T r = 0$. So, $\mathcal{X}^{Proj}_{k|y} \perp \text{null}(C)$. Then $\dim\left(\mathcal{X}^{Proj}_{k|y} \oplus \mathcal{R}\right) \geq \dim\left(\mathcal{X}^{Proj}_{k|y}\right) + \dim(\text{null}(C)) = \dim\left(\mathcal{X}^{Proj}_{k|y}\right) + m_x - m_y$. It's easy to check that $\dim\left(\mathcal{X}^{Proj}_{k|y}\right) = m_y$. Since $\dim\left(\mathcal{X}^{Proj}_{k|y} \oplus \mathcal{R}\right) \leq \dim\left(\mathcal{X}^{Proj}_{k|y}\right) + \dim(\mathcal{R}) \leq m_x$, we have $\dim(\mathcal{R}) = m_x - m_y$. So, $\mathcal{R} = \text{null}(C)$, that is, $\text{null}(C)$ is the direction of $\mathcal{X}_{k|y}$.

If $\mathcal{X}^{Proj}_{k|y} \oplus \text{null}(C) \subset \mathcal{X}_{k|y}$, there exists $x = C^-\left(G(Q^v)^{1/2} s + y(k) - Gc^v\right) \in \mathcal{X}_{k|y}$, where $s^T s > 1$. It follows $Cx = G(Q^v)^{1/2} s + y(t) - Gc^v$. Since $-(Q^v)^{1/2} s + c^v \notin \mathcal{E}(Q^v, c^v)$, $x \notin \mathcal{X}_{k|y}$, which is a contradiction. So, $\mathcal{X}^{Proj}_{k|y} \oplus \text{null}(C) = \mathcal{X}_{k|y}$. Thus, $\mathcal{X}^{Proj}_{k|y}$ is a minimal intersection of $\mathcal{X}_{k|y}$. ∎

Proposition 2: If $\mathcal{E}(Q, c')$ is a minimal intersection of $\mathcal{C}(P, c)$, then $c - c' \in \mathcal{R}$, where $\mathcal{R}$ is the direction of $\mathcal{C}(P, c)$, and Q and P have same number of zero eigenvalues, i.e. $\dim(\mathcal{R})$. If λ is a nonzero eigenvalue of Q, $1/\lambda$ is a nonzero eigenvalue of P with the same eigenvector. □

Proof: It is obvious that $c \in \mathcal{C}(P, c)$. Since $\mathcal{E}(Q, c')$ is a minimal intersection of $\mathcal{C}(P, c)$, there exists an $r \in \mathcal{R}$ such that $c \in \mathcal{E}(Q, c') \oplus r = \mathcal{E}(Q, c' + r)$. It's easy to verify that $\mathcal{E}(Q, c' + r)$ is a minimal intersection of $\mathcal{C}(P, c)$. If $c \neq c' + r$, there exists $x \in \mathcal{E}(Q, c' + r)$ such that $2c - x \notin \mathcal{E}(Q, c' + r)$. We have $(2c - x - c)^T P (2c - x - c) \leq 1$. So, $2c - x \in \mathcal{C}(P, c)$. Since $2c - x \in span(\mathcal{E}(Q, c' + r))$, $\mathcal{E}(Q, c') \oplus \mathcal{R} \subset \mathcal{C}(P, c)$, we have a contradiction. Therefore, $c = c' + r$, that is, $c - c' \in \mathcal{R}$.

Since $\mathcal{E}(Q, c')$ is a minimal intersection of $\mathcal{C}(P, c)$, there exists $x \in \mathcal{E}(Q, c')$ such that $(x - c)^T P (x - c) = 1$. For any $r \in \mathcal{R}$, $x + r \in \mathcal{C}(P, c)$, i.e., $(x - c)^T P (x - c) = 1 + r^T P r \leq 1$. So $r^T P r = 0$. This implies that $\mathcal{R} \subseteq \text{null}(P)$.

For any $r \in \text{null}(P)$ and $x \in \mathcal{C}(P, c)$, we have $(x + r - c)^T P (x + r - c) = (x - r)^T P (x - r) \leq 1$. So, $r \in \mathcal{R}$, that is, $\text{null}(P) \subseteq \mathcal{R}$. Thus, $\text{null}(P) = \mathcal{R}$, i.e. P has $\dim(\mathcal{R})$ zero eigenvalues.

Since $\mathcal{E}(Q, c')$ is a minimal intersection of $\mathcal{C}(P, c)$, $\mathcal{E}(Q, c') \perp \mathcal{R}$. So $\mathcal{E}(Q, c')$ has $\dim(\mathcal{R})$ axes of zero radius. This means that Q has $\dim(\mathcal{R})$ zero eigenvalues.

Let $\lambda_1, \lambda_2, \cdots, \lambda_{m_y}$ be nonzero eigenvalues of P and $x_1, x_2, \cdots, x_{m_y}$ be corresponding orthogonal eigenvectors. Since $Px_i = \lambda_i x_i$, $x_i \in span(P)$. We have $x_i \perp \text{null}(P)$. It is easy to verify that

$$\mathcal{C}(P, c) = \left\{ \sum_{i=1}^{m_y} \frac{\alpha_i x_i}{\sqrt{\lambda_i}} + c' + r \,\middle|\, \sum_{i=1}^{m_y} \alpha_i^2 \leq 1, r \in \mathcal{R} \right\}.$$

Define Q' to be the matrix that has $m_x - m_y$ zero eigenvalues and has $1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_{m_y}$ as nonzero eigenvalues with $x_1, \cdots, x_{m_y}$ as corresponding eigenvectors. Since

$$\left(\sum_{i=1}^{m_y} \alpha_i x_i\right)^T \left(\sum_{i=1}^{m_y} \alpha_i x_i\right) \leq 1 \text{ for } \sum_{i=1}^{m_y} \alpha_i^2 \leq 1,$$

the ellipsoid $\mathcal{E}(Q', c')$ is

$$\mathcal{E}(Q', c') = \left\{ \sum_{i=1}^{m_y} \frac{\alpha_i x_i}{\sqrt{\lambda_i}} + c' \middle| \sum_{i=1}^{m_y} \alpha_i^2 \leq 1 \right\}.$$

Obviously, $\mathcal{E}(Q', c')$ is a minimal intersection of $\mathcal{C}(P, c)$. Since $\mathcal{E}(Q, c')$ is also a minimal intersection, Q and Q' have the same eigenvalues and the corresponding eigenvectors. So, $1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_{m_y}$ nonzero eigenvalues of Q and $x_1, \cdots, x_{m_y}$ are corresponding eigenvectors. ■

Based on the above two propositions, we construct $\mathcal{X}_{k|y} = \mathcal{C}(P, c)$ in the following way. Let $c = C^-(y(k) - Gc^v)$. Compute eigenvalues and eigenvectors of $C^- G(P^v)^{-1} G^T (C^-)^{-T}$. Denote the eigenvalues by $\lambda_1, \cdots, \lambda_{m_y}, 0, \cdots, 0$ and the corresponding eigenvectors by $\phi_1, \phi_2, \cdots, \phi_{m_x}$. Then, we compute P by

$$P = \Phi \text{diag}\left(1/\lambda_1, \cdots, 1/\lambda_{m_y}, 0, \cdots, 0\right) \Phi^{-1},$$

where $\Phi = \left[\begin{array}{ccc} \phi_1 & \cdots & \phi_{m_x} \end{array}\right]$.

The set-membership state estimation needs to satisfy $\mathcal{X}(k) \subseteq \mathcal{X}_{k|k-1}$ to guarantee stability of the min-max MPC optimization with state estimation. Since $\mathcal{X}_{k|y}$ is a cylinder, the optimal ellipsoids for $\mathcal{X}(k)$ is $\mathcal{X}_{k|k-1}$. To handle this issue, we propose that if $\mathcal{X}_{k|k-1}$ is the optimal solution, we compute an optimal approximation $\tilde{\mathcal{X}}(k)$ of $\mathcal{X}_{k+1|k} \cap \mathcal{X}_{k+1|y}$ without the constraint $\mathcal{X}_{k+1|k}$ and propagate it for future computation of the set-membership state estimation. We modify recursive set-membership state estimator as

$$\begin{aligned} \mathcal{X}_{k|k-1} &\supseteq \mathcal{X}(k) \\ &\supseteq \mathcal{X}_{k|k-1} \cap \mathcal{X}_{k|y} \\ &\bigcap_{\tilde{\mathcal{X}}(l) \neq \emptyset} \left[A^{k-l} \tilde{\mathcal{X}}(l) \bigoplus_{m=l}^{k-1} \left(A^{k-m-1} Bu(m) \oplus F\mathcal{W} \right) \right]. \end{aligned}$$

The optimal ellipsoidal approximation subject to this constraint can be computed through LMI techniques. To save memory and reduce computation load, once the optimal approximation is inside $\mathcal{X}_{k|k-1}$, all $\tilde{\mathcal{X}}(l)$ for $l \leq k$ are set to be empty sets.

V. Example

We consider the following discrete-time linear time-invariant system:

$$x(k+1) = \left[\begin{array}{cc} 0.6 & 0.4 \\ 0.8 & 0.5 \end{array}\right] x(k) + \left[\begin{array}{c} 0.6 \\ 1.0 \end{array}\right] u(k) + \left[\begin{array}{c} 0.1 \\ 0.1 \end{array}\right] w(k),$$

$$y(k) = \left[\begin{array}{cc} 1 & 0 \end{array}\right] \left[\begin{array}{c} x^1(k) \\ x^2(k) \end{array}\right] + 0.1 v(k)$$

where x, u and y are the state, control output of the system, respectively. w is the external disturbance to the system and v is the measure noise. Both are bounded by the range $[-1, 1]$. The system state needs to satisfy an ellipsoidal constraint defined by $\mathcal{E}(Q^x, c^x)$, where

$$Q^x = \left[\begin{array}{cc} 1 & 0 \\ 0 & 4 \end{array}\right] \text{ and } c^x = \left[\begin{array}{c} 0 \\ 0 \end{array}\right].$$

The system with eigenvalues of 1.1179 and -0.0179 is not stable. The objective is stabilize the system and to satisfy the state constraints. The weighting matrices in the cost function are assigned as

$$\Gamma^x = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \text{ and } \Gamma^u = 100.$$

We define a state estimator of the form (4) and a ellipsoidal constraint $\mathcal{E}\left(Q^{\tilde{x}}, c^{\tilde{x}}\right)$ for the augmented system (5) as

$$Q^{\tilde{x}} = \left[\begin{array}{cccc} 1 & 0 & 0.6 & 0 \\ 0 & 4 & 0 & 3.6 \\ 0.6 & 0 & 1 & 0 \\ 0 & 3.6 & 0 & 4 \end{array}\right] \text{ and } c^{\tilde{x}} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

We choose the feedback matrix for the state observer (4) as $L = \left[\begin{array}{cc} 0.5033 & 1.0599 \end{array}\right]^T$. An ellipsoidal control invariant set for the augmented system (5) is given by $\mathcal{E}(Q_{\mathcal{RI}}, c_{\mathcal{RI}})$ as

$$\begin{aligned} Q_{\mathcal{RI}} &= \left[\begin{array}{cccc} 0.6548 & 0.7872 & 0.3341 & 0.5698 \\ 0.7872 & 1.0120 & 0.3411 & 0.6743 \\ 0.3341 & 0.3411 & 0.2633 & 0.3463 \\ 05698 & 0.6743 & 0.3463 & 0.5792 \end{array}\right] \\ c_{\mathcal{RI}} &= \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}\right]^T, \end{aligned}$$

and the corresponding control parameters are

$$K_{\mathcal{RI}} = \left[\begin{array}{cc} -0.7356 & -0.3823 \end{array}\right] \text{ and } u^0_{\mathcal{RI}} = 0.$$

The system response by the min-max MPC with state estimation is shown in Figures 1–3, where the initial state of the system is $\left[\begin{array}{cc} 1 & 0 \end{array}\right]^T$ and the prediction horizon is $N = 2$. The initial state is bounded by $\mathcal{E}\left(Q^x_{0|0}, c^x_{0|0}\right)$, where

$$Q^x_{0|0} = \left[\begin{array}{cc} 0.01 & 0 \\ 0 & 0.01 \end{array}\right] \text{ and } c^x_{0|0} = \left[\begin{array}{c} 0.9 \\ 0 \end{array}\right].$$

We set the initial state for the state estimator as $\hat{x}(0) = \left[\begin{array}{cc} 0.6 & 0.0 \end{array}\right]^T$. Figures 1 and 2 show the state trajectory of the system. For the system with perfect state feedback, we use the min-max MPC scheme proposed in [7]. The performance of the system with state estimation is close to that of the system with state feedback. Figure 3 illustrates the system response in state space. The dashed ellipsoid is the ellipsoidal constraint on the system state and the dash-dotted ellipsoid is the projection of the control invariant set used in the min-max MPC with state estimation. The dotted ellipsoid is the control invariant set used in the min-max MPC optimization for the system with perfect measurement, which is defined as $\mathcal{E}(Q^x_{\mathcal{RI}}, c^x_{\mathcal{RI}})$,

$$Q^x_{\mathcal{RI}} = \left[\begin{array}{cc} 0.0203 & 0.0188 \\ 0.0188 & 0.0247 \end{array}\right] \text{ and } c^x_{\mathcal{RI}} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right],$$

with the corresponding control parameters

$$K_{\mathcal{RI}} = \left[\begin{array}{cc} -0.9412 & -0.6176 \end{array} \right] \text{ and } u^0_{\mathcal{RI}} = 0.$$

The projection of the control invariant set to the state space of the system is much larger than the control invariant set for the system with perfect state measurement because of the noise in the output measurement and the imperfect state feedback given by the state estimator (4). The simulation shows that the state of the system goes into the control invariant set within $N = 2$ steps and the state trajectory converges to a set smaller than the control invariant set used in the min-max MPC with state estimation.

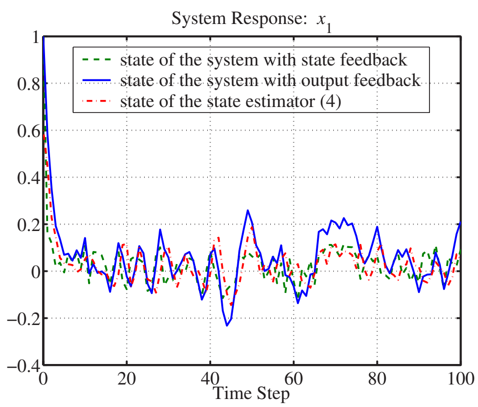

Fig. 1. System Response: x_1

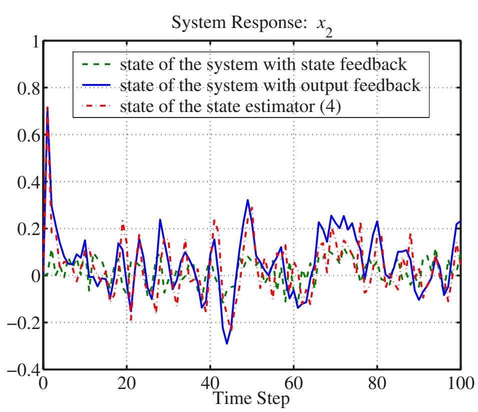

Fig. 2. System Response: x_2

VI. Conclusion

This paper presents a state estimation scheme for min-max model predictive control where the state of the system cannot be measured perfectly. This scheme uses two state estimators. A standard state estimator provides a state estimate for the controller and a set-membership state estimator gives a bound on the system state. A recursive method is provided for computing ellipsoidal set-membership state estimation. In contrast to previous schemes for recursive

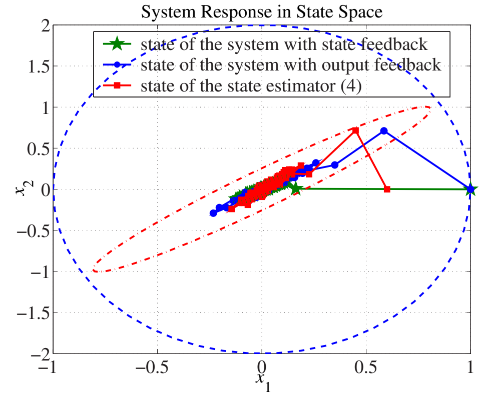

Fig. 3. System Response in State Space

set-membership state estimation, the estimation subject to constraints based on previous predictions. The proposed min-max MPC with state estimation guarantees the bounded stability of the system. Future work will address the design of the L matrix in the state estimator (4) and the computation of the control invariant set.

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