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Schur Stability Radius Bounds for Robust Iterative Learning Controller Design

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***Abstract*— This paper computes bounds on the Schur stability radius for use in designing robust iterative learning controllers. The discrete Lyapunov equation is used to compute the Schur stability radius of the ILC system for the case of parametric or interval perturbations in the system Markov parameters. This paper is distinct from the existing ILC works in that the interval robustness concepts are integrated with asymptotic stability and monotonic convergence conditions to suggest design of the learning gain matrix that makes the system robust against interval model uncertainties up to a computable bound. After derivations of the analytical stability radius, optimization schemes are suggested to design the learning gain matrix. The proposed approach allows design of causal/noncausal time-varying learning matrix gains.**

***Index Terms*— Iterative Learning Control, Interval Model Uncertainties, Schur Stability Radius, Monotone Convergence.**

I. Introduction

Iterative learning control (ILC) is an effective technique for improving the transient performance of systems whose model is uncertain but which have a repetitive task. ILC has a well-established research history, described in, for example, [1], [2]. Of particular interest in this paper, the super-vector approach to ILC has been used to analyze convergence in the iteration domain [3], [4], [5], [6], [7], with asymptotic stability and monotonic convergence conditions investigated in [8], [9], [10], and feedback control and quadratic optimal control methods suggested to design the super-vector learning gain matrix in [11], [12], [13], [14]. In this paper we seek to extend these previous results by considering model uncertainty within the supervector framework. With regard to model uncertainties or noise considerations, H_∞-based ILC has been used in [15], [16], [17], while stochastic-control-based ILC was used in [18]. Recently, an algebraic analysis in super-vector ILC was performed in [19] that suggests the possibility of applying interval robustness concepts to the ILC problem to make the system robust against the parameter uncertainties, assuming there exist interval uncertainties in the system Markov parameters. For such a situation, it is desirable to design the ILC learning gain matrix such that the system is stable for the largest possible range of interval uncertainties on the nominal plant. This is the ultimate aim of this paper.

We begin in Section II by describing the concept of interval ILC. In Section III the matrix norm inequality relationship based on the discrete Lyapunov equation is used to find the (sufficient) Schur stability radius of an interval Markov matrix for both asymptotic and monotonic ILC convergence. In Section IV two optimization schemes are suggested to maximize the stability radius under both asymptotic stability and monotonic convergence conditions. The learning matrices we optimize over include linear, time-varying, Arimoto-like, causal, and non-causal ILC control laws. The effectiveness of the suggested methods is verified through simulation in Section V and conclusions are given in Section VI.

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II. Interval Iterative Learning Control and Stability Analysis

In this section, the super-vector approach to ILC is briefly reviewed and the notion of interval ILC is introduced. The objective of ILC is to track a repeating reference signal assuming an unknown or uncertain system model. Assume the nominal system is given by:

$$\begin{aligned} x_k(t+1) &= Ax_k(t) + Bu_k(t) \\ y_k(t) &= Cx_k(t) \\ x_k(0) &= x_0 \end{aligned} \quad (1)$$

where A, B, and C are the matrices describing the system in the state space and $x_k(t), u_k(t)$, and $y_k(t)$ are the state, input, and output variables, respectively. The subscript "k" is used to denote iteration and "t" is used to denote time. The plant is defined as follows: taking z-transforms in time, define the plant to be $H(z)$, so that

$$\begin{aligned} Y(z) &= H(z)U(z) = (h_m z^{-m} + h_{m+1} z^{-(m+1)} \\ &\quad + h_{m+2} z^{-(m+2)} + \cdots)U(z), \end{aligned} \quad (2)$$

where m is the relative degree of the system, z^{-1} is the standard delay operator in time, and the parameters h_i are the standard Markov parameters of the system $H(z)$, defined by $h_i = CA^{i-1}B$. Note that if we define:

$$\begin{aligned} U_k &= (u_k(0), u_k(1), \cdots, u_k(n-1)), \\ Y_k &= (y_k(m), y_k(m+1), \cdots, y_k(n-1+m)), \\ Y_d &= (y_d(m), y_d(m+1), \cdots, y_d(n-1+m)), \end{aligned} \quad (3)$$

and $m = 1$, then the linear plant can be described by $Y_k = HU_k$, where H is a matrix of rank n whose elements are the Markov parameters of the plant $H(z)$:

$$H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & \cdots & h_1 \end{bmatrix}.$$

We call this matrix the Markov matrix. In this paper, we consider the case where the Markov parameters have interval uncertainties[1] given by $h_i \in [\underline{h}_i, \overline{h}_i]$. We call the ILC problem for such a plant the interval ILC problem.

From the interval Markov parameters we can now define an interval Markov matrix. In the following the superscript I is used to denote interval uncertainty and no subscript or superscript denotes a nominal value. We then make the following definitions:

Definition 2.1: An *interval Markov matrix* H^I is defined as a matrix that is a member of the interval plant $\mathcal{H}^I$ defined by:

$$\mathcal{H}^I = \{H^I : h_{ij}^I \in [\underline{h_{ij}}, \overline{h_{ij}}], i, j = 1, \cdots, n\},$$

where $\overline{h_{ij}}$ is maximum extreme value of the i^{th} row, j^{th} column element of Markov matrix, and $\underline{h_{ij}}$ is minimum extreme value of the i^{th} row, j^{th} column element of Markov matrix. The *upper bound matrix* ($\overline{H}$) is a matrix whose elements are $\overline{h_{ij}}$; and the *lower bound matrix* ($\underline{H}$) is a matrix whose elements are $\underline{h_{ij}}$.

Definition 2.2: The *nominal Markov matrix* is related by the upper and lower bound matrices as:

$$H = \frac{\overline{H} + \underline{H}}{2}$$

Definition 2.3: The *interval radius matrix* (ΔH^r) is defined as:

$$\Delta H^r = \frac{\overline{H} - \underline{H}}{2}$$

The maximum value of ΔH^r for which a given ILC law has guaranteed asymptotic convergence for all $H_I \in \mathcal{H}^I$ satisfying $-\Delta H^r < H^I - H < \Delta H^r$ is called the maximum Schur asymptotic stability radius and is denoted ΔH_a^r. The maximum value of ΔH^r for which a given ILC law has guaranteed monotonic convergence for all $H_I \in \mathcal{H}^I$ satisfying $-\Delta H^r < H^I - H < \Delta H^r$ is called the maximum Schur monotonic stability radius and is denoted ΔH_m^r.

In this paper, first-order ILC is considered, using the following ILC update equation:

$$U_{k+1} = U_k + \Gamma E_k, \tag{4}$$

which gives the following evolution of the error vector in iterations:

$$E_{k+1} = (I - H\Gamma)E_k, \tag{5}$$

where $E_k = Y_d - Y_k$ and Γ is the learning gain matrix defined as

$$\Gamma = \{\gamma_{ij}\}, i, j = 1, \cdots, n. \tag{6}$$

The ILC algorithm is called Arimoto-like when the ILC gains are $\gamma_{ij} = 0, i \neq j$ and $\gamma_{ij} = \gamma, i = j$. The gains γ_{ij} are called causal ILC gains for $i > j$ and non-causal ILC gains for $i < j$. If the gains do not exhibit Toeplitz-like symmetry we call the learning algorithm time-varying. We also refer to the band size of Γ. For example, if band size is 1, only a diagonal line composed of Arimoto-like gains is used. If band size is 2, then one causal diagonal line and one noncausal diagonal line are used in addition to Arimoto-like gains.

In ILC, there are two stability concepts: asymptotic stability and monotonic convergence. The asymptotic stability condition is defined as:

$$\rho(I - H\Gamma) < 1, \tag{7}$$

where ρ represents the spectral radius of $(1 - H\Gamma)$. The monotonic convergence condition is defined in an appropriate l_p-norm topology of E_k as:

$$\|(I - H\Gamma)\|_i < 1, \tag{8}$$

[1]More generally, the model uncertainties may be considered to be in A, B, and C and from these uncertainties one might compute corresponding uncertainty bounds on h_i. However, in this paper we assume the bounds on h_i are known directly.

where $\|\cdot\|_i$ is the appropriate induced operator norm. In this paper, for monotonic convergence the 2-norm topology is used used, based the following definition (see page 295 of [20]):

Definition 2.4: If $\|I - H\Gamma\|_2 < 1$, then $\|E_k\|$ is monotonically converging to zero in l_2-norm topology.

For convenience, the symbols T and T^I are introduced, where

$$T \equiv I - H\Gamma,$$

and

$$T^I \equiv I - H^I\Gamma.$$

Then, the following notation is defined:

$$\Delta T = I - H\Gamma - (I - H^I\Gamma) = (H^I - H)\Gamma = \Delta H\Gamma,$$

where ΔT is the interval uncertainty of the iterative learning control system, and ΔH is the interval uncertainty of the nominal Markov matrix. Using ΔT, optimization schemes will be suggested to maximize $\|\Delta H\|_2$ in Section IV. First, however, in the next section we compute the asymptotic and monotonic stability radii for the interval ILC problem based on the interval concepts and stability conditions introduced above.

III. Stability Radius of Interval ILC

In this section, the Schur stability radius under sufficient stability conditions is calculated using the discrete Lyapunov equation. We introduce the symbol, $\langle\cdot\rangle$ to represent the bigger norm value between a matrix and its transpose:

$$\langle \Delta T \rangle \equiv \max\{\|(\Delta T)^T\|, \|\Delta T\|\},$$

where $\|\cdot\|$ denotes any kind of matrix norm. With this notation we can present our first result:

Theorem 3.1: Given Γ designed for the nominal plant H, if there exists a symmetric, positive definite matrix P that satisfies the constraint

$$(I - H\Gamma)^T P (I - H\Gamma) - P = -I,$$

then the maximum allowable interval uncertainty (AIU), ΔH, for which $(I - (H \pm \Delta H)\Gamma)$ is guaranteed to have asymptotic stability is bigger than ΔH that satisfies

$$\langle \Delta H \rangle \equiv \frac{-\langle I - H\Gamma \rangle + \sqrt{\langle I - H\Gamma \rangle^2 + \frac{1}{\|P\|}}}{\langle \Gamma \rangle}.$$

Proof: Let us assume that $(I - H\Gamma)$ is Schur stable, so T is Schur stable. Then there exists $P = P^T > 0$ such that

$$T^T P T - P = -I. \tag{9}$$

If the following inequality is true with $P = P^T > 0$ of (9):

$$(T^I)^T P T^I - P < 0, \tag{10}$$

then T^I is Schur. Using $T^I = T - \Delta T$, (10) is changed as:

$$\begin{aligned} &(T - \Delta T)^T P (T - \Delta T) - P < 0 \\ \Leftrightarrow\ &T^T P T - T^T P \Delta T - (\Delta T)^T P T \\ &+ (\Delta T)^T P \Delta T - P < 0 \end{aligned} \tag{11}$$

Substituting (9) into (11), we have:

$$\begin{aligned} &-T^T P \Delta T - (\Delta T)^T P T + (\Delta T)^T P \Delta T - I < 0 \\ \Leftrightarrow\ &-T^T P \Delta T - (\Delta T)^T P T + (\Delta T)^T P \Delta T < I, \end{aligned} \tag{12}$$

where the left-hand side is a symmetric matrix. Therefore, if (12) is satisfied with $P = P^T$ determined by (9), then T^I is Schur stable. Now, taking the matrix norm of both sides of (12), we have

$$\| -T^T P \Delta T - (\Delta T)^T P T + (\Delta T)^T P \Delta T \| < 1. \tag{13}$$

Recall that if (13) is true, then (12) is true; not vice-versa (see Appendix for more explanation). Now we change (13) like:

$$\|T^T\|\|P\|\|\Delta T\| + \|(\Delta T)^T\|\|P\|\|T\| + \|(\Delta T)^T\|\|P\|\|\Delta T\| < 1. \quad (14)$$

Notice that (14) is a sufficient condition for (13). Using $\Delta T = \Delta H\Gamma$ and the $\langle\cdot\rangle$ operator, the above inequality is changed as

$$\begin{aligned}&\langle T\rangle\|P\|\langle\Delta T\rangle + \langle\Delta T\rangle\|P\|\langle T\rangle + \langle\Delta T\rangle\|P\|\langle\Delta T\rangle < 1.\\ &\Leftrightarrow [2\langle\Delta T\rangle\langle T\rangle + \langle\Delta T\rangle^2]\|P\| < 1\\ &\Leftrightarrow [2\langle\Delta H\rangle\langle\Gamma\rangle\langle T\rangle + \langle\Delta H\rangle^2\langle\Gamma\rangle^2]\|P\| < 1.\end{aligned} \quad (15)$$

Also, notice that (15) is a sufficient condition for (14). Let $\alpha \equiv \langle\Gamma\rangle^2$; and $\beta \equiv \langle\Gamma\rangle\langle T\rangle$; and $x \equiv \langle\Delta H\rangle$. Then, (15) is of the form:

$$\begin{aligned}&(2\beta x + \alpha x^2)\|P\| < 1\\ &\Rightarrow \alpha x^2 + 2\beta x < \tfrac{1}{\|P\|}.\end{aligned} \quad (16)$$

Here, since $\alpha > 0$,

$$\frac{-\beta - \sqrt{\beta^2 + \frac{\alpha}{\|P\|}}}{\alpha} < x < \frac{-\beta + \sqrt{\beta^2 + \frac{\alpha}{\|P\|}}}{\alpha}.$$

Using $x > 0$,

$$x < \frac{-\beta + \sqrt{\beta^2 + \frac{\alpha}{\|P\|}}}{\alpha}.$$

Therefore, the following inequality is satisfied:

$$\langle\Delta H\rangle < \frac{-\langle\Gamma\rangle\langle T\rangle + \sqrt{[\langle\Gamma\rangle\langle T\rangle]^2 + \frac{\langle\Gamma\rangle^2}{\|P\|}}}{\langle\Gamma\rangle^2}. \quad (17)$$

Using $\langle T\rangle = \langle I - H\Gamma\rangle$, (17) is changed as

$$\langle\Delta H\rangle < \frac{-\langle I - H\Gamma\rangle + \sqrt{\langle I - H\Gamma\rangle^2 + \frac{1}{\|P\|}}}{\langle\Gamma\rangle}. \quad (18)$$

Finally, if there exists $P = P^T$ such that (9) is satisfied and inequality (18) is true, then T^I is Schur (but still, vice-versa is not true). Therefore, the maximum interval uncertainty is allowed more than $\langle\Delta H\rangle_{\max}$, which is defined as

$$\langle\Delta H\rangle_{\max} \equiv \frac{-\langle I - H\Gamma\rangle + \sqrt{\langle I - H\Gamma\rangle^2 + \frac{1}{\|P\|}}}{\langle\Gamma\rangle}. \quad (19)$$

■

From Theorem 3.1, the following corollary is immediate:

Corollary 3.1: If $\Gamma = H^{-1}$, the maximum AIU of the interval ILC is $\frac{1}{\|\Gamma\|} = \frac{1}{\|H^{-1}\|}$.

Proof: When $\Gamma = H^{-1}$, $I - H\Gamma = 0$; and from (9), since T is zero, P equals to I. Also, since $\|\Gamma\| = \|\Gamma^T\|$, from (19), the maximum AIU is

$$\|\Delta H\| = \frac{1}{\|\Gamma\|} = \frac{1}{\|H^{-1}\|}.$$

■

Now, let us consider the monotonic convergence condition. For this we need the following definitions:

Definition 3.1:

$$T_s^I = \begin{bmatrix} 0 & (I - H^I\Gamma)^T \\ (I - H^I\Gamma) & 0 \end{bmatrix};$$

$$T_s = \begin{bmatrix} 0 & (I - H\Gamma)^T \\ (I - H\Gamma) & 0 \end{bmatrix}.$$

Also let $\|\cdot\|_2$ be the matrix 2-norm; $\langle\Gamma\rangle_k \equiv \max\{\|\Gamma\|_k, \|\Gamma^T\|_k\}$; and $\langle\Delta H\rangle_k \equiv \max\{\|\Delta H\|_k, \|\Delta H^T\|_k\}$. Then we have the following result.

Theorem 3.2: Given Γ designed for the nominal plant H, if there exists a symmetric, positive definite matrix P_s that satisfies the constraint

$$T_s^T P_s T_s - P_s = -I_{2n\times 2n}$$

then the maximum AIU, ΔH, for which $(I - (H \pm \Delta H)\Gamma)$ is guaranteed to have monotonic stability is bigger than ΔH that satisfies

$$\langle\Delta H\rangle_k \equiv \frac{-\|T_s\|_2 + \sqrt{\|T_s\|_2^2 + \frac{1}{\|P_s\|_2}}}{\langle\Gamma\rangle_k},$$

where k is 1 or ∞.

Proof: The ILC system is given as:

$$E_{k+1} = (I - H^I\Gamma)E_k \quad (20)$$

In Theorem 3.1, the condition for guaranteeing $\rho(I - H^I\Gamma) < 1$ (i.e., spectral radius less than 1) using the discrete Lyapunov inequality was found. The maximum singular value is defined as:

$$\overline{\sigma}(I - H^I\Gamma) = \sqrt{\rho[(I - H^I\Gamma)^T(I - H^I\Gamma)]}.$$

So, the following relationship is true:

$$[\overline{\sigma}(I - H^I\Gamma)]^2 = \overline{\lambda}\begin{bmatrix} 0_{n\times n} & (I - H^I\Gamma)^T \\ (I - H^I\Gamma) & 0_{n\times n} \end{bmatrix} = \overline{\lambda}(T_s^I),$$

where $\overline{\lambda}$ is the maximum eigenvalue. Therefore, since the maximum eigenvalue of the right-hand side equals the spectral radius, if $\rho[(I - H^I\Gamma)^T(I - H^I\Gamma)] < 1$, then $\overline{\sigma}(I - H^I\Gamma) < 1$. So, the singular value stability problem is changed to eigenvalue problem by Definition 2.1. Since the eigenvalues of $(I - H^I\Gamma)^T(I - H^I\Gamma)$ are equal to the eigenvalues of T_s^I, the discrete Lyapunov inequality can be applied to T_s^I. If T_s is Schur stable, then the following is true:

$$T_s^T P_s T_s - P_s = -I_{2n\times 2n}, \quad (21)$$

with $P_s = P_s^T > 0$. If T_s^I is stable, following is also true:

$$(T_s^I)^T P_s T_s^I - P_s < 0. \quad (22)$$

Thus the following relationships can be derived:

$$\begin{aligned}T_s - T_s^I &= \Delta T_s\\ &= \begin{bmatrix} 0_{n\times n} & (I - H\Gamma - (I - H^I\Gamma))^T \\ (I - H\Gamma - (I - H^I\Gamma)) & 0_{n\times n} \end{bmatrix}\\ &= \begin{bmatrix} 0_{n\times n} & (\Delta H\Gamma)^T \\ (\Delta H\Gamma) & 0_{n\times n} \end{bmatrix}.\end{aligned} \quad (23)$$

Note, T_s and ΔT_s are symmetric matrices. So, $\|T_s\|_2 = \|T_s^T\|_2$ and $\|\Delta T_s\|_2 = \|\Delta T_s^T\|_2$. Now, let us change (22) to be:

$$\begin{aligned}&(T_s - \Delta T_s)^T P_s (T_s - \Delta T_s) - P_s < 0\\ &\Leftrightarrow T_s^T P_s T_s - T_s^T P_s \Delta T_s - (\Delta T_s)^T P_s T_s\\ &\quad + (\Delta T_s)^T P_s \Delta T_s - P_s < 0\end{aligned} \quad (24)$$

Using (21), the above inequality is changed as:

$$-T_s^T P_s \Delta T_s - (\Delta T_s)^T P_s T_s + (\Delta T_s)^T P_s \Delta T_s < I_{2n\times 2n} \quad (25)$$

and taking the 2-norm to both sides, we get

$$\left[2\|\Delta T_s\|_2\|T_s\|_2 + \|\Delta T_s\|_2^2\right]\|P_s\|_2 < 1. \quad (26)$$

Here, it is necessary to separate $\|\Delta T_s\|_2$ into $\|\Delta H\|$ and $\|\Gamma\|$. For this purpose, the following inequality is used:

$$\|\Delta T_s\|_2 \leq \sqrt{\|\Delta T_s\|_1 \|\Delta T_s\|_\infty} \tag{27}$$

So, from the following relationship:

$$\begin{aligned} \|\Delta T_s\|_1 &= \left\| \begin{bmatrix} 0_{n\times n} & (\Delta H\Gamma)^T \\ (\Delta H\Gamma) & 0_{n\times n} \end{bmatrix} \right\|_1 \\ &= \max\{\|\Delta H\Gamma\|_1, \|(\Delta H\Gamma)^T\|_1\} \end{aligned} \tag{28}$$

$$\begin{aligned} \|\Delta T_s\|_\infty &= \left\| \begin{bmatrix} 0_{n\times n} & (\Delta H\Gamma)^T \\ (\Delta H\Gamma) & 0_{n\times n} \end{bmatrix} \right\|_\infty \\ &= \max\{\|\Delta H\Gamma\|_\infty, \|(\Delta H\Gamma)^T\|_\infty\}, \end{aligned} \tag{29}$$

and using the $\langle\cdot\rangle$ operator, we have

$$\begin{aligned} \langle \Delta H\Gamma \rangle &= \max\{\|\Delta H\Gamma\|_1, \|(\Delta H\Gamma)^T\|_1\} \\ &= \max\{\|\Delta H\Gamma\|_\infty, \|(\Delta H\Gamma)^T\|_\infty\} \end{aligned} \tag{30}$$

However, notice that in $\langle \Delta H\Gamma \rangle$ only the 1 and ∞ norms are effective. Thus, (27) is changed as:

$$\begin{aligned} \|\Delta T_s\|_2 &\leq \sqrt{\|\Delta T_s\|_1 \|\Delta T_s\|_\infty} \\ &= \sqrt{\langle \Delta H\Gamma \rangle \langle \Delta H\Gamma \rangle} \\ &= \langle \Delta H\Gamma \rangle \\ &\leq \langle \Delta H \rangle \langle \Gamma \rangle \end{aligned} \tag{31}$$

Now, let us substitute $\langle \Delta H \rangle \langle \Gamma \rangle$ into $\|\Delta T_s\|_2$ of (26) to obtain the following sufficient inequality:

$$\left[2\langle \Delta H \rangle \langle \Gamma \rangle \|T_s\|_2 + (\langle \Delta H \rangle \langle \Gamma \rangle)^2\right] \|P_s\|_2 < 1. \tag{32}$$

Then, using the same procedure as in the Theorem 3.1, we have

$$\langle \Delta H \rangle < \frac{-\|T_s\|_2 + \sqrt{\|T_s\|_2^2 + \frac{1}{\|P_s\|_2}}}{\langle \Gamma \rangle}.$$

Therefore, the maximum allowable interval uncertainty is calculated as:

$$\langle \Delta H \rangle_{\max} = \frac{-\|T_s\|_2 + \sqrt{\|T_s\|_2^2 + \frac{1}{\|P_s\|_2}}}{\langle \Gamma \rangle}, \tag{33}$$

where $\langle \Delta H \rangle_{\max}$ and $\langle \Gamma \rangle$ are restricted to 1 and ∞ norms. ■

In Theorem 3.2, the AIU, $\langle \Delta H \rangle_{\max}$, is calculated in 1 or ∞ norms. If the following relationship is used, then the AIU can be calculated using the 2 norm:

$$\begin{aligned} \langle \Gamma \rangle_1 &= \max\{\|\Gamma\|_1, \|\Gamma^T\|_1\} \\ &= \max\{\|\Gamma\|_\infty, \|\Gamma^T\|_\infty\} \\ &= \langle \Gamma \rangle_\infty. \end{aligned} \tag{34}$$

Thus we have the following corollary:

Corollary 3.2: The 2-norm based AIU is calculated as:

$$\langle \Delta H \rangle_2 \equiv \frac{-\|T_s\|_2 + \sqrt{\|T_s\|_2^2 + \frac{1}{\|P_s\|_2}}}{\langle \Gamma \rangle_k}, \tag{35}$$

where $k = 1$ or ∞. Notice that the right-hand side of (35) is equivalent to the right-hand side of (33).

Proof: By Theorem 3.2 and (34), the following is true:

$$\langle \Delta H \rangle_1 \equiv \langle \Delta H \rangle_\infty$$

Also, the following inequality is satisfied:

$$\langle \Delta H \rangle_2 \leq \sqrt{\langle \Delta H \rangle_1 \langle \Delta H \rangle_\infty}$$

and the followig relationship is immediate:

$$\langle \Delta H \rangle_2 \leq \frac{-\|T_s\|_2 + \sqrt{\|T_s\|_2^2 + \frac{1}{\|P_s\|_2}}}{\langle \Gamma \rangle_k}. \tag{36}$$

■

For convenience, let us denote the right-hand side of (19) as ΔH_{asym} and the right-hand side of (35) as ΔH_{mono}. Then, it is our conclusion that if the interval uncertainty in ILC is less than ΔH_{asym}, the ILC system is asymptotically stable and, if the interval uncertainty in ILC is less than ΔH_{mono}, the ILC system is monotonically convergent in the l_2-norm topology of E_k.

Remark 3.1: Theorem 3.1 and Corollary 3.1 are satisfied with any kind of norms. In Theorem 3.2, $\langle \Delta H \rangle$ is the 1-norm or the ∞-norm, and in Corollary 3.2, $\langle \Delta H \rangle$ is 2-norm. In Theorem 3.2 and Corollary 3.2, $\|T_s\|$ and $\|P_s\|$ are 2-norms, and $\langle \Gamma \rangle$ is the 1 or ∞ norm.

Remark 3.2: In Definition 2.3, the maximum Schur stability radius was defined. ΔH_{asym} and ΔH_{mono} provide sufficient stability radii for the interval ILC. "Sufficient" means that the actual maximum stability radii, ΔH_a^r and ΔH_m^r, may be bigger than the calculated stability radius, ΔH_{asym} and ΔH_{mono}, respectively. Thus, following inequalities should be noticed:

$$\Delta H_{asym} \leq \|\Delta H_a^r\|; \ \Delta H_{mono} \leq \|\Delta H_m^r\|_2.$$

So, ΔH_{asym} and ΔH_{mono} will be conservative compared with the actual maximum stability radius.

So far, we have found AIUs for both the asymptotic stability and the monotone convergence. In next section, optimization methods are used to design Γ in order to maximize the stability radius.

IV. Optimization

In this section, two optimization schemes are designed based on Section III. The purpose of optimization is to maximize $\|\Delta H_{asym}\|$ and $\|\Delta H_{mono}\|$ by designing Γ, with which the ILC system is either asymptotically stable or monotonically convergent.

To find the optimal Γ that allows more interval uncertainties in terms of asymptotic stability, the following optimization scheme is suggested:

$$\max_{\Gamma} \Delta H_{asym} \tag{37}$$

$$\text{s.t. } (I - H\Gamma)^T P (I - H\Gamma) - P = -I.$$

The same optimization idea can be used for increasing the uncertainty interval of the system in terms of monotonic convergence. It is designed as:

$$\max_{\Gamma} \Delta H_{mono} \tag{38}$$

$$\text{s.t. } T_s^T P_s T_s - P_s = -I_{2n\times 2n}.$$

Remark 4.1: It is easy to see that the maximum interval uncertainties occur when $\Gamma = 0$, because when $\Gamma = 0$, there could be infinity interval uncertainties in H. From

$$I - H^I\Gamma,$$

it is easy to observe that as $\Gamma \to 0$, even though $H^I \to \infty$, the following is true:

$$\|I - H^I\Gamma\| < 1.$$

Thus, in the optimization problem, the required maximum spectral radius and singular value should be fixed. In other words, we should add one more constraint in the optimization schemes such as

$$\rho(I - H\Gamma) < \rho_{max}$$

or

$$\overline{\sigma}(I - H\Gamma) < \overline{\sigma}_{max},$$

where $\rho_{max} < 1$ and $\overline{\sigma}_{max} < 1$.

Note, in above two optimization schemes, ρ_{max} and $\overline{\sigma}_{max}$ are the design parameters. If these values are near to zero, the system converges quickly, but with the trade-off that there could be a small AIU. On the contrary, if these values are near to 1, the system is converges slowly, but allows a large AIU. Thus we must choose ρ_{max} and $\overline{\sigma}_{max}$ before applying optimization schemes.

V. Simulation Illustration

The following discrete system is considered in this paper:

$$x_{k+1} = \begin{bmatrix} -0.50 & 0.00 & 0.00 \\ 1.00 & 1.24 & -0.87 \\ 0.00 & 0.87 & 0.00 \end{bmatrix} x_k + \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \quad (39)$$

$$y_k = [\,2.0 \quad 2.6 \quad -2.8\,]\, x_k, \quad (40)$$

which has poles at $[\,0.62 + j0.62, 0.62 - j0.62, -0.50\,]$ and zeros at $[\,0.65, -0.71\,]$. We assume a zero initial condition.

A. Test Setup

The simulation test is performed with the following reference sinusoidal signal:

$$Y_d = \sin(8.0j/n),$$

where $n = 10$ and $j = 1, \cdots, n$. The band size is fixed at 3, and learning gains are determined by optimization problems described in Section IV. Since the gains of each band are not fixed at the same value, the ILC algorithm is considered to be linear, time-varying, and non-causal. The uniformly distributed random number generator of MATLAB was used to make interval uncertainties in Markov parameters according to:

$$h_i = h_i + \delta |h_i| w,$$

where $w \in [-1, 1]$ is a uniformly distributed random number; and δ is tuned to limit the interval amount (in matrix 2 norm). First the optimal learning gain matrices are designed from the optimization problems suggested in Section IV. In MATLAB, the nonlinear optimization command $fmincon$ was used to solve these problems. Then, using the resulting learning gain matrix, an ILC experiment was performed with each of 1000 different random plants. For each random plant, 20 iterations were carried out as shown in Table I. The design parameters ρ_{max} and $\overline{\sigma}_{max}$ were selected as 0.9. The monotonic convergence optimization scheme was designed assuming an l_2-norm topology for E_k. So, if there exists an optimization solution for $\|\Delta H_{mono}\|$, then $(\sum_{i=1}^{n} |E_k(i)|^2)^{1/2}$ will be monotonically convergent.

TABLE I
SIMULATION TEST SETUP

```
for i = 1 : 1 : 1000
   Pick a random plant
      for j = 1 : 1 : 20
         Repeat iterative test
      end
end
```

B. Test Results

From the optimization problems of (37) and (38), the learning gain matrices were designed using the nominal plant such that the calculated maximum AIUs become $\Delta H_{asym} = 0.737$, and $\Delta H_{mono} = 0.6954$. The physical meaning of $\Delta H_{asym} = 0.737$ is that ILC gain matrix designed from optimization (37) allows interval uncertainty for the nominal Markov matrix by the amount of $\|\Delta H\|_2 < 0.737$ while ensuring asymptotic stability. The physical meaning of $\Delta H_{mono} = 0.6954$ is that ILC gain matrix designed from optimization (38) allows interval uncertainty for the nominal Markov matrix by the amount of $\|\Delta H\|_2 < 0.6954$ while still ensuring monotonic convergence.

The results are illustrated in Fig.1 for the asymptotic convergence case and Fig. 2 for the monotonic convergence case. In both figures, the left-hand side of the figures show plots of the 2-norm of the various random plants used (the index of the 1000 different plants is shown on the horizontal axis with the resulting 2-norm of the plant given on the vertical axis). The right-hand side of the figures shows the maximum and minimum l_2 norm of the super-vector error plotted as a function of iteration.

First, let us check the validity of $\Delta H_{asym} = 0.737$. To check the validity of this value, we gave random intervals to each Markov parameter, and we selected interval plants $\|\Delta H\|_2$ are less than 0.737. The results shown in Fig. 1.1.a meet our expectation that all these plants should converge asymptotically. However, as commented in Remark 3.2, there could exist interval plants with $\|\Delta H\|_2 > 0.737$ that are asymptotically stable with the ILC gain matrix designed from (37), because our result is only sufficient. Fig. 1.2.a and 1.2.b show such a situation. But, as the perturbation grows beyond our bound ΔH_{asym} eventually we encounter plants for which our designed learning gain no longer gives asymptotic stability. This is shown in Fig. 1.3.a and 1.3.b.

Similarly, we can check the validity of $\Delta H_{mono} = 0.6954$. Fig. 2.1.a and Fig. 2.1.b show the situation when $\|\Delta H\|_2 < \Delta H_{mono} = 0.6954$. We see that the ILC gain matrix designed from (38) guarantees the monotone convergence. As in the asymptotic stability example, the remaining plots in Fig. 2 show the sufficiency of the condition and the final instability.

VI. Concluding Remarks

In this paper we have considered the interval ILC problem. We calculated bounds on the the maximum allowable uncertainty in the plant Markov parameters for both asymptotic stability and monotonic convergence. These bounds were then used to design the ILC learning gain matrix to maximize the asymptotic and monotonic stability radii of the nominal plant. Simulation results illustrated the ideas. This approach, though conservative, provides an effective scheme for designing robust ILC system.

VII. Appendix

Here we prove why (13) is a sufficient condition for (12). We use the following simple inequalities:

$$A < I \text{ and } \|A\| < 1, \quad (41)$$

where A is a symmetric matrix. Note that $A < I$ can be written as $0 < I - A$. Then, since $I - A$ is a positive definite matrix, the eigenvalues of $I - A$ are all positive by Theorem 3.7 of [21]. However, from the following relationship: $\lambda(I - A) = 1 - \lambda(A)$, if and only if $\lambda(A) < 1$, then again we have $I - A$ is positive definite. Now consider $\|A\| < 1$. From the fact that A is symmetric and $\rho(A) \leq \|A\|$, if $\|A\| < 1$, then $\rho(A) < 1 \Leftrightarrow \max\{|\lambda(A)|\} < 1$. Therefore, if $\|A\| < 1$ is true, then $A < I$. However, vice-versa is not true. So, (13) is a sufficient condition for (12).

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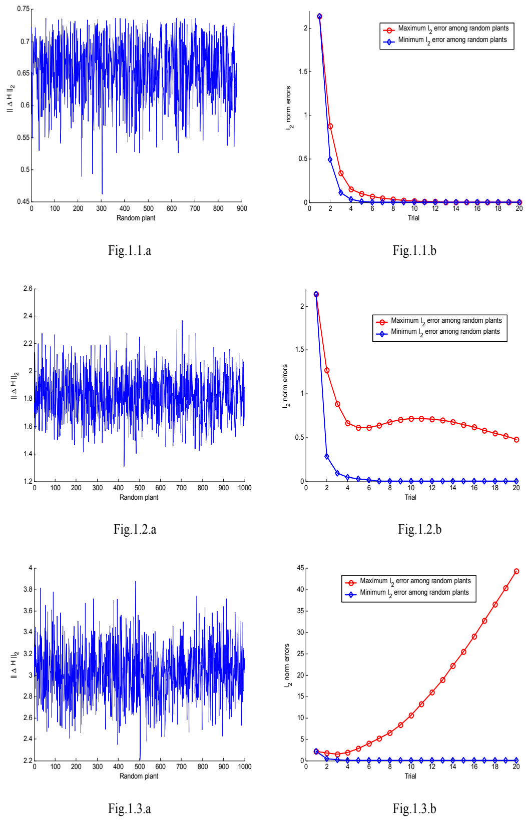

Fig. 1. Asymptotic stability test: tests were performed using the ILC learning gain matrix designed from optimization (37); left figures show the interval amount of random plants in matrix 2-norms; right figures show ILC performance corresponding to left figures; circle marked lines are the l_2-norm errors vs. ILC iteration number corresponding to the plant with the maximum matrix 2-norm and diamond marked lines are l_2-norm errors vs. ILC iteration number corresponding to the plant with the minimum matrix 2-norm, among random plants.

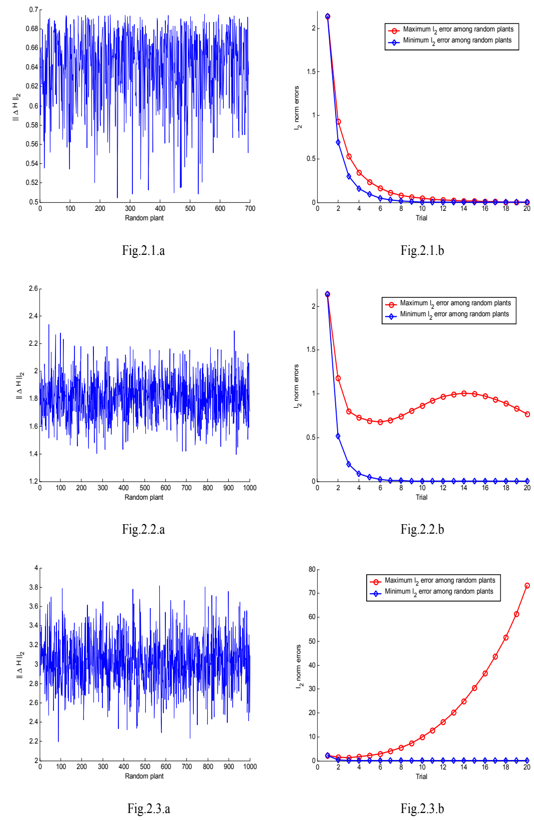

Fig. 2. Monotone convergence test: tests were performed using the ILC learning gain matrix designed from optimization (38); left figures show the interval amount of random plants in matrix 2-norms; right figures show ILC performance corresponding to left figures; circle marked lines are the l_2-norm errors vs. ILC iteration number corresponding to the plant with the maximum matrix 2-norm and diamond marked lines are l_2-norm errors vs. ILC iteration number corresponding to the plant with the minimum matrix 2-norm, among random plants.

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