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Robust Stability Analysis of Nonlinear Uncertain Singularly Impulsive Dynamical Systems

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Abstract— **Singularly impulsive (or generalized impulsive) dynamical systems are systems which dynamics are characterized by the set of differential, difference and algebraic equations. They represent the class of hybrid systems, where algebraic equations represent constraints that differential and difference equations need to satisfy. For the class of nonlinear uncertain singularly impulsive dynamical systems we present robust stability analyze results.**

I. Introduction

Singularly impulsive or generalized impulsive dynamical systems has been recently presented in [1]. Dynamics of this systems is characterized with the set of differential, difference and algebraic equations,wherein algebraic equations represents constraints that differential and difference equations need to satisfy. Applications of this class of systems can be found in contact problems.

For the class of nonlinear uncertain singularly impulsive dynamical systems in this paper we give robust stability analyze results. In doing so, we generalize robust stability results developed in [2]. At first, for the class of nonlinear uncertain singularly impulsive dynamical systems, given a hybrid performance functional, we develop sufficient conditions for robust stability. Next, for the dynamics of the system written in form of the nominal dynamics plus perturbation, we derive robust stability results. Then, we specialize results to the linear uncertain singularly impulsive case.

Finally, in this paper we use the following standard notation. Let R denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let R^n denote the set of $n\times 1$ real column vectors, let $R^{n\times m}$ denote the set of $n\times m$ real matrices, let S^n denote the set of $n\times n$ symmetric matrices, and let N^n (resp., P^n) denote the set of $n\times n$ nonnegative (resp., positive) definite matrices, and let I_n or I denote the $n\times n$ identity matrix. Furthermore, $A\geq 0$ (resp., $A>0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A\geq B$ (resp., $A>B$) denotes the fact that $A-B\geq 0$ (resp., $A-B>0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x, and $\partial\mathcal{S},\dot{\mathcal{S}},\bar{\mathcal{S}}$ denote the boundary, the interior, and a closure of the subset $\mathcal{S}\subset R^n$, respectively. Finally, let C^0 denote the set of continuous functions and C^{r} denote the set of functions with r continuous derivatives.

II. Robust Stability Analysis of Nonlinear Uncertain Singularly Impulsive Dynamical Systems

In this section we present sufficient conditions for robust stability for a class of nonlinear uncertain singularly impulsive dynamical systems. We consider the problem of evaluating a performance bound for a nonlinear-nonquadratic hybrid cost functional depending upon a class of nonlinear uncertain singularly impulsive dynamical systems. It turns out that the cost bound can be evaluated in closed-form as long as the hybrid cost functional is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainity set. Here, we restrict our attention to nonlinear state-dependent singularly impulsive dynamical system [1] $\mathcal{G}$ given by

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t)),\quad x(0)=x_0,\quad x(t)\notin\mathcal{Z}, \tag{1}$$
$$E_{\rm d}\Delta x(t) = f_{\rm d}(x(t)),\quad x(t)\in\mathcal{Z}, \tag{2}$$

where $t\geq 0$, $x(t)\in\mathcal{D}\subseteq R^n$, $\mathcal{D}$ is an open set with $0\in\mathcal{D}$, t_k denotes the $k^{\rm th}$ instant of time at which $x(t)$ intersects $\mathcal{Z}$, $f_{\rm c}:\mathcal{D}\to R^n$ is Lipschitz continuous and satisfies $f_{\rm c}(0)=0$, $f_{\rm d}:\mathcal{D}\to R^n$ is continuous and satisfies $f_{\rm d}(0)=0$, and $\mathcal{Z}\subset R^n$ is the resetting set. Matrices $E_{\rm c}$, $E_{\rm d}$ may be singular matrices. From this we can explore different models:

In case $E_{\rm c}=I$, $E_{\rm d}=I$, (1), (2) represent standard impulsive dynamical systems described in [3], [4]; in absence of discrete dynamics they specialize to singular continuous-time systems, with further specialization $E_{\rm c}=I$ to standard continuous-time systems; if only discrete dynamics is present they specialize to singular discrete-time systems, with further specialization $E_{\mathrm{d}} = I$ to standard discrete-time systems. Therefore, theory of the singularly impulsive or generalized impulsive dynamical systems can be viewed as a generalization of the singular and impulsive dynamical system theory.

We refer to the differential equation (1) as the *continuous-time dynamics*, and we refer to the difference equation (2) as the *resetting law*. Furthermore, $\mathcal{F}_{\mathrm{c}}$ and $\mathcal{F}_{\mathrm{d}}$ denote the class of nonlinear uncertain singularly impulsive dynamical systems with $f_{\mathrm{c}0}(\cdot) \in \mathcal{F}_{\mathrm{c}}$ and $f_{\mathrm{d}0}(\cdot) \in \mathcal{F}_{\mathrm{d}}$ defining the nominal nonlinear singularly impulsive dynamical system for the continuous-time and the resetting dynamics, respectively. Note that since the resetting set $\mathcal{Z}$ is a subset of the state space $\mathcal{D}$ and is independent of time, state-dependent singularly impulsive dynamical systems are time-invariant. In this paper we assume that existence and uniqueness properties of a given state-dependent singularly impulsive dynamical systems are satisfied in forward time. For details see [5].

We make the following additional assumptions:

A1. If $x(t) \in \bar{\mathcal{Z}}\backslash\mathcal{Z}$, then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$, $x(\delta, x(t)) \notin \mathcal{Z}$.

A3. If $x \in \mathcal{Z}$, then $E_{\mathrm{d}}x + f_{\mathrm{d}}(x) \notin \mathcal{Z}$, $f_{\mathrm{d}}(\cdot) \in \mathcal{F}_{\mathrm{d}}$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$, that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore A2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exists $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $E_{\mathrm{d}}x_0^+ = E_{\mathrm{d}}x_0 + f_{\mathrm{d}}(x_0)$ which serves as the initial condition for the continuous dynamics (1). It follows from A1 and A2 that $\partial\mathcal{Z} \cap \mathcal{Z}$ is closed and hence the resetting times $\tau_k(x_0)$ are well defined and distinct. Furthermore, it follows from A2 that if $x^* \in \mathcal{D}$ satisfies $f_{\mathrm{d}}(x^*) = 0$, then $x^* \notin \mathcal{Z}$. In particular, we note $0 \notin \mathcal{Z}$. For further insights on Assumptions the interested reader is referred to [1], [3].

For the following result let $L_{\mathrm{c}}, L_{\mathrm{d}} : \mathcal{D} \to R$, and let $\mathcal{F}_{\mathrm{c}} \subset \{f_{\mathrm{c}} : \mathcal{D} \to R^n : f_{\mathrm{c}}(0) = 0\}$ and $\mathcal{F}_{\mathrm{d}} \subset \{f_{\mathrm{d}} : \mathcal{D} \to R^n : f_{\mathrm{d}}(0) = 0\}$ denote the class of nonlinear uncertain singularly impulsive dynamical systems with $f_{\mathrm{c}0}(\cdot) \in \mathcal{F}_{\mathrm{c}}$ and $f_{\mathrm{d}0}(\cdot) \in \mathcal{F}_{\mathrm{d}}$ defining the nominal nonlinear singularly impulsive dynamical system on continuous-time and discrete-time dynamics, respectively. For the following result and a remainder of the paper we denote the resetting times $\tau_k(x_0)$ by t_k, and for simplicity of exposition, we also define $(f_{\mathrm{c}}(\cdot), f_{\mathrm{d}}(\cdot)) \in \mathcal{F}_{\mathrm{c}} \times \mathcal{F}_{\mathrm{d}} = \mathcal{F}$ and $\mathcal{N}_{[0,t)} = \{k : 0 \le t_k < t\}$. Within the context of robustness analysis, it is assumed that the zero solution $x(t) \equiv 0$ to the nominal nonlinear singularly impulsive dynamical system (1), (2) is asymptotically stable. Furthermore, we assume that an infinite number of resetting occurs.

Theorem 2.1: Consider the nonlinear uncertain singularly impulsive dynamical system $\mathcal{G}$ given by (1), (2), where $(f_{\mathrm{c}}(\cdot), f_{\mathrm{d}}(\cdot)) \in \mathcal{F}$, with performance functional

$$J(x_0) = \int_0^\infty L_{\mathrm{c}}(x(t))\mathrm{d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} L_{\mathrm{d}}(x(t_k)). \tag{3}$$

Furthermore, assume there exist functions $\Gamma_{\mathrm{c}}, \Gamma_{\mathrm{d}} : \mathcal{D} \to R$ and $V : \mathcal{D} \to R$, where $V(\cdot)$ is C^1 function, such that

$$V(0) = 0, \tag{4}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{5}$$

$$V'(x)f_{\mathrm{c}}(x) \le V'(x)f_{\mathrm{c}0}(x) + \Gamma_{\mathrm{c}}(x), \quad x \notin \mathcal{Z}, f_{\mathrm{c}}(\cdot) \in \mathcal{F}_{\mathrm{c}}, \tag{6}$$

$$V'(x)f_{\mathrm{c}0}(x) + \Gamma_{\mathrm{c}}(x) < 0, \quad x \notin \mathcal{Z}, \quad x \neq 0, \tag{7}$$

$$L_{\mathrm{c}}(x) + V'(x)f_{\mathrm{c}0}(x) + \Gamma_{\mathrm{c}}(x) = 0, \quad x \notin \mathcal{Z}, \tag{8}$$

$$V(E_{\mathrm{d}}x + f_{\mathrm{d}}(x)) - V(x) \le V(E_{\mathrm{d}}x + f_{\mathrm{d}0}(x)) - V(x) + \Gamma_{\mathrm{d}}(x), \quad x \in \mathcal{Z}, \ f_{\mathrm{d}}(\cdot) \in \mathcal{F}_{\mathrm{d}}, \tag{9}$$

$$V(E_{\mathrm{d}}x + f_{\mathrm{d}0}(x)) - V(x) + \Gamma_{\mathrm{d}}(x) \le 0, \quad x \in \mathcal{Z}, \tag{10}$$

$$L_{\mathrm{d}}(x) + V(E_{\mathrm{d}}x + f_{\mathrm{d}0}(x)) - V(x) + \Gamma_{\mathrm{d}}(x) = 0, \quad x \in \mathcal{Z}, \tag{11}$$

where $(f_{\mathrm{c}0}(\cdot), f_{\mathrm{d}0}(\cdot)) \in \mathcal{F}$ defines the nominal nonlinear singularly impulsive dynamical system. Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (1), (2) is locally asymptotically stable for all $(f_{\mathrm{c}}(\cdot), f_{\mathrm{d}}(\cdot)) \in \mathcal{F}$, and

$$\sup_{(f_{\mathrm{c}}(\cdot), f_{\mathrm{d}}(\cdot))\in\mathcal{F}} J(x_0) \le \mathcal{J}(x_0) = V(x_0), \tag{12}$$

where

$$\mathcal{J}(x_0) = \int_0^\infty [L_{\mathrm{c}}(x(t)) + \Gamma_{\mathrm{c}}(x(t))]\mathrm{d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\mathrm{d}}(x(t_k)) + \Gamma_{\mathrm{d}}(x(t_k))], \tag{13}$$

and where $x(t)$, $t \ge 0$, is a solution to (1), (2) with $(f_{\mathrm{c}}(x(t)), f_{\mathrm{d}}(x(t_k))) = (f_{\mathrm{c}0}(x(t)), f_{\mathrm{d}0}(x(t_k)))$. Finally, if $\mathcal{D} = R^n$ and

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \tag{14}$$

then the zero solution $x(t) \equiv 0$ to (1), (2) is globally asymptotically stable for all $(f_{\mathrm{c}}(\cdot), f_{\mathrm{d}}(\cdot)) \in \mathcal{F}$.

Proof: Let $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$ and $x(t)$, satisfies (1), (2). Then,

$$\dot{V}(x(t)) = \frac{\rm d}{{\rm d}t}V(x(t)) = V'(x(t))f_{\rm c}(x(t)),\\ x(t) \notin \mathcal{Z}, t_k < t \le t_{k+1}. \quad (15)$$

Hence, it follows from (6) and (7) that

$$\dot{V}(x(t)) < 0, \quad x(t) \notin \mathcal{Z},\, x(t) \neq 0,\, t_k < t \le t_{k+1}. \quad (16)$$

Furthermore,

$$\Delta V(x(t_k)) = V(E_{\rm d}x(t_k) + f_{\rm d}(x(t_k))) \\ -V(x(t_k)), \qquad x(t_k) \in \mathcal{Z}. \quad (17)$$

Hence, it follows from (9) and (10) that

$$\Delta V(x(t_k)) \le 0, \qquad x(t_k) \in \mathcal{Z}. \quad (18)$$

Thus, from (4), (5), (16), and (18) it follows from Theorem 3.2 of [6] that $V(\cdot)$ is Lyapunov function for (1), (2), which proves local asymptotic stability of the zero solution $x(t) \equiv 0$ for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$. Consequently, $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x_0 \in \mathcal{D}_0$ for some neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin. Now, (15) and (17) imply that

$$0 = -\dot{V}(x(t)) + V'(x(t))f_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z},\\ t_k < t \le t_{k+1} \quad (19)$$

$$0 = -\Delta V(x(t_k)) + V(E_{\rm d}x(t_k) + f_{\rm d}(x(t_k)))\\ -V(x(t_k)), x(t_k) \in \mathcal{Z}. \quad (20)$$

From (19), using (6) and (8),

$$\begin{aligned} L_{\rm c}(x(t)) &= -\dot{V}(x(t)) + L_{\rm c}(x(t)) + V'(x(t))f_{\rm c}(x(t)) \\ &\le -\dot{V}(x(t)) + L_{\rm c}(x(t)) + V'(x(t))f_{\rm c0}(x(t)) + \Gamma_{\rm c}(x(t)) \\ &= -\dot{V}(x(t)), \qquad x(t) \notin \mathcal{Z}, \quad t_k < t \le t_{k+1}. \quad (21)\end{aligned}$$

From (20), using (9) and (11),

$$\begin{aligned} L_{\rm d}(x(t_k)) &= -\Delta V(x(t_k)) + L_{\rm d}(x(t_k)) + V(E_{\rm d}x(t_k) \\ &\quad + f_{\rm d}(x(t_k))) - V(x(t_k)) \\ &\le -\Delta V(x(t_k)) + L_{\rm d}(x(t_k)) + V(E_{\rm d}x(t_k) \\ &\quad + f_{\rm d0}(x(t_k))) - V(x(t_k)) + \Gamma_{\rm d}(x(t_k)) \\ &= -\Delta V(x(t_k)), \qquad x(t_k) \in \mathcal{Z}. \quad (22)\end{aligned}$$

Now, integrating over the interval $[0,t)$ with $\mathcal{N}_{[0,t)} = \{1, 2, \ldots, i, \ldots, j\}$, (21) and (22) yield

$$\begin{aligned} &\int_0^t L_{\rm c}(x(s)){\rm d}s + \sum_{k\in\mathcal{N}_{[0,t)}} L_{\rm d}(x(t_k)) = \int_0^{t_i} L_{\rm c}(x(s)){\rm d}s \\ &+L_{\rm d}(x(t_i)) + \int_{t_i^+}^{t_{i+1}} L_{\rm c}(x(s)){\rm d}s + L_{\rm d}(x(t_{i+1})) \\ &+ \ldots + \int_{t_{j-1}^+}^{t_j} L_{\rm c}(x(s)){\rm d}s + L_{\rm d}(x(t_j)) + \int_{t_{j+}}^{t} L_{\rm c}(x(s)){\rm d}s \\ &\le -V(x(t_i)) + V(x_0) - V(E_{\rm d}x(t_i) + f_{\rm d}(x(t_i))) \\ &\quad +V(x(t_i)) - V(x(t_{i+1})) + V(x(t_i^+)) - V(E_{\rm d}x(t_{i+1}) \\ &\quad +f_{\rm d}(x(t_{i+1}))) + V(x(t_{i+1})) + \ldots - V(x(t_j)) \\ &\quad +V(x(t_{j-1}^+)) - V(E_{\rm d}x(t_j) + f_{\rm d}(x(t_j))) + V(x(t_j)) \\ &\quad -V(x(t)) + V(x(t_j^+)) \\ &\le -V(x(t_i)) + V(x_0) - V(x(t_i^+)) + V(x(t_i)) \\ &\quad -V(x(t_{i+1})) + V(x(t_i^+)) - V(x(t_{i+1}^+)) + V(x(t_{i+1})) \\ &\quad + \ldots - V(x(t_j)) + V(x(t_{j-1}^+)) - V(x(t_j^+)) \\ &\quad +V(x(t_j)) - V(x(t)) + V(x(t_j^+)) \\ &\le -V(x(t)) + V(x_0). \qquad (23)\end{aligned}$$

Letting $t \to \infty$ and noting that $V(x(t)) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $J_{(f_{\rm c}, f_{\rm d})}(x_0) \le V(x_0)$. Next, let $x(t)$, $t \ge 0$, satisfy (1), (2) with $(f_{\rm c}(x(t)), f_{\rm d}(x(t))) = (f_{\rm c0}(x(t)), f_{\rm d0}(x(t)))$. Now, from (19), using (8),

$$\begin{aligned} L_{\rm c}(x(t)) + \Gamma_{\rm c}(x(t)) &= -\dot{V}(x(t)) + L_{\rm c}(x(t)) \\ &\quad +V'(x(t))f_{\rm c0}(x(t)) + \Gamma_{\rm c}(x(t)) \\ &= -\dot{V}(x(t)), \qquad x(t) \notin \mathcal{Z}, \quad t_k < t \le t_{k+1}. (24)\end{aligned}$$

From (20), using (11),

$$\begin{aligned} L_{\rm d}(x(t_k)) + \Gamma_{\rm d}(x(t_k)) &= -\Delta V(x(t_k)) + L_{\rm d}(x(t_k)) + \\ &\quad V(E_{\rm d}x(t_k) + f_{\rm d0}(x(t_k))) - V(x(t_k))) + \Gamma_{\rm d}(x(t_k)) \\ &= -\Delta V(x(t_k)), \qquad x(t_k) \in \mathcal{Z}. \quad (25)\end{aligned}$$

Now, integrating over the interval $[0,t)$ with $\mathcal{N}_{[0,t)} = \{1, 2, \ldots, i, \ldots, j\}$, (24) and (25) yield

$$\int_0^t [L_{\rm c}(x(t)) + \Gamma_{\rm c}(x(t))]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,t)}} [L_{\rm d}(x(t_k)) + \\ \Gamma_{\rm d}(x(t_k))] = -V(x(t)) + V(x_0). \quad (26)$$

Letting $t \to \infty$ and noting that $V(x(t)) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $\mathcal{J}(x_0) = V(x_0)$. Finally, for $\mathcal{D} = R^n$ and for all $(f_{\rm c}(\cdot), f_{\rm d}(\cdot))$, global asymptotic stability of the zero solution $x(t) \equiv 0$ to (1), (2) is a direct consequence of Theorem 3.2 of [6] using radially unbounded condition (14) on $V(x)$, $x \in R^n$.

Remark 2.1: Theorem 2.1 provides sufficient conditions for robust stability of a class of nonlinear uncertain singularly impulsive dynamical systems $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$.

Specifically, (4) and (5) assume that $V(x)$ is a Lyapunov function candidate for the nonlinear uncertain singularly impulsive dynamical system (1), (2). Conditions (6), (7), (9), and (10) imply $\dot{V}(x(t)) < 0$, $x(t) \notin \mathcal{Z}$, $t > 0$, and $\Delta V(x(t_k)) \leq 0$, $x(t_k) \in \mathcal{Z}$, $k \in \mathcal{N}$, for $x(\cdot)$ satisfying (1), (2) for all $f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$, and hence $V(\cdot)$ is a Lyapunov function guaranteeing robust stability of the nonlinear uncertain singularly impulsive dynamical system (1), (2). It is important to note that Conditions (7) and (10) are *verifiable* conditions since they are independent of the uncertain system parameters $(f_{\rm c}(\cdot), f_{\rm d}(\cdot)) \in \mathcal{F}$. To apply Theorem 2.1 we specify the bounding functions $\Gamma_{\rm c}(\cdot)$ and $\Gamma_{\rm d}(\cdot)$ for the uncertain set $\mathcal{F}_{\rm c}\times\mathcal{F}_{\rm d}$ such that $\Gamma_{\rm c}(\cdot)$ and $\Gamma_{\rm d}(\cdot)$ bound $\mathcal{F}_{\rm c}\times\mathcal{F}_{\rm d}$. For further details see [7]. If $\mathcal{F}$ consists only of the nominal nonlinear singularly impulsive dynamical system $(f_{\rm c0}(\cdot), f_{\rm d0}(\cdot))$, then $\Gamma_{\rm c}(x) = 0$ and $\Gamma_{\rm d}(x) = 0$ for all $x \in \mathcal{D}$ satisfy (6) and (9), respectively, and hence $J_{(f_{\rm c0}, f_{\rm d0})} = \mathcal{J}(x_0)$. Finally, a worst-case upper bound to the nonlinear-nonquadratic hybrid performance functional is given in terms of a Lyapunov function which can be interpreted in terms of an auxiliary cost defined for the nominal singularly impulsive dynamical system.

Next, we specialize Theorem 2.1 to nonlinear uncertain singularly impulsive dynamical systems of the form

$$E_{\rm c}\dot{x}(t) = f_{\rm c0}(x(t)) + \Delta f_{\rm c}(x(t)), \;\; x(0) = x_0,\; x(t) \notin \mathcal{Z}, \tag{27}$$

$$E_{\rm d}\Delta x(t) = f_{\rm d0}(x(t)) + \Delta f_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{28}$$

where $t \geq 0$, $f_{\rm c0}, f_{\rm d0} : \mathcal{D} \to R^n$ and satisfies $f_{\rm c0}(0) = 0, f_{\rm d0}(0) = 0$ and $(\Delta f_{\rm c}, \Delta f_{\rm d}) \in \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d} = \mathcal{F}$, where

$$\mathcal{F}_{\rm c} \subset \{\Delta f_{\rm c} : \mathcal{D} \to R^n : \Delta f_{\rm c}(0) = 0\}, \tag{29}$$

$$\mathcal{F}_{\rm d} \subset \{\Delta f_{\rm d} : \mathcal{D} \to R^n : \Delta f_{\rm d}(0) = 0\}. \tag{30}$$

Corollary 2.1: Consider the nonlinear uncertain singularly impulsive dynamical system (27), (28) with performance functional (3). Furthermore, assume there exists functions $\Gamma_{\rm c}, \Gamma_{\rm d} : \mathcal{D} \to R$, $P_{1f_{\rm d}} : \mathcal{D} \to R^{1\times n}$, $P_{2f_{\rm d}} : \mathcal{D} \to N^n$, and $V : \mathcal{D} \to R$, where $V(\cdot)$ is a C^1 function, such that (4), (5), (10), and (11) hold and

$$V'(x)\Delta f_{\rm c}(x) \leq \Gamma_{\rm c}(x), \quad x \notin \mathcal{Z}, \Delta f_{\rm c}(\cdot) \in \mathcal{F}_{\rm c}, \tag{31}$$

$$V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x) < 0, \quad x \notin \mathcal{Z}, \quad x \neq 0, \tag{32}$$

$$L_{\rm c}(x) + V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x) = 0, \quad x \notin \mathcal{Z}, \tag{33}$$

$$P_{1f_{\rm d}}(0) = 0, \tag{34}$$

$$\begin{gathered}\Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}(x) + P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x) \\ +\Delta f_{\rm d}^{\rm T}(x)P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x) \leq \Gamma_{\rm d}(x), \\ x \in \mathcal{Z},\; \Delta f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d},\end{gathered} \tag{35}$$

$$\begin{gathered}V(E_{\rm d}x + f_{\rm d0}(x) + \Delta f_{\rm d}(x)) - V(x) \leq \\ V(E_{\rm d}x + f_{\rm d0}(x)) - V(x) + \Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}^{\rm T}(x) \\ +P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x) + \Delta f_{\rm d}^{\rm T}(x)P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x), \\ x \in \mathcal{Z}, \quad \Delta f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d}.\end{gathered} \tag{36}$$

Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (27), (28) is locally asymptotically stable for all $(\Delta f_{\rm c}(\cdot), \Delta f_{\rm d}(\cdot)) \in \mathcal{F}$, and the hybrid performance functional (3) satisfies

$$\sup_{(\Delta f_{\rm c}(\cdot),\Delta f_{\rm d}(\cdot))\in\mathcal{F}} J(x_0) \leq \mathcal{J}(x_0) = V(x_0), \tag{37}$$

where

$$\begin{aligned}\mathcal{J}(x_0) \;\; &= \;\; \int_0^\infty [L_{\rm c}(x(t)) + \Gamma_{\rm c}(x(t))]{\rm d}t \\ &\quad + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d}(x(t_k)) + \Gamma_{\rm d}(x(t_k))],\end{aligned} \tag{38}$$

and where $x(t)$, $t \geq 0$, is a solution to (27), (28) with $(\Delta f_{\rm c}(x(t)), \Delta f_{\rm d}(x(t))) = (0, 0)$. Finally, if $\mathcal{D} = R^n$ and $V(x)$, $x \in R^n$, satisfies (14), then the zero solution $x(t) \equiv 0$ to (27), (28) is globally asymptotically stable for all $(\Delta f_{\rm c}(\cdot), \Delta f_{\rm d}(\cdot)) \in \mathcal{F}$.

Proof: The result is direct consequence of Theorem 2.1 with $f_{\rm c}(x) = f_{\rm c0}(x) + \Delta f_{\rm c}(x)$, for $x \notin \mathcal{Z}$ and $f_{\rm d}(x) = f_{\rm d0}(x) + \Delta f_{\rm d}(x)$, for $x \in \mathcal{Z}$. Specifically, in this case it follows from (31) and (32) that $V'(x)f_{\rm c}(x) \leq V'(x)f_{\rm c0}(x) + \Gamma_{\rm c}(x) < 0$ for all $x \neq 0$, $x \notin \mathcal{Z}$, and $\Delta f_{\rm c}(\cdot) \in \mathcal{F}_{\rm c}$. Further, it follows from (35) and (36) that $V(E_{\rm d}x+f_{\rm d}(x))-V(x) \leq V(E_{\rm d}x+f_{\rm d0}(x))-V(x)+\Gamma_{\rm d}(x)$ for all $x \in \mathcal{Z}$ and $\Delta f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d}$. Hence, all the conditions of Theorem 2.1 are satisfied.

The following corollary specializes Theorem 2.1 to a class of linear uncertain singularly impulsive dynamical systems. Specifically, we consider $\mathcal{F} = \mathcal{F}_{\rm c}\times\mathcal{F}_{\rm d}$ to be the set of linear uncertain singularly impulsive dynamical systems, with

$$\mathcal{F}_{\rm c} = \{(A_{\rm c} + \Delta A_{\rm c})x : x \in R^n, A_{\rm c} \in R^{n\times n}, \Delta A_{\rm c} \in \mathbf{\Delta}_{A_{\rm c}}\},$$

$$\mathcal{F}_{\rm d} = \{(A_{\rm d}+\Delta A_{\rm d})x : x \in R^n, A_{\rm d} \in R^{n\times n}, \Delta A_{\rm d} \in \mathbf{\Delta}_{A_{\rm d}}\},$$

where $\mathbf{\Delta}_{A_{\rm c}}, \mathbf{\Delta}_{A_{\rm d}} \subset R^{n\times n}$ are given bounded uncertainty sets of uncertain perturbations $\Delta A_{\rm c}, \Delta A_{\rm d}$ of the nominal asymptotically stable system matrices $A_{\rm c}, A_{\rm d}$ such that $0 \in \mathbf{\Delta}_{A_{\rm c}}$ and $0 \in \mathbf{\Delta}_{A_{\rm d}}$. For simplicity of exposition, we also define $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}_{A{\rm c}} \times \mathbf{\Delta}_{A{\rm d}} = \mathbf{\Delta}$.

Corollary 2.2: Let $R_{\rm c} \in P^n$ and $R_{\rm d} \in N^n$. Consider the linear state-dependent uncertain impulsive dynamical system

$$E_c\dot{x}(t) = (A_c+\Delta A_c)x(t),\ x(0)=x_0, t \geq 0, x(t) \notin \mathcal{Z}, \quad (39)$$

$$E_d \Delta x(t) = (A_d + \Delta A_d - E_d)x(t), \quad x(t) \in \mathcal{Z}, \quad (40)$$

with performance functional

$$J_{\Delta A_c, \Delta A_d}(x_0) = \int_0^\infty x^T(t) R_c x(t) dt + \sum_{k\in\mathcal{N}_{[0,\infty)}} x^T(t_k) R_d x(t_k), \quad (41)$$

where $(\Delta A_c, \Delta A_d) \in \mathbf{\Delta}$. Let $\Omega_c, \Omega_d : \mathcal{N}_P \subseteq S^n \to N^n$, $P \in \mathcal{N}_P$, be such that

$$x^T(\Delta A_c^T P E_c + E_c^T P \Delta A_c)x \leq x^T \Omega_c(P) x, \quad x \notin \mathcal{Z}, \quad \Delta A_c \in \mathbf{\Delta}_{Ac}, \quad (42)$$

$$x^T E_d^T(\Delta A_d^T P \Delta A_d + \Delta A_d^T P A_d + A_d^T P \Delta A_d) E_d x \leq x^T \Omega_d(P) x, \quad x \in \mathcal{Z}, \quad \Delta A_d \in \mathbf{\Delta}_{Ad}. \quad (43)$$

Furthermore, suppose there exist $P \in P^n$ satisfying

$$0 = x^T(A_c^T P E_c + E_c^T P A_c + \Omega_c(P) + R_c)x, x \notin \mathcal{Z}, \quad (44)$$

$$0 = x^T(A_d^T P A_d - E_d^T P E_d + \Omega_d(P) + R_d)x, x \in \mathcal{Z}. \quad (45)$$

Then the zero solution $x(t) \equiv 0$ to (39), (40) is globally asymptotically stable for all $(\Delta A_c, \Delta A_d) \in \mathbf{\Delta}$, and the hybrid performance functional (41) satisfies

$$\sup_{(\Delta A_c, \Delta A_d)\in\mathbf{\Delta}} J_{\Delta A_c, \Delta A_d}(x_0) \leq \mathcal{J}(x_0) = x_0^T E_c^T P E_c x_0, x_0 \in R^n, \quad (46)$$

where

$$\mathcal{J}(x_0) = \int_0^\infty x^T(t)(\Omega_c(P) + R_c)x(t) dt + \sum_{k\in\mathcal{N}_{[0,\infty)}} x^T(t_k)(\Omega_d(P) + R_d)x(t_k), \quad (47)$$

and where $x(t)$, $t \geq 0$, is a solution to (39), (40) with $(\Delta A_c, \Delta A_d) = (0, 0)$.

Proof: The result is direct consequence of Theorem 2.1 with $f_c(x) = (A_c + \Delta A_c)x$, $f_{c0}(x) = A_c x$, $L_c(x) = x^T R_c x$, $\Gamma_c(x) = x^T \Omega_c(P) x$, for $x \notin \mathcal{Z}$, $f_d(x) = (A_d + \Delta A_d - E_d)x$, $f_{d0}(x) = (A_d - E_d)x$, $L_d(x) = x^T R_d x$, $\Gamma_d(x) = x^T \Omega_d(P) x$, for $x \in \mathcal{Z}$, $V(x) = x^T P x$, with arguments $E_c x$ and $E_d x$, and $\mathcal{D} = R^n$. Specifically, conditions (4) and (5) are trivially satisfied. Now, for the argument $E_c x$, $V'(x) f_c(x) = x^T(A_c^T P E_c + E_c^T P A_c)x + x^T(\Delta A_c^T P E_c + E_c^T P \Delta A_c)x$, for all $x \neq 0$, $x \notin \mathcal{Z}$ and $\Delta A_c \in \mathbf{\Delta}_{Ac}$, and hence it follows from (42) that $V'(x) f_c(x) \leq V'(x) f_{c0}(x) + \Gamma_c(x) = x^T(A_c^T P E_c + E_c^T P A_c + \Omega_c(P))x$, for all $x \neq 0$, $x \notin \mathcal{Z}$. Similarly, for the argument $E_d x$, $V(E_d x + f_d(x)) - V(x) = x^T(A_d^T P A_d - E_d^T P E_d)x + x^T E_d^T(\Delta A_d^T P \Delta A_d + \Delta A_d^T P A_d + \Delta A_d^T P \Delta A_d) E_d x$, $x \in \mathcal{Z}$ and $\Delta A_d \in \mathbf{\Delta}_{Ad}$, and hence it follows from (43) that $V(E_d x + f_d(x)) - V(x) \leq V(E_d x + f_{d0}(x)) - V(x) + \Gamma_d(x) = x^T(A_d^T P A_d - E_d^T P E_d + \Omega_d(P))x$, for all $x \in \mathcal{Z}$. Furthermore, it follows from (44) that $L_c(x) + V'(x) f_{c0}(x) + \Gamma_c(x) = 0$, $x \notin \mathcal{Z}$ and hence $V'(x) f_{c0}(x) + \Gamma_c(x) < 0$, for all $x \neq 0$, $x \notin \mathcal{Z}$. Similarly, it follows from (45) that $L_d(x) + V(E_d x + f_{d0}(x)) - V(x) + \Gamma_d(x) = 0$, $x \in \mathcal{Z}$ and hence $V(E_d x + f_{d0}(x)) - V(x) + \Gamma_d(x) \leq 0$, $x \in \mathcal{Z}$, so that all the conditions of Theorem 2.1 are satisfied. Finally, since $V(x)$, $x \in R^n$, is radially unbounded, (39), (40) is globally asymptotically stable for all $(\Delta A_c, \Delta A_d) \in \mathbf{\Delta}$.

III. Conclusion

In this paper we have developed robust stability analyze results for the class of nonlinear uncertain singularly impulsive dynamical systems [1]. Results are based on Lyapunov and asymptotic stability theorems developed in [6]. Presented results will be base for further work on optimal nonlinear robust *control* that is under the development.

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