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Dither Shape in the Averaging of Switched Systems

Luigi Iannelli Karl Henrik Johansson Ulf T. Jönsson Francesco Vasca

***Abstract*— It was shown by Zames and Shneydor that a high-frequency dither of a quite arbitrary shape can be used to smooth the effective nonlinear sector of Lipschitz continuous feedback systems. Here it is shown that also systems with discontinuous nonlinearities can be smoothed using dither signals, as long as the amplitude distribution function of the dither is Lipschitz continuous.**

I. Introduction

A frequently used technique to stabilize a nonlinear feedback system in Lur'e form is by injecting a high-frequency dither signal to decrease the effective nonlinear sector. If the dither frequency is sufficiently high, the dithered system behaves qualitatively the same as an averaged system, in which the dither and the nonlinearity are replaced by another nonlinearity. Control design is then done for the averaged system, but applied to the dithered. For Lipschitz continuous systems this scheme can be justified rigorously by using classical averaging theory [1]. In [2], [3], Zames and Shneydor discussed how the behavior of the system depends on the amplitude distribution function of the dither for the case when the nonlinearities are Lipschitz continuous. For systems with nonsmooth nonlinearities there are only rigorous treatments of systems with particular nonlinearities and dithers, such as pulse-width modulated systems [4], [5], power converters [6], and relay systems [7], [8]. The current paper provides an averaging theorem for a general class of switched systems with a quite arbitrary periodic dither. Our main result states that the averaged and the dithered system have qualitatively the same behavior when the nonlinearities have bounded variation and the dither has an absolutely continuous amplitude distribution function with bounded derivative and a sufficiently high frequency. The result appears to be fairly tight, because examples suggest that dithering might loose its effect when any of the assumptions are violated. The outline of the paper is as follows. The dithered system and its averaged counterpart are introduced in Section II. The main result on the approximation error between the dithered and the averaged systems is presented and proved in Section III. Some comments and remarks are finally given in Section IV.

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L. Iannelli is with the Dept. of Computer and Systems Engineering, University of Napoli Federico II, 80125 Napoli, Italy, e-mail: luiannel@unina.it

K. H. Johansson is with the Dept. of Signals, Sensors & Systems, Royal Institute of Technology, 100 44 Stockholm, Sweden, e-mail: kallej@s3.kth.se

U. Jönsson is with the Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden, e-mail: ulfj@math.kth.se

F. Vasca is with the Dept. of Engineering, University of Sannio in Benevento, 82100 Benevento, Italy, e-mail: vasca@unisannio.it

II. Preliminaries

The dithered and averaged systems are defined in this section, together with the amplitude distribution function of the dither signal.

A. Dithered System

The dithered feedback system is defined as

$$\dot{x} = f_0(x,t) + \sum_{i=1}^{m} f_i(x,t) n_i(g_i(x,t) + \delta_i), \quad x(0) = x_0. \quad (1)$$

The state x belongs to $\mathbb{R}^q$. The functions $f_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$, $i = 1, \ldots, m$, are assumed to be globally Lipschitz with respect to both x and t, i.e., there exists a positive constant L_f such that for all $x_1, x_2 \in \mathbb{R}^q$ and $t_1, t_2 \geq 0$,

$$|f_i(x_1,t_1) - f_i(x_2,t_2)| \leq L_f \left(|x_1 - x_2| + |t_1 - t_2|\right).$$

We further assume that f_0 is piece-wise continuous in t, $f_0(0,t) = 0$, $\forall t \geq 0$ and

$$|f_0(x_1,t) - f_0(x_2,t)| \leq L_f |x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^q$ and $t \geq 0$. Similarly, the functions $g_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$ are assumed to have a common Lipschitz constant $L_g > 0$, i.e.,

$$|g_i(x_1,t_1) - g_i(x_2,t_2)| \leq L_g \left(|x_1 - x_2| + |t_1 - t_2|\right)$$

for all $x_1, x_2 \in \mathbb{R}^q$, $t_1, t_2 \geq 0$. The nonlinearities $n_i : \mathbb{R} \to \mathbb{R}$ are assumed to be functions of bounded variation[1]. Hence, n_i can be a discontinuous function, but it is necessarily bounded. Let $M_n := \max_i(|n_i(0)| + TV(n_i))$ denote such a bound, where $TV(n_i)$ is the total variation of n_i. Each dither signal $\delta_i : [0,\infty) \to \mathbb{R}$ is supposed to be a p-periodic measurable function bounded by M_δ.

When the differential equation (1) has a discontinuous right-hand side (due to that some n_i are discontinuous), existence and uniqueness of solutions depend critically on the considered definition of solution [10]. In the following we assume that the differential equation (1) has at least one absolutely continuous solution $x(t,x_0)$ on $[0,\infty)$. Moreover, by using Lipschitz conditions on f_i, it is possible to show that there exists a positive constant L_x such that $|x(t_1) - x(t_2)| \leq L_x |t_1 - t_2|$ for almost all $0 \leq t_1 \leq t_2 < \infty$. Estimates of the Lipschitz constant L_x can easily be obtained on any compact interval by using the assumptions above. Analogously it can be computed an upper bound on f_i: $|f_i(x,t)| \leq M_f$ (see [11] for details).

[1] See [9] for an introduction to measure theory.

B. Amplitude Distribution Function

Definition 2.1: The amplitude distribution function of a dither signal $\delta : [0,\infty) \to \mathbb{R}$ is the function $F_\delta : \mathbb{R} \to [0,1]$ defined as

$$F_\delta(\xi) = \frac{1}{p}\mu\{t \in [0,p) : \delta(t) \le \xi\}, \tag{2}$$

where μ denotes the Lebesgue measure.

When the amplitude distribution function is absolutely continuous (with respect to its Lebesgue measure), the amplitude density function $f_\delta(\xi)$ is defined as

$$f_\delta(\xi) = \frac{\mathrm{d}F_\delta}{\mathrm{d}\xi}(\xi). \tag{3}$$

The amplitude density and amplitude distribution functions play in a deterministic framework the same role as probability density and cumulative distribution functions play in a stochastic framework. In particular the amplitude distribution function is bounded, monotonously increasing, continuous from the right, and, if it is differentiable, its derivative obviously corresponds to the amplitude density function.

C. Averaged System

The averaged system is given by

$$\dot{w} = f_0(w,t) + \sum_{i=1}^{m} f_i(w,t)N_i(g_i(w,t)), \quad w(0) = w_0, \tag{4}$$

where N_i is called the averaged nonlinearity,[2] and it is derived from the original nonlinearity n_i and the dither signal δ_i. We assume that there exists at least one absolutely continuous solution $w(t,w_0)$ defined on $[0,\infty)$. We will later show that this is the case when the amplitude distribution function is absolutely continuous and has a bounded derivative, because then the averaged nonlinearity is Lipschitz continuous.

The averaged nonlinearity is defined as follows[3].

Definition 2.2: For a dither signal $\delta : [0,\infty) \to \mathbb{R}$ and a nonlinearity $n : \mathbb{R} \to \mathbb{R}$ the averaged nonlinearity $N : \mathbb{R} \to \mathbb{R}$ is defined as

$$N(z) \triangleq \int_{\mathbb{R}} n(z+\xi)\mathrm{d}F_\delta(\xi) \tag{5}$$

where the integral is a Lebesgue–Stieltjes integral.

In many cases the averaged nonlinearity can be formulated as a time average, as the following lemma states.

Lemma 2.1: [12], [13] The following equality holds provided that either side exists:

$$N(z) = \frac{1}{p}\int_{[0,p)} n(z+\delta(s))\mathrm{d}s. \tag{6}$$

[2]In the literature, N_i is sometimes called the smoothed nonlinearity, which however seems less appropriate here since N_i can be a discontinuous function, as shown in Section IV.

[3]For the sake of simplicity from now on we will omit subscript i when not necessary.

Two cases are of particular interest in the paper. First, when the amplitude distribution function is absolutely continuous, we have

$$N(z) = \int_{\mathbb{R}} n(z+\xi)\mathrm{d}F_\delta(\xi) = \int_{\mathbb{R}} n(z+\xi)f_\delta(\xi)\mathrm{d}\xi,$$

which is always well defined under the given assumptions on n.

Second, when the Lebesgue–Stieltjes measure corresponding to the amplitude distribution function has a decomposition (relative to the Lebesgue measure) into an absolutely continuous part and a singular part, we have (if n is continuous at $z+\xi_k$)

$$\begin{aligned} N(z) &= \int_{\mathbb{R}} n(z+\xi)\mathrm{d}F_\delta(\xi) \\ &= \int_{\mathbb{R}} n(z+\xi)f_\delta(\xi)d\xi + \sum_{k=1}^{Q} n(z+\xi_k)f_k, \end{aligned}$$

where $f_k \neq 0$ are the jump discontinuities corresponding to the singular parts of the amplitude distribution function. Note that square wave and trapezoidal dither signals have this kind of amplitude distribution functions.

III. Averaging Theorem

The next theorem states conditions under which the averaged system approximates the behavior of the dithered system for a sufficiently high dither frequency. The theorem is stated for a finite time-horizon, but can be used to derive infinite horizon stability results, cf., [7].

Theorem 3.1: Consider the dithered system (1) and the averaged system (4) under the assumptions introduced in the previous section, i.e., for each $i = 1,\dots,m$,

- (i) f_i and g_i are globally Lipschitz with Lipschitz constants, respectively, L_f and L_g,
- (ii) f_0 is globally Lipschitz with respect to x with Lipschitz constant L_f, and $f_0(0,t) = 0$,
- (iii) n_i is a function of bounded variation and sup-norm $\|n_i\|_\infty \le M_n$,
- (iv) each dither δ_i is p-periodic with absolutely continuous amplitude distribution function F_{δ_i} and $L_F \triangleq \sup_{\zeta\in\mathbb{R}} |f_{\delta_i}(\zeta)| < \infty$.

Then the averaged nonlinearities N_i are Lipschitz continuous and the averaged system (4) has a unique absolutely continuous solution on $[0,\infty)$. Moreover, for any given $T > 0$ and $x_0 \in \mathbb{R}^n$, it holds that

$$|x(t,x_0) - w(t,x_0)| = O(p), \quad \forall t \in [0,T].$$

The proof of the theorem is based on three lemmas. In the first lemma we show that the averaged nonlinearity under our assumptions is Lipschitz continuous, which implies that there exists a unique absolutely continuous solution of the averaged system on any finite time-horizon.

Lemma 3.1: Suppose n is of bounded variation and that F_δ is absolutely continuous with $L_F = \sup_{\xi\in\mathbb{R}} |f_\delta(\xi)| < \infty$. Then

$$N(z) = \int_{\mathbb{R}} n(z+\xi)f_\delta(\xi)d\xi$$

is Lipschitz continuous with $L_N \leq L_F \cdot TV(n)$ and bounded with $\|N\|_\infty \leq \|n\|_\infty$.

Proof: We have

$$\begin{aligned}|N(z_1) - N(z_2)| &= \left|\int_{\mathbb{R}} [n(z_1+\xi) - n(z_2+\xi)]\, dF_\delta(\xi)\right| \\ &= \left|\int_{\mathbb{R}} n(\xi)\,[dF_\delta(\xi - z_1) - dF_\delta(\xi - z_2)]\right|\end{aligned}$$

Let $V(\xi) = F_\delta(\xi - z_1) - F_\delta(\xi - z_2)$. We have $V(\xi) = 0$ for $\xi \notin S = [-M_\delta - \max(z_1,z_2), M_\delta - \min(z_1,z_2)]$. Hence, for any $I = [a,b] \supset S$ integration by parts gives

$$\begin{aligned}|N(z_1) - N(z_2)| &= \left|\int_I n(\xi)[dF_\delta(\xi - z_1) - dF_\delta(\xi - z_2)]\right| \\ &= \left|n(b)V(b) - n(a)V(a) - \int_I V(\xi)dn(\xi)\right| \\ &\leq \sup_{\xi\in I}|V(\xi)| \int_I |dn(\xi)| \leq L_F\,|z_1 - z_2| \cdot TV(n)\end{aligned}$$

where the last inequality follows because $V(a) = V(b) = 0$ and

$$|V(\xi)| = \left|\int_{z_2}^{z_1} f_\delta(\xi - \sigma)d\sigma\right| \leq L_F|z_1 - z_2|$$

The boundedness follows since

$$|N(z)| = \left|\int_{\mathbb{R}} n(z+\xi)f_\delta(\xi)d\xi\right| \leq \|n\|_\infty \int_{\mathbb{R}} f_\delta(\xi)d\xi = \|n\|_\infty$$

since f_δ is nonnegative. ■

It should be noticed that if n is Lipschitz then the corresponding averaged nonlinearity N will be Lipschitz independently on the characteristic of F_δ, see [2]. Lemma 3.1 states that in order to have N Lipschitz when n is not Lipschitz, we need further assumptions on the dither. The next lemma is the key to the proof of Theorem 3.1.

Lemma 3.2: Suppose the signal $y : [0,p] \to \mathbb{R}$ has Lipschitz constant L_y. Introduce a constant $\tilde{y}$ satisfying

$$\min_{s\in[0,p]} y(s) \leq \tilde{y} \leq \max_{s\in[0,p]} y(s).$$

Suppose that F_δ is absolutely continuous with $L_F = \sup_{\xi\in\mathbb{R}}|f_\delta(\xi)| < \infty$. Then,

$$\begin{aligned}E &\triangleq \left|\int_0^p n(-y(s)+\delta(s))\mathrm{d}s - \int_0^p n(-\tilde{y}+\delta(s))\mathrm{d}s\right| \\ &\leq 2L_F L_y TV(n)p^2,\end{aligned} \tag{7}$$

where $TV(n)$ is the total variation of n.

Proof: From the previous definition of F_δ we introduce

$$F_{-y+\delta}(\xi) = \frac{1}{p}\mu(\{s\in[0,p):\ -y(s)+\delta(s) \leq \xi\}),$$

so that

$$E = p\left|\int_{\mathbb{R}} n(\xi)\mathrm{d}F_{-y+\delta}(\xi) - \int_{\mathbb{R}} n(\xi)\mathrm{d}F_{-\tilde{y}+\delta}(\xi)\right|.$$

By hypothesis

$$\tilde{y} - L_y p \leq y(s) \leq \tilde{y} + L_y p, \quad \forall s \in [0,p]$$

and thus it follows that for any $\xi \in \mathbb{R}$,

$$F_{-\tilde{y}+\delta}(\xi - L_y p) \leq F_{-y+\delta}(\xi) \leq F_{-\tilde{y}+\delta}(\xi + L_y p). \tag{8}$$

On the other hand, since $F_{-\tilde{y}+\delta}$ is nondecreasing,

$$F_{-\tilde{y}+\delta}(\xi - L_y p) \leq F_{-\tilde{y}+\delta}(\xi) \leq F_{-\tilde{y}+\delta}(\xi + L_y p). \tag{9}$$

By combining that (8) and (9) and using that $F_{-\tilde{y}+\delta}(\xi) = F_\delta(\xi + \tilde{y})$ is Lipschitz and non-negative, we get

$$\begin{aligned}F_{-y+\delta}(\xi) - F_{-\tilde{y}+\delta}(\xi) &\leq F_{-\tilde{y}+\delta}(\xi + L_y p) - F_{-\tilde{y}+\delta}(\xi - L_y p) \\ &\leq 2L_F L_y p.\end{aligned}$$

In an analogous way,

$$-2L_F L_y p \leq F_{-y+\delta}(\xi) - F_{-\tilde{y}+\delta}(\xi).$$

So we can write

$$F_{-y+\delta}(\xi) = F_{-\tilde{y}+\delta}(\xi) + V(\xi) \tag{10}$$

with $|V(\xi)| \leq 2L_F L_y p$ and thus

$$E = p\left|\int_{\mathbb{R}} n(\xi)\mathrm{d}V(\xi)\right|. \tag{11}$$

Since for $s \in [0,p]$ we have $|y(s) - \tilde{y}| \leq L_y p$ and $|\delta(s)| \leq M_\delta$,

$$V(\xi) = 0, \quad \forall \xi \notin [-\tilde{y} - L_y p - M_\delta, -\tilde{y} + L_y p + M_\delta] \triangleq S.$$

The function $V(\xi)$ is of bounded variation and continuous from the right, since it is the difference of two functions that satisfy both these properties. By hypothesis n is of bounded variation with total variation $TV(n)$ so we can integrate by parts [14]:

$$\int_{[a,b]} n(\xi)\mathrm{d}V(\xi) = n(b)V(b) - n(a)V(a) - \int_{[a,b]} V(\xi)\mathrm{d}n(\xi), \tag{12}$$

where right and left limits of n and V are used in order to cope with discontinuities. If $[a,b] \supset S$ then $V(a) = V(b) = 0$, and thus

$$E = p\left|\int_S V(\xi)\mathrm{d}n(\xi)\right| \leq 2p^2 L_F L_y TV(n), \tag{13}$$

which proves the lemma. ■

Lemmas 3.1 and 3.2 are used to prove the following result (see [11] for the proof).

Lemma 3.3: If the assumptions of Theorem 3.1 hold, then there exist constants $\bar{K}, \tilde{K} > 0$ such that

$$\begin{aligned}&\left|\int_0^p f_i(x,s)n_i(g_i(x,s)+\delta_i)\mathrm{d}s - \int_0^p f_i(w,s)N_i(g_i(w,s))\mathrm{d}s\right| \\ &\qquad \leq \bar{K}\int_0^p |x(s) - w(s)|\mathrm{d}s + \tilde{K}p^2.\end{aligned} \tag{14}$$

Now we can proceed by showing that the approximation error between the dithered and the averaged system can be arbitrarily small by increasing the dither frequency, as stated in the theorem.

Proof of Theorem 3.1. Consider the dithered system (1) and the averaged system (4) on the time interval $[0,T]$ with $w_0 = x_0$. By integrating the right-hand sides of (1) and (4), we can write

$$\begin{aligned}|x(t)-w(t)| \le \int_0^t |f_0(x,s)-f_0(w,s)|\mathrm{d}s \\ +\sum_{i=1}^m \left| \int_0^t [f_i(x,s)\cdot n_i(g_i(x,s)+\delta_i) \right. \\ \left. - f_i(w,s)\cdot N_i(g_i(w,s))\,]\,\mathrm{d}s \right|\end{aligned} \tag{15}$$

for all $t \in [0,T]$.

If we introduce $l = \lfloor T/p \rfloor$, the largest integer such that $lp \le T$, then by using the periodicity of δ_i,

$$\begin{aligned}|x(t)-w(t)| &\le \int_0^t |f_0(x(s),s)-f_0(w(s),s)|\mathrm{d}s \\ &+\sum_{k=0}^{l-1}\sum_{i=1}^m \left| \int_{kp}^{(k+1)p} [f_i(x(s),s)n_i(g_i(x(s),s)+\delta_i(s)) \right. \\ &\left. - f_i(w(s),s)N_i(g_i(w(s),s))\,]\,\mathrm{d}s \right| \\ &+V_1(p), \quad \forall t \in [0,T],\end{aligned} \tag{16}$$

where the last term is bounded as

$$|V_1(p)| \le 2mM_fM_np.$$

The Lipschitz property of f_0 gives

$$\left| \int_0^t [f_0(x(s),s)-f_0(w(s),s)]\mathrm{d}s \right| \le L_f \int_0^t |x(s)-w(s)|\,\mathrm{d}s.$$

Next we notice that each integral in the sum of (16) can be written

$$\int_0^p [f_i(x_k(s),s_k)n_i(g_i(x_k(s),s_k)+\delta_i(s)) - f_i(w_k(s),s_k)N_i(g_i(w_k(s),s_k))]ds$$

where the subscript k denotes a time translation: $s_k = s+kp$, $x_k(s) = x(s+kp)$ and similarly for w. Then applying Lemma 3.3, each integral

$$\int_0^p (f_i(x_k,s_k)n_i(g_i(x_k,s_k)+\delta_i(s)))\mathrm{d}s$$

can be approximated by

$$\int_0^p f_i(w_k,s_k)N_i(g_i(w_k,s_k))\mathrm{d}s.$$

Indeed, the Lipschitz assumptions on the f_i and g_i are uniform in t so Lemma 3.3 can be applied to all functions x_k. The approximation error has an upper bound $\bar{K}\int_0^p |x_k - w_k|\mathrm{d}s + \tilde{K}p^2$. By summing all the contributions given by the time intervals $[kp,(k+1)p] \subset [0,T]$, we get

$$\begin{aligned}|x(t)-w(t)| \le K\int_0^t |x(s)-w(s)|\mathrm{d}s \\ +m\tilde{K}pT + V_1(p)+V_2(p), \quad \forall t \in [0,T],\end{aligned}$$

where $K = L_f + m\bar{K} = L_f + m(M_nL_f + M_fL_NL_g)$ and $V_2(p)$ is bounded by

$$|V_2(p)| \le m\bar{K}(M_x+M_w)p,$$

where M_x and M_w are easily derived upper bounds in $[0,T]$ of x and w, respectively, see [11] for details.

By applying Grönvall-Bellman Lemma [15], the theorem follows since

$$|x(t)-w(t)| \le (m\tilde{K}Tp + V_1(p)+V_2(p))\,e^{KT} \quad \forall t \in [0,T] \tag{17}$$

where the right hand side is of order p. ■

IV. Discussion

In this section we make some remarks on the assumptions and applications of Theorem 3.1.

A. Smoothness of Amplitude Distribution Function

The assumption on absolute continuity of the dither amplitude distribution function F_{δ_i} in Theorem 3.1 seems to be necessary (when n_i is not Lipschitz continuous). Recall that for example a square wave violates this assumption. Indeed, an example in [7] shows that a dithered and an averaged relay system can have very different qualitative behaviors, when a square wave dither is used, even if the frequency of the dither is high. We showed in [8] that this phenomenon can be detected also in real applications. Next we provide a new example, which illustrates the importance of having dither signals with absolutely continuous amplitude distribution functions in nonsmooth systems.

Let us consider the dithered relay feedback system

$$\dot{x}(t) = Ax(t) + b\,\mathrm{sgn}\,(bx(t)+R+\delta(t)), \quad x(0)=x_0, \tag{18}$$

with

$$A = \begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix},\ b = \begin{bmatrix}0\\1\end{bmatrix},\ c = \begin{bmatrix}-1 & 0\end{bmatrix}, x \tag{19}$$

$$\mathrm{sgn}(z) = \begin{cases}+1, & z>0\\ 0 & z=0\\ -1, & z<0.\end{cases} \tag{20}$$

Let the external reference be constant $R = 0.5$ and let $\delta(t)$ be a square wave dither of amplitude $M_\delta = 0.5$, i.e.,

$$R+\delta(t) = \begin{cases}1 & \mathrm{mod}\,(t,p) \in [0,\frac{p}{2})\\ 0 & \mathrm{mod}\,(t,p) \in [\frac{p}{2},p).\end{cases} \tag{21}$$

In this case the averaged system is equal to

$$\dot{w}(t) = Aw(t) + bN(cw(t)+R), \quad w(0)=w_0 \tag{22}$$

with

$$N(z) = \begin{cases}-1, & z<-0.5\\ -0.5 & z=-0.5\\ 0, & |z|<0.5\\ 0.5 & z=0.5\\ +1, & z>0.5.\end{cases} \tag{23}$$

To argue about the evolution of the dithered and averaged systems, let us partition the state space into the following three regions and comment on the dynamics in each of them:

- Region $\Omega_1 = \{x : x_1 < 0\}$: In this region $n(cx+R+\delta) = 1$, so $\dot{x} = Ax + b$ and the vector field of the dithered system coincides with the vector field of the averaged system. The equilibrium point of this system is given by $P_1 = -A^{-1}b = (-1,1)^T$.
- Region $\Omega_2 = \{x : x_1 > 1\}$. In this region $n(cx+R+\delta) = -1$, so $\dot{x} = Ax - b$ and the dithered system coincides also here with the averaged system. The equilibrium point of the system is $P_2 = A^{-1}b = (1,-1)^T$.
- Region $\Omega_0 = \{x : 0 < x_1 < 1\}$. In Ω_0 the state does not affect the output of the relay, so the dithered system can be represented by the following linear system:

$$\dot{\zeta}(t) = A\zeta(t) + bu(t), \tag{24}$$

where u is a periodic signal that switches between -1 (when $R+\delta(t) = 0$) and $+1$ (when $R+\delta(t) = 1$). The averaged system has an input equal to zero in this region, i.e., $\dot{w}(t) = Aw(t)$. Note that if $w_0 \in \Omega_0$, the solution $w(t) = (w_1(t), w_2(t))^T$ is given by

$$w_1(t) = e^{-t}\left[w_1(0) - tw_2(0)\right], \quad w_2(t) = e^{-t}w_2(0),$$

as long as $0 < w_1(t) < 1$.

Now, consider solutions of the dithered and averaged systems as shown in Fig. 1. Let the initial conditions be equal $x(0) = w(0)$ and such that $x_2(0) = 0$ and $0 < x_1(0) < 1$. Then, the averaged system has a solution that lies on the x_1 axis:

$$w_1(t) = e^{-t}x_1(0), \quad w_2(t) = 0.$$

Hence, the averaged system tends to the origin.

For the solution of the dithered system, note that in $\Omega_0^+ = \{x \in \Omega_0 : 0 < x_1 < 1,\ x_2 > 1\}$ we have $\dot{x}_1 < 0$ and $\dot{x}_2 < 0$ while in $\Omega_0^- = \{x \in \Omega_0 : 0 < x_1 < 1,\ x_2 < -1\}$ we have $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$, cf., vector fields indicated in Ω_0^- and Ω_0^+ in Fig. 1. It then follows that the trajectory of the dithered system must leave $\Omega_0 \setminus (\Omega_0^+ \bigcup \Omega_0^-)$ by crossing the line segment $\{x : x_1 = 0,\ 0 \le x_2 \le 1\}$. Moreover, in Ω_0 the dithered solution can be represented as

$$x(t) = e^{At}(x(0) - \zeta_0) + \zeta_{ss}(t)$$

where ζ_{ss} is the steady state p-periodic solution of (24) and

$$\zeta_0 = (I - e^{Ap})^{-1}\int_0^p e^{A(p-s)}bu(s)ds.$$

Since A is Hurwitz, $x(t)$ will converge to $\zeta_{ss}(t)$. It is possible to choose a sufficiently small dither period p such that $\zeta_{ss}(t) \notin \Omega_2$, because $\zeta_{ss}(t) \to 0$ as $p \to 0$. It is thus clear that $x(t)$ eventually will cross the x_2 axis for some $0 \le x_2 \le 1$. From Fig. 1, we conclude that the second orthant is an invariant set for the dynamics in region Ω_1. Moreover, since the system matrix A is Hurwitz, the dithered solution $x(t)$ will tend toward the equilibrium point P_1. Recall that the

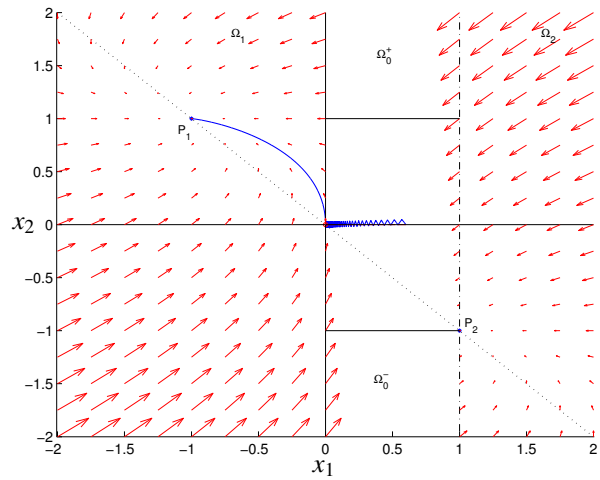

Fig. 1. Phase plane of a system with square wave dither. The indicated trajectory of the dithered system tends to the equilibrium point P_1. The corresponding trajectory of the averaged system tends to the origin. Hence, their characteristics are different.

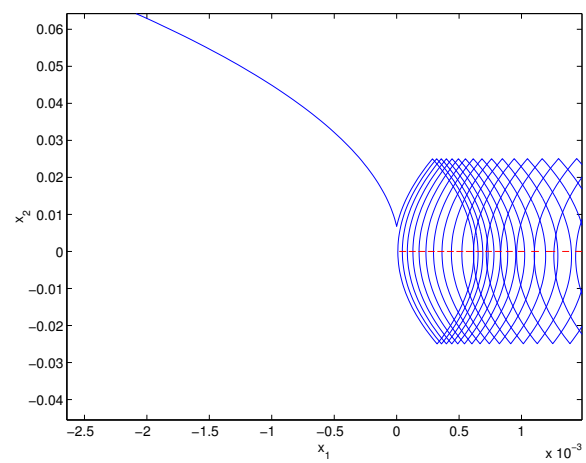

Fig. 2. Zoom of the phase plane in Fig. 1. Solutions of the dithered system (*solid* line) and averaged system (*dashed* line).

averaged system converges to the origin. The behaviors of the dithered system and the averaged system are hence quite different. See simulations in Fig. 2 and Fig. 3 for further comparison.

Even though we showed above that the dithered and the averaged system can behave qualitatively different and that this holds for any small dither period p, the example does not prove that the conclusion of Theorem 3.1 does not hold if the amplitude distribution function is not absolutely continuous. The reason for this is that it does not follow that the dithered state x crosses the vertical axis $x_1 = 0$ in $[0,T]$. In particular, when increasing the dither frequency (as required by Theorem 3.1), the crossing is not guaranteed on a fixed interval $[0,T]$. The example does illustrate, however, how the dithered and the averaged solution bifurcate due to the fact that the averaged nonlinearity is discontinuous. We have discovered similar phenomena for limit cycles of the averaged and the dithered systems [7], [16], [17].

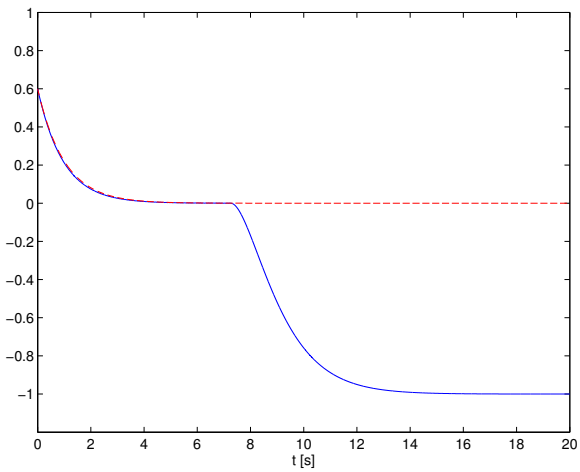

Fig. 3. Time evolution of x_1 (*solid* line) and w_1 (*dashed* line) of the dithered and averaged systems, respectively.

If assumption (iv) weakened such that the boundedness assumption on the density function is removed then the amplitude distribution function is no longer Lipschitz continuous and the conclusions of the theorem do not hold. For example, in [18] we have an example where an absolutely continuous dither with unbounded density function gives an averaged system that does not have a unique solution.

B. Applications

One of the main applications of Theorem 3.1 is to derive stability results for the dithered system. For example, if the averaged system is globally asymptotically stable, then it is possible to show that the dithered system is practically stable. Hence, a dither signal can be injected in order to (practically) stabilize a system. We refer to [7], [4], [19] for various results of this type.

In some applications it might be desirable to consider more general dither signals than periodic ones. It is possible to relax the assumption of Theorem 3.1 on the dither and consider so called F-repetitive dither as defined in [2]. Recall that a dither signal δ is F-repetitive if there exists an unbounded sequence $\{t_k\}$, $0 = t_0 < t_1 \ldots$, such that the maximal repetition $p = \sup_k (t_k - t_{k-1})$ is bounded and the amplitude distribution function of δ on (t_{k-1}, t_k), is equal to the amplitude distribution function on (t_0, t_1).

Dither period, and amplitude and shape of the amplitude distribution function can be used as design variables. Theorem 3.1 shows that the dither period determines the approximation between the dithered and averaged systems. Moreover, for desired performance of the dithered system, since each amplitude distribution function defines a class of dither signals which gives rise to the same averaged nonlinearity, it is more appropriate to design the amplitude and the shape of the amplitude distribution function instead of looking at the dither amplitude and shape. For instance, the shape of the amplitude distribution function can be critical for determining a compromise between the maximal reduction of the nonlinear sector and the local attractivity of an equilibrium. We will discuss this further in our future work.

V. Conclusions

We have proved that dither can be efficiently used for the averaging of a quite general class of nonsmooth nonlinear systems, provided that the amplitude distribution function of the dither is Lipschitz. This condition, which is not needed for smooth Lipschitz nonlinear systems, suggests that the dither shape must be carefully chosen for smoothing common discontinuous systems such as mechanical systems with friction, power electronic converters and other hybrid systems.

References

[1] S. Mossaheb, "Application of a method of averaging to the study of dither in non-linear systems," *International Journal of Control*, vol. 38, no. 3, pp. 557–576, September 1983.

[2] G. Zames and N. A. Shneydor, "Dither in non-linear systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 5, pp. 660–667, October 1976.

[3] ——, "Structural stabilization and quenching by dither in non-linear systems," *IEEE Transactions on Automatic Control*, vol. 22, no. 3, pp. 352–361, June 1977.

[4] A. K. Gelig and A. Churilov, *Stability and Oscillations of Nonlinear Pulse Modulated Systems*. Berlin: Birkhäuser, 1998.

[5] A. Teel, L. Moreau, and D. Nesic, "Input-to-state set stability of pulse width modulated systems with disturbances," *Systems and Control Letters*, vol. 51, January 2004.

[6] B. Lehman and R. Bass, "Extensions of averaging theory for power electronics systems," *IEEE Transactions on Power Electronics*, vol. 11, no. 4, pp. 542–553, 1996.

[7] L. Iannelli, K. Johansson, U. Jönsson, and F. Vasca, "Dither for smoothing relay feedback systems: an averaging approach," *IEEE Transactions on Circuits and Systems, Part I*, vol. 50, no. 8, pp. 1025–1035, August 2003.

[8] ——, "Practical stability and limit cycles of dithered relay feedback systems," in *Proc. of IEE European Control Conference*, Cambridge, UK, September 2003.

[9] R. Wheeden and A. Zygmund, *Measure and Integral*. New York, NY, USA: Marcel Deeker Inc., 1977.

[10] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, 1988.

[11] L. Iannelli, K. Johansson, U. Jönsson, and F. Vasca, "Conditions on the dither shape in the averaging of switched systems," Department of Mathematics, Royal Institute of Technology, Tech. Rep. TRITA/MAT-03-OS08, September 2003.

[12] K. L. Chung, *A Course in Probability Theory*. Orlando, FL, USA: Academic Press Inc., 1974.

[13] S. Taylor, *Introduction to Measure and Integration*. Cambridge University Press, 1966.

[14] Riesz and Sz-Nagy, *Functional Analysis*. Dover Pubns, 1990.

[15] S. Sastry, *Nonlinear Systems: Analysis, Stability and Control*. New York: Springer-Verlag, 1999.

[16] L. Iannelli, "Dither for smoothing relay feedback systems: an averaging approach," Ph.D. dissertation, University of Napoli "Federico II", Napoli, Italy, Nov. 2002, http://cds.unina.it/∼luiannel/eng/public.htm.

[17] L. Iannelli, K. Johansson, U. Jönsson, and F. Vasca, "Effects of dither shapes in nonsmooth feedback systems: Experimental results and theoretical insight," in *Proc. of IEEE Conference on Decision and Control*, Maui, Hawaii (USA), December 2003.

[18] L. Iannelli, K. H. Johansson, U. Jönsson, and F. Vasca, "Averaging of nonsmooth systems through dithered switchings," 2004, in preparation.

[19] L. Moreau and D. Aeyels, "Practical stability and stabilization," *IEEE Trans. Automat. Control*, vol. 45, no. 8, pp. 1554–1558, august 2000.