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Minimizing Weighted Earliness and Tardiness Penalties about a Common Due Date on Single Machine with Exponential Processing Times

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***Abstract*— A problem of scheduling n jobs with exponential processing times on a single machine is discussed, the objective is to find an optimal schedule to minimize the expectation of total weighted absolute deviations of completion times about a deterministic common due date. This problem is a typical scheduling model in Just-In-Time manufacturing system where both earliness penalties and tardiness penalties are considered. The deterministic equivalent of the objective function is derived, and Λ-shaped property, with respect to the products of jobs' weights and their processing time rates, of the optimal schedules of this problem is established. The Λ-shaped property of the optimal schedule can reduce the candidates of optimal schedule from $n!$ to 2^n.**

Keywords: Scheduling; Single machine; Exponential processing time; Common due date; Λ-shaped schedule

I. INTRODUCTION

The single machine scheduling problem to minimize total weighted absolute deviations of completion times about a common due date is a typical Earliness/Tardiness scheduling model [1], where both earliness penalties and tardiness penalties are considered. This problem is raised from Just-In-Time (JIT) manufacturing system. In JIT manufacturing environment, on one hand, a job completed after the assigned due date will incur much higher delivery cost, because an alternative, substantially more expensive mean will have to be used to deliver the job. On the other hand, the job completed ahead of the due date will have to be stored in a warehouse to wait for the due date, which will lead to inventory cost. This problem is described as follows.

Given a set $N = \{1, 2, \dots, n\}, n > 1$, of independent and simultaneously available jobs which are to be processed non-preemptively on a machine. Job i $(i \in N)$ requires a nonnegative processing time p_i and is assigned a positive weight w_i. d is the common due date of the jobs. Let Π be the set of all $n!$ possible permutations of the integers $1, 2, \dots, n$, $\pi = ([1], [2], \dots, [n]) \in \Pi$ which denotes that job $[k]$ $(k = 1, 2, \dots, n)$ is the kth job to be processed on the machine. Assume that there is no inserted idle time in schedule $\pi \in \Pi$, and the start time of the schedule is 0. Then, under schedule π, the completion time $C_{[i]}(\pi)$ of job $[i]$ is $C_{[i]}(\pi) = \sum_{k=1}^{i} p_{[k]}$. The objective is to find a schedule $\pi^* \in \Pi$ to minimize

$$D(\pi) = \sum_{i=1}^{n} w_{[i]} |C_{[i]}(\pi) - d|. \qquad (1)$$

Hall and Posner [2], Hall, Kubiak and Sethi [3] and Hoogeveen and Van de Velde [4] studied this problem and proved that the problem is NP-hard.

Recently, many researchers studied the scheduling problems where the processing times are random. With the assumption that the processing times are exponentially distributed random variables, Pinedo [5] and Pinedo and Rammouz [6] studied several regular scheduling problems in which the objective functions are increasing functions of completion times, such as the total tardiness or the number of tardy jobs. Frenk [7] established a general framework for the single machine stochastic scheduling problems. Forst [8] addressed a stochastic single machine scheduling problem with earliness and tardiness penalties. Manna [9] discussed a due date assignment and scheduling problem on a single machine which is the stochastic version of the problem discussed by Panwalkar and Smith [10]. Manna [9] first derived the distribution and density functions of the completion times when the processing times are exponentially distributed, and then developed several properties of the optimal schedules, such as V-shaped property. Jia [11] studied the stochastic counterpart of the problem represented by (1) where the processing times and the due date are exponentially distributed. Further, Jia [12] discussed the case with proportional weights and derived the optimal schedule.

In this paper, we also discuss the stochastic counterpart of the problem described by (1), the processing times are exponentially distributed, but the common due date is deterministic. The objective function is written as

$$D_S(\pi) = E(\sum_{i=1}^{n} w_{[i]} |C_{[i]}(\pi) - d|). \qquad (2)$$

By using the distribution functions of the completion times given by Manna [9], we derive the deterministic equivalent of (2), and then develop the Λ-shaped property of the optimal schedules.

II. MAIN RESULTS

Let $p_i (i \in N)$ be exponentially distributed random variables with rate λ_i, and let d be the deterministic common

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due date of all the jobs. $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. Under schedule $\pi = ([1],[2],\ldots,[n])$, the distribution function of completion time $C_{[i]}(\pi) = \sum_{k=1}^{i} p_{[k]}$ is

$$\begin{aligned} G_{[i]}(u,\pi) &= Pr(C_{[i]}(\pi) \le u) \\ &= 1 - \sum_{k=1}^{i} \frac{\prod_{j=1,j\neq k}^{i} \lambda_{[j]}}{\prod_{j=1,j\neq k}^{i} (\lambda_{[j]} - \lambda_{[k]})} e^{-\lambda_{[k]} u}, \end{aligned} \quad (3)$$

which is derived by induction [9]. Let $\bar{G}_{[i]}(u,\pi) = 1 - G_{[i]}(u,\pi)$, then with some efforts, we can get the deterministic equivalent of (2) as

$$\begin{aligned} D_S(\pi) &= \sum_{i=1}^{n} (w_{[i]} \sum_{j=1}^{i} \frac{1}{\lambda_{[j]}}) + d \sum_{i=1}^{n} w_{[i]} \\ & \quad -2 \sum_{i=1}^{n} w_{[i]} \int_0^d \bar{G}_{[i]}(u,\pi) \mathrm{d}u. \end{aligned} \quad (4)$$

A Λ-shaped schedule with respect to processing times is that the jobs are arranged in ascending order of their processing times if they are placed before the largest job, but in descending order if placed after it. To prove Λ-shaped optimality, one should show that a schedule with three consecutive jobs r, s and t such that the processing time of job s is smaller than the processing times of jobs r and t cannot be optimal.

Theorem. The optimal schedules that minimize (4) are Λ-shaped in terms of $w_i\lambda_i$ $(i \in N)$, where $w_i\lambda_i$=$w_i/E(p_i)$.

Proof. Let $\pi^* = ([1],[2],\ldots,[m-1],r,s,t,[m+3],\ldots,[n])$, $1 \le m \le n-2$, be an optimal schedule to minimize (4), where jobs r, s and t are three consecutive jobs such that $w_r\lambda_r > w_s\lambda_s$ and $w_t\lambda_t > w_s\lambda_s$. Let $\pi_1 = ([1],[2],\ldots,[m-1],s,r,t,[m+3],\ldots,[n])$ and $\pi_2 = ([1],[2],\ldots,[m-1],r,t,s,[m+3],\ldots,[n])$. From (3) and (4), by using job interchanging argument, we can get

$$D_S(\pi_1) - D_S(\pi^*) = \frac{w_r\lambda_r - w_s\lambda_s}{\lambda_r\lambda_s} \Big\{ 1 - 2G_s(d,\pi^*) \Big\}$$

and

$$D_S(\pi_2) - D_S(\pi^*) = \frac{w_s\lambda_s - w_t\lambda_t}{\lambda_s\lambda_t} [1 - 2G_t(d,\pi^*)].$$

Therefore,

$$\begin{aligned} & \frac{D_S(\pi_1) - D_S(\pi^*)}{(w_r\lambda_r - w_s\lambda_s)/(\lambda_r\lambda_s)} + \frac{D_S(\pi_2) - D_S(\pi^*)}{(w_t\lambda_t - w_s\lambda_s)/(\lambda_s\lambda_t)} \\ = & \quad 2\{G_t(d,\pi^*) - G_s(d,\pi^*)\} \\ < & \quad 0, \end{aligned}$$

because $C_r(\pi^*) < C_s(\pi^*) < C_t(\pi^*)$, and consequently, $G_t(d,\pi^*) < G_s(d,\pi^*) < G_r(d,\pi^*)$.

Since $w_r\lambda_r > w_s\lambda_s$ and $w_t\lambda_t > w_s\lambda_s$, we get $D_S(\pi_1) - D_S(\pi^*) < 0$ or $D_S(\pi_2) - D_S(\pi^*) < 0$, which is contrary to the optimality of π^*. The proof of the theorem is complete.

Corollary. (1) The job schedule in decreasing order with respect to $w_i\lambda_i$ $(i \in N)$ is optimal to minimize the expected total weighted earliness penalties about common due date d, i.e., $E_S(\pi) = E(\sum_{i=1}^{n} w_{[i]} \max\{d - C_{[i]}, 0\})$. (2) The job schedule in increasing order with respect to $w_i\lambda_i$ $(i \in N)$ is optimal to minimize the expected total weighted tardiness penalties about common due date d, i.e., $T_S(\pi) = E(\sum_{i=1}^{n} w_{[i]} \max\{C_{[i]} - d, 0\})$.

III. CONCLUSIONS

In this paper, a single machine scheduling problem with exponential processing times is discussed, the objective is to find an optimal schedule to minimize the expected total weighted deviations of completion times from a deterministic common due date. It is shown that the optimal schedules of the problem are Λ-shaped with respect to the products of the job weights and the processing time rates.

The Λ-shaped property of optimal schedules reduces the candidates of optimal schedules from $n!$ to 2^n. This property is similar to that of deterministic case where the processing times are deterministic [2]. Based on this property, we can follow the methods by Hall and Posnter [2] and Mittenthal et al. [13] to establish the optimal solution.

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