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Perturbation Analysis of Stochastic Flow Systems with Multiplicative Feedback

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***Abstract*— This paper uses Stochastic Flow Models (SFMs) for control and optimization (rather than performance analysis) of queueing systems with multiplicative feedback. Unlike earlier work based on additive feedback, the *multiplicative* feedback scheme considered here requires minimal state information and bypasses the problem of delayed state information. Using Infinitesimal Perturbation Analysis (IPA), we derive gradient estimators for loss and workload related performance metrics with respect to a *feedback gain* parameter, in contrast to previous work where threshold parameters were considered. The unbiasedness of these estimators is also established.**

***Index Terms*— Stochastic Flow Model, Discrete Event System, Hybrid System, Perturbation Analysis**

I. Introduction

Fluid models have been long adopted as a modeling technique of queueing theory, for applications such as communication networks and manufacturing systems. Introduced in [1] and later proposed in [2] for the analysis of multiplexed data streams and network performance [3], fluid models have been shown to be especially useful for simulating various kinds of high speed networks [4],[5],[6], as well as manufacturing systems [7]. Stochastic Flow Models (SFM) have the extra feature that the flow rates are treated as *stochastic* processes. Under this modelling technique, a new approach for sensitivity analysis has been recently proposed, based on Infinitesimal Perturbation Analysis (IPA) [8],[9],[10],[11]. The essence of this approach is the on-line estimation of gradients (sensitivities) of certain performance measures, such as average workloads and loss rates, as functions of various controllable parameters. These gradient estimates may be incorporated in standard stochastic approximation algorithms to optimize the parameter settings.

Queueing networks have been studied largely based on the assumption that system state, typically queue length information, has no effect on arrival or service processes. If these processes are regarded as input to the queueing system, their independence from system information implies the absence of feedback. Thus, we may ignore a potentially important feature of actual system design and operation. For example the Random Early Detection (RED) algorithm in TCP congestion control provides some form of feedback for network management. Unfortunately, the presence of feedback significantly complicates analysis. For instance, it is extremely difficult to derive closed-form expressions of performance metrics such as average queue length or mean waiting time, unless stringent assumptions are made [12],[13], let alone developing analytical schemes for performance optimization. It is equally difficult to extend the theory of PA for Discrete Event Systems (DES) in the presence of feedback.

Motivated by the importance of incorporating feedback to stochastic DES as well as their SFM counterparts, and the effectiveness of IPA methods applied to SFMs to date, we have been studying the problem of deriving IPA gradient estimators for SFMs with feedback mechanisms. In [14], an *additive* feedback mechanism was introduced by setting the inflow rate to $\sigma(t) - p(x(t))$ where $\sigma(t)$ is the maximal external incoming flow rate, $x(t)$ is the buffer content (state), and $p(x)$ is a feedback function. For the problem of determining a threshold that minimizes a weighted sum of loss volume and average workload, it was shown that IPA yields simple nonparametric sensitivity estimators for these performance metrics with respect to threshold parameters. Moreover, the estimators are unbiased under weak structural assumptions on the defining traffic process. However, this feedback mechanism implies that system information, i.e., buffer content, is instantaneously available to the controller (this is true in situations such as manufacturing systems, but unlikely to hold in high-speed distributed environments such as communication networks). This stringent requirement, together with a natural interest in feedback mechanisms which are readily applicable to real-world DES, leads to the present paper which tackles the problem of deriving IPA gradient estimators for SFMs with *multiplicative* feedback mechanisms. Consider a single-node SFM with threshold-based buffer control as in [10]. We define $\sigma(t)$ as the maximal external incoming flow rate and introduce a feedback mechanism by setting the inflow rate to $c \cdot \sigma(t)$ when the buffer content $x(t)$ is greater than a certain threshold ϕ. Compared with [14], the current mechanism has two main advantages: (i) system information is needed only when the buffer content reaches or leaves the threshold ϕ; while in [14] it has to be continuously available. As a result, communication costs are greatly reduced; (ii) the multiplicative feedback mechanism can be easily implemented in real-world DES, for example via probabilistic dropping.

The main contribution of the paper is the derivation of

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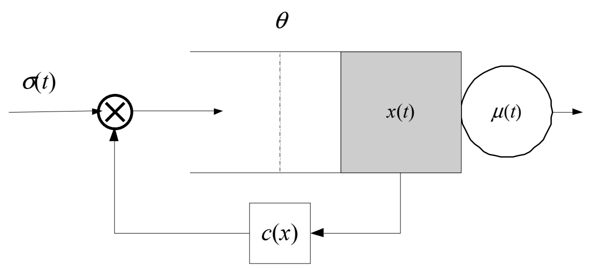

Fig. 1. A SFM with multiplicative feedback

IPA gradient estimators for performance metrics related to loss and workload levels with respect to the control parameter c. It is worth mentioning that in most papers applying IPA to SFM, i.e., [10], [15], buffer capacity parameters were of primary interest. To the best of our knowledge, the present paper is the first attempt to use IPA in the SFM context to obtain sensitivity information with respect to feedback control parameters. Even though the presence of feedback in the SFM considerably complicates the task of carrying out IPA, we are able to obtain such IPA estimators. Despite these complications, we are also able to prove that the estimators are unbiased under mild technical conditions, similar to those in [14]. Because of feedback, however, it is harder to apply these estimators using only data observable on a sample path of the actual DES. It is also worth reiterating that in the SFM we consider in this paper, as well as in [10] and [14], all flow rates are treated as random processes without distributional assumptions. This is different from the models adopted in [16] and [17]. Treating rates as random processes allows us to capture randomness in arrival and service processes as well as the time-varying behavior of the system.

The paper is organized as follows. First in Section 2, we present the feedback-based buffer control problem in the SFM setting and define the performance metrics and parameters of interest. In Section 3, we carry out IPA by first deriving sample derivatives of event times in our model and then obtaining the IPA estimators for the gradients of the expected loss rate and average workload with respect to feedback control parameters. Section 4 is devoted to proofs of unbiasedness. Finally in Section 5 we outline a number of open problems and future research directions.

II. SFM Setting

The SFM we consider consists of a server with a buffer fed by a source as shown in Fig. 1. The buffer content at time t is denoted by $x(t)$ and it is limited to θ, which may be viewed as a capacity or as a threshold parameter used for buffer control as described in [10]. Thus, $0 \leq x(t) \leq \theta$. When the buffer level reaches θ the underlying queueing system starts to drop customers. The maximal processing rate of the server is generally time-varying and denoted by $\mu(t)$. The maximal rate of the source at time t is denoted by $\sigma(t)$, but the actual incoming rate is $\sigma(t) \cdot c(x(t))$, where $c(x) \leq 1$ is a *feedback gain function* defined upon $x \in [0, \theta]$. In this paper, we shall concentrate on the following form of $c(x)$, and note that it is independent of $\sigma(t)$ or $\mu(t)$:

$$c(x) = \begin{cases} c & \text{if } \phi \leq x \leq \theta \\ 1 & \text{if } 0 \leq x < \phi \end{cases} \tag{1}$$

where $\phi < \theta$ is an intermediate threshold. We assume $0 < c \leq 1$, which ensures that the effect of feedback is more pronounced when $x > \phi$. This feedback mechanism implies that the supply source is instantaneously informed of the event that $x(t)$ reaches or leaves ϕ. It is also assumed that the stochastic processes $\{\sigma(t)\}$ and $\{\mu(t)\}$ are independent of the buffer level $x(t)$, c, ϕ or θ. Finally, we assume that the real-valued parameter c is confined to a closed and bounded (compact) interval $\mathcal{C}$ and that $c > 0$ for all $c \in \mathcal{C}$.

Given (1), we can see that the dynamics of the buffer content are given by

$$\frac{dx(t)}{dt^+} = \begin{cases} 0 & \text{when } x(t) = 0 \text{ and} \\ & \sigma(t) - \mu(t) \leq 0 \\ 0 & \text{when } x(t) = \theta \text{ and} \\ & c\sigma(t) - \mu(t) \geq 0 \\ 0 & \text{when } x(t) = \phi \text{ and} \\ & c\sigma(t) \leq \mu(t) \leq \sigma(t) \\ \sigma(t) - \mu(t) & \text{when } x(t) = \phi \text{ and} \\ & \sigma(t) < \mu(t) \\ c(x)\sigma(t) - \mu(t) & \text{otherwise} \end{cases} \tag{2}$$

with the initial condition $x(0; c) = x_0$ for some given x_0; for simplicity, we set $x_0 = 0$ throughout the paper. Note that the cases when $x(t) = \phi$ are included in the above equation to prevent "chattering" of the inflow rate between $c\sigma(t)$ and $\sigma(t)$ when $x(t) = \phi$. Such chattering behavior is due to the nature of the SFM and does not occur in the actual DES where buffer levels are maintained for finite periods of time (for details, see [18]).

Similar to [10], our purpose is to obtain sensitivity information of some performance metrics with respect to key parameters. In this paper, we limit ourselves to the feedback gain c as the controllable parameter of interest. For a finite time horizon $[0, T]$, we define the *Average Workload* as:

$$Q_T(c) = \frac{1}{T} \int_0^T x(t)dt \tag{3}$$

and the *Loss Rate* as:

$$L_T(c) = \frac{1}{T} \int_0^T \mathbf{1}[x(t) = \theta](c\sigma(t) - \mu(t))dt \tag{4}$$

where $\mathbf{1}[\cdot]$ is the usual indicator function. Accordingly, the main objective of the following sections is the derivation of $dQ_T(c)/dc$ and $dL_T(c)/dc$, which we will pursue through Infinitesimal Perturbation Analysis (IPA) techniques. For any sample performance metric $\mathcal{L}(c)$, the IPA gradient estimation technique computes $d\mathcal{L}(c)/dc$ along an observed sample path ω. If the IPA-based estimate $d\mathcal{L}(c)/dc$ satisfies $dE[\mathcal{L}(c)]/dc = E[d\mathcal{L}/dc]$, it is unbiased. Unbiasedness

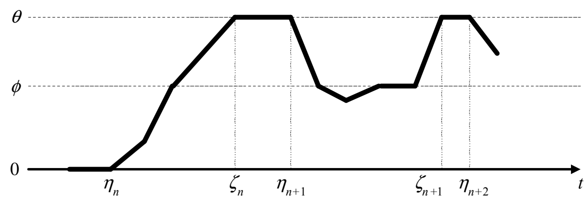

Fig. 2. A typical sample path

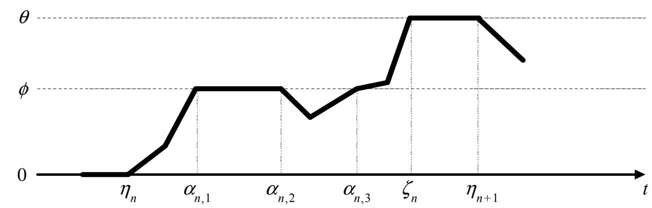

Fig. 3. A typical NBP

is the principal condition for making the application of IPA practical, since it enables the use of the IPA sample derivative in stochastic gradient-based algorithms. A comprehensive discussion of IPA and its applications can be found in [19], [20] and [21].

III. Infinitesimal Perturbation Analysis

A. Sample Path Decomposition and Event Definition

As already mentioned, our objective is to estimate the derivatives $dE[Q_T(c)]/dc$ and $dE[L_T(c)]/dc$ through the sample derivatives $dQ_T(c)/dc$ and $dL_T(c)/dc$, which are commonly referred to as IPA estimators. In the process, however, it will be necessary to identify events of interest and decompose the sample path.

For a fixed c, the interval $[0, T]$ is divided into alternating boundary periods and non-boundary periods. A *Boundary Period* (BP) is defined as the time interval during which $x(t) = \theta$ or $x(t) = 0$, and a *Non-Boundary Period* (NBP) is defined as the time interval during which $0 < x(t) < \theta$. BPs are further classified as *Empty Periods* (EP) and *Full Periods* (FP). An EP is the interval such that $x(t) = 0$; a FP is the interval such that $x(t) = \theta$. We assume that there are N NBPs in the interval $[0, T]$, where N is a random number, and index these NBPs by $n = 1, \ldots, N$. The starting and ending points of a NBP are denoted by η_n and ζ_n respectively. Fig. 2 shows a typical sample path of the SFM. We define the following random index set: $\Psi_F(c) = \{n : x(t) = \theta \text{ for all } t \in [\zeta_{n-1}, \eta_n), n = 1, \ldots, N\}$. Clearly, if $n \in \Psi_F$, the nth BP (which immediately precedes the nth NBP) is a FP; if $n \notin \Psi_F$, the nth BP (which immediately precedes the nth NBP) is an EP.

Next we will identify events of interest. To do so, we view the SFM as a DES in which we define the following types of events: (i) A jump in $\sigma(t)$ or $\mu(t)$, which is termed an *exogenous* event, reflecting the fact that its occurrence time is independent of the controllable parameter c, and (ii) The buffer content $x(t)$ reaches any one of the critical values $0, \phi$ or θ; this is termed an *endogenous* event, to reflect the fact that its occurrence time generally depends on c. Note that the combination of these events and the continuous dynamics in (2) gives rise to a stochastic hybrid system model of the underlying DES of Fig. 1.

Finally, we further decompose the sample path according to the events defined above. Let us consider a typical NBP $[\eta_n, \zeta_n)$ as shown in Fig. 3. Let $\alpha_{n,i}$ denote the ith time when $x(t)$ reaches or leaves ϕ in this NBP, where $i = 1, \ldots, I_n - 1$, in which $I_n - 1$ is the number of such events. It is possible that $I_n - 1 = 0$ for a NBP, so to maintain notational consistency we set $\eta_n = \alpha_{n,0}$ and $\zeta_n = \alpha_{n,I_n}$. We can now see that a sample path is decomposed into five sets of intervals that we shall refer to as *modes*: (i) Mode 0 is the set M_0 of all EPs contained in the sample path, (ii) Mode 1 is the set M_1 of intervals $[\alpha_{n,i}, \alpha_{n,i+1})$ such that $x(\alpha_{n,i}) = 0$ or ϕ and $0 < x(t) < \phi$ for all $t \in (\alpha_{n,i}, \alpha_{n,i+1})$, $n = 1, \ldots, N$, (iii) Mode 2 is the set M_2 of intervals $[\alpha_{n,i}, \alpha_{n,i+1})$ such that $x(t) = \phi$ for all $t \in [\alpha_{n,i}, \alpha_{n,i+1})$, $n = 1, \ldots, N$, (iv) Mode 3 is the set M_3 of intervals $[\alpha_{n,i}, \alpha_{n,i+1})$ such that $x(\alpha_{n,i}) = \phi$ or θ and $\phi < x(t) < \theta$ for all $t \in (\alpha_{n,i}, \alpha_{n,i+1})$, $n = 1, \ldots, N$ and (v) Mode 4 is the set M_4 of all FPs contained in the sample path. Note that the events occurring at times $\alpha_{n,i}$ are all endogenous for $i = 1, \ldots, I_n$ and we should express them as $\alpha_{n,i}(c)$ to stress this fact; for notational economy, however, we will only write $\alpha_{n,i}$. Finally, recall that for $i = 0$, we have $\alpha_{n,0} = \eta_n$, corresponding to an exogenous event starting the nth NBP. In the sequel, we will also denote the buffer content as $x(t; c)$ in order to specify its dependence on c.

B. Boundedness of Buffer Level Perturbations

In this section we establish an important boundedness property for *buffer level perturbations*

$$\Delta x(t; c) = x(t; c + \Delta c) - x(t; c)$$

with respect to a perturbation Δc. We assume that $0 < c + \Delta c \leq 1$ to ensure that it is consistent with the definition in (1). For simplicity, let us limit ourselves to a perturbation $\Delta c > 0$. The case where $\Delta c < 0$ can be similarly analyzed. We use SFM_N and SMF_P to denote the state trajectory of the nominal sample path and the perturbed sample path respectively and state the boundedness property of $\Delta x(t; c)$ in the following series of lemmas (proofs of all lemmas and theorems in the paper may be found in [18].):

Lemma 1: $\Delta x(t; c) \geq 0$, for all $t, 0 \leq t \leq T$.

Under the following:

Assumption 1: W.p.1, $\sigma(t) \leq \sigma_{\max} < \infty$, $\mu(t) \leq \mu_{\max} < \infty$

we are then able to prove the following result:

Lemma 2: Under Assumption 1, for all $t \in [0, T)$,

$$\Delta x(t; c) \leq K \Delta c \tag{5}$$

where $K = T \cdot \sigma_{\max}$.

C. Event Time Sample Derivatives

Our main objective is to derive IPA estimators for certain performance metrics, which will be presented in the next section. In this section, we present sample derivatives for the event times.

First we make the following additional assumptions:

Assumption 2: For every c, w.p.1, no two events (either exogenous or endogenous) occur at the same time.

This assumption precludes a situation where the queue content reaches one of the critical threshold values $0, \phi$ or θ at the same time α_i as an exogenous event which might cause it to leave the threshold; this would prevent the existence of event time sample derivative $\partial\alpha_i/\partial c$ which will be presented in the following derivation (however, one could still carry out perturbation analysis with one-sided derivatives as in [10]). Moreover, by Assumption 2, N, the number of NBPs in the sample path, is locally independent of c (since no two events may occur simultaneously, and the occurrence of exogenous events does not depend on c, there exists a neighborhood of c within which, w.p.1, the number of NBPs in $[0, T]$ is constant). Hence, the random index set Ψ_F is also locally independent of c. Similarly, the decomposition of the sample path into modes is also locally independent of c.

Assumption 3: $\sigma(t)$ and $\mu(t)$ are piecewise constant functions that can take a finite number of values.

Due to this assumption and recalling the dynamics of (2), $x(t; c)$ has to be a piecewise linear function of time t, as shown in Fig. 2.

Assumption 4: W.p.1, there exists an arbitrarily small positive constant ϵ such that for all t, $|\sigma(t) - \mu(t)| \geq \epsilon > 0$ and c satisfies

$$|c\sigma(t) - \mu(t)| \geq \epsilon > 0$$

Combining the above two assumptions, we obtain $\left|c\sigma_i - \mu_j\right| \geq \epsilon$ for every pair of possible values of $\sigma(t)$ and $\mu(t)$, which is equivalent to $c\sigma_i - \mu_j \geq \epsilon$ or $c\sigma_i - \mu_j \leq -\epsilon$. Therefore we obtain $c \geq \frac{\mu_j + \epsilon}{\sigma_i}$ or $c \leq \frac{\mu_j - \epsilon}{\sigma_i}$ which implies an "invalid interval" of $\left(\frac{\mu_j - \epsilon}{\sigma_i}, \frac{\mu_j + \epsilon}{\sigma_i}\right)$ for c. According to Assumption 3, there is a finite number of invalid intervals. We shall also refer to a *valid interval* as the maximal interval between two adjacent invalid intervals.

In what follows, we shall concentrate on a typical NBP $[\eta_n, \zeta_n(c))$ and drop the index n from the event times $\alpha_{n,i}$ in order to simplify notation.

In the rest of this section, we derive the sample derivative $\partial\alpha_i/\partial c$ through a series of lemmas which cover all possible values that $x(\alpha_i; c)$ can take in an interval $[\alpha_i, \alpha_{i+1})$.

Lemma 3: Under Assumptions $2-4$, if a FP ends at time η_n, i.e., $x(\eta_n; c) = \theta$, then $\partial\eta_n/\partial c = 0$.

Lemma 4: Under Assumptions $2-4$, if an EP ends at time η_n, i.e., $x(\eta_n; c) = 0$, then $\partial\eta_n/\partial c = 0$.

The above two lemmas show that an event time perturbation will be eliminated after a NBP ends.

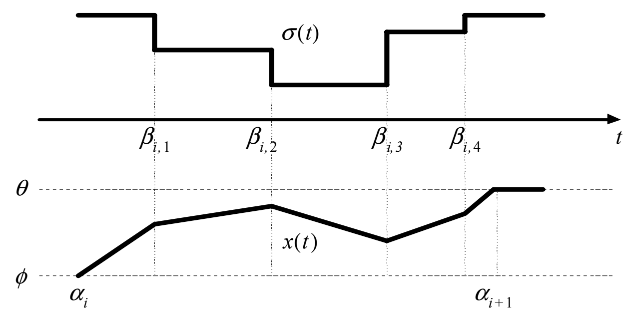

Fig. 4. The Decomposition of an M_3 Interval

Lemma 5: Under Assumptions $2-4$, if an M_2 interval ends at time α_i, i.e., $x(\alpha_i; c) = \phi$, then $\partial\alpha_i/\partial c = 0$.

We define the following shorthand notations: $A(t) = c\sigma(t) - \mu(t)$ and $B(t) = \sigma(t) - \mu(t)$.

Lemma 6: Under Assumptions $2-4$, if $i \in M_3$, then

$$\frac{\partial\alpha_{i+1}}{\partial c} = \frac{A(\alpha_i^+)}{A(\alpha_{i+1}^-)} \cdot \frac{\partial\alpha_i}{\partial c} - \frac{[x(\alpha_{i+1}; c) - x(\alpha_i; c)] + T(\alpha_i, \alpha_{i+1})}{cA(\alpha_{i+1}^-)} \quad (6)$$

where

$$T(\alpha_i, t) = \int_{\alpha_i}^{t} \mu(t)dt \quad (7)$$

is the server throughput during time interval $[\alpha_i, t)$ and $[x(\alpha_{i+1}; c) - x(\alpha_i; c)] \in \{\phi - \theta, 0, \theta - \phi\}$

According to Assumption 3, $\sigma(t)$ and $\mu(t)$ are piecewise constant functions. The interval $[\alpha_i, \alpha_{i+1})$ can be then decomposed by exogenous events occurring when $\sigma(t)$ jumps from one value to another. As shown in Fig. 4, we use $\beta_{i,k}, k = 1, \ldots S_i$ to denote the kth $\sigma(t)$ exogenous jump event and let $\alpha_i = \beta_{i,0}$ and $\alpha_{i+1} = \beta_{i,S_i+1}$ in order to maintain notational consistency. Moreover we define the value of $\sigma(t)$ in the interval $\left[\beta_{i,k}, \beta_{i,k+1}\right)$ as $\sigma_{i,k}$. It follows that

$$\int_{\alpha_i}^{\alpha_{i+1}} \sigma(t)dt = \sum_{k=0}^{S_i} \sigma_{i,k}\left(\beta_{i,k+1} - \beta_{i,k}\right)$$

If we use the shorthand $b_{i,k} = \beta_{i,k+1} - \beta_{i,k}$ for all i, k, to define the length of interval between two exogenous $\sigma(t)$ jump events, we get $\int_{\alpha_i}^{\alpha_{i+1}} \sigma(t)dt = \sum_{k=0}^{S_i} \sigma_{i,k} b_{i,k}$. Then, (6) becomes

$$\frac{\partial\alpha_{i+1}}{\partial c} = \frac{A(\alpha_i^+)}{A(\alpha_{i+1}^-)} \cdot \frac{\partial\alpha_i}{\partial c} - \frac{\sum_{k=0}^{S_i} \sigma_{i,k} b_{i,k}}{A(\alpha_{i+1}^-)} \quad (8)$$

Similar to the work in [14], our ultimate purpose is to apply the IPA estimators (which we will derive in the next section based on event time sample derivatives) to an actual underlying DES. The two expressions (6) and (8) provide alternative ways to evaluate the event time sample derivative which are equivalent in the SFM context. In the discrete-event setting, however, some of the information required by IPA estimation may be more difficult to obtain than other. For example, (8) depends on the evaluation of $\sigma_{i,k}$, the maximal incoming rate and $b_{i,k}$, the length of intervals between two $\sigma(t)$ jump events. This information may be difficult to acquire or measure if the supply source is remote. On the other hand, (6) requires a throughput evaluation during the time interval $[\alpha_i, \alpha_{i+1})$, which may be much easier to obtain, i.e., in an actual DES, it can be done by simply counting processed customers. In summary, we want to remind our readers that different forms of IPA estimators exist and that one should select the appropriate one based on implementation considerations.

Lemma 7: If $i \in M_1$,

$$\frac{\partial \alpha_{i+1}}{\partial c} = \frac{B(\alpha_i^+)}{B(\alpha_{i+1}^-)} \cdot \frac{\partial \alpha_i}{\partial c} \tag{9}$$

The combination of Lemmas 3-7 provides a linear recursive relationship for obtaining the event time sample derivative $\partial\alpha_i/\partial c$, and the coefficients involved are based on information directly available from a sample path of the SFM and the throughput calculators in (7). Moreover, $\frac{\partial \eta_n}{\partial c}$ and $\frac{\partial \zeta_n}{\partial c}$, the event time sample derivatives for the starting and ending time in a NBP $[\eta_n, \zeta_n)$, can also be derived from the above lemma combination. Recall that $\eta_n = \alpha_{n,0}$ and $\zeta_n = \alpha_{n,I_n}$. Since η_n, the start of the NBP, is the end of an EP or a FP, from Lemmas 4 and 3 we obtain $\frac{\partial \eta_n}{\partial c} = 0$. Using the previous lemmas, we can also obtain a recursive expression for $\frac{\partial \zeta_n}{\partial c}$ as follows:

Lemma 8: For a NBP $[\eta_n, \zeta_n)$,

$$\frac{\partial \zeta_n}{\partial c} = \begin{cases} \frac{B(\alpha_{n,I_n-1}^+)}{B(\alpha_{n,I_n}^-)} \cdot \frac{\partial \alpha_{n,I_n-1}}{\partial c} & \text{if } x(\zeta_n) = 0 \\ -\frac{T(\eta_n,\zeta_n)}{cA(\alpha_{n,I_n}^-)} & \text{if } x(\zeta_n) = \theta \text{ and } x(\alpha_{n,I_n-1}) = \theta \\ \frac{A(\alpha_{n,I_n-1}^+)}{A(\alpha_{n,I_n}^-)} \cdot \frac{\partial \alpha_{n,I_n-1}}{\partial c} - \frac{\theta-\phi+T(\alpha_{n,I_n-1},\alpha_{n,I_n})}{cA(\alpha_{n,I_n}^-)} & \text{if } x(\zeta_n) = \theta \text{ and } x(\alpha_{n,I_n-1}) = \phi \end{cases}$$

With the help of these lemmas, we are now able to derive IPA estimators for various performance metrics in the following section.

D. IPA Sample Derivative of Average Queue Length

Recalling the definition of the average workload Q_T in (3), and making use of the lemmas previously derived, we obtain the following IPA estimator.

Theorem 9: The IPA estimator of $dE[Q_T(c)]/dc$ is:

$$\frac{dQ_T}{dc} = \frac{1}{T} \sum_{i \in M_3} \left\{ \int_{\alpha_i}^{\alpha_{i+1}} \frac{x(t;c) - x(\alpha_i;c) + T(\alpha_i,t)}{c} dt - (c\sigma(\alpha_i^+) - \mu(\alpha_i^+))(\alpha_{i+1} - \alpha_i)\frac{\partial \alpha_i}{\partial c} \right\} \tag{10}$$

E. IPA Sample Derivative of Loss Rate

Recalling the definition of the loss rate L_T in (4), we have

$$\begin{aligned} L_T(c) &= \frac{1}{T}\int_0^T 1[x(t;c) = \theta]\,(c\sigma(t) - \mu(t))\,dt \\ &= \frac{1}{T}\sum_{n\in\Psi_F}\int_{\zeta_{n-1}}^{\eta_n} (c\sigma(t) - \mu(t))\,dt \end{aligned} \tag{11}$$

We then establish the following:

Theorem 10: The IPA estimator of $dE[L_T(c)]/dc$ is:

$$\frac{dL_T(c)}{dc} = \frac{1}{T}\sum_{n\in\Psi_F}\left\{\frac{1}{c}T(\eta_n, \zeta_{n-1}) - A(\zeta_{n-1})\frac{\partial \zeta_{n-1}}{\partial c}\right\} + \frac{L_T}{c} \tag{12}$$

Similar to our discussion on the IPA event time sample derivative, the interval $[\zeta_{n-1}, \eta_n)$ can be decomposed by $\sigma(t)$ exogenous jump events. Let us use $\beta_{i,k}, k = 1, \ldots S_i$ to denote the kth $\sigma(t)$ exogenous jump event and let $\zeta_{n-1} = \beta_{i,0}$ and $\eta_n = \beta_{i,S_i+1}$ for notational consistency. We also define the value of $\sigma(t)$ in the interval $[\beta_{i,k}, \beta_{i,k+1})$ as $\sigma_{i,k}$ and write $b_{i,k} = \beta_{i,k+1} - \beta_{i,k}$. It follows that

$$\int_{\zeta_{n-1}}^{\eta_n} \sigma(t)dt = \sum_{k=0}^{S_i} \sigma_{i,k}\left(\beta_{i,k+1} - \beta_{i,k}\right) = \sum_{k=0}^{S_i}\sigma_{i,k}b_{i,k}$$

from which we obtain:

$$\frac{dL_T(c)}{dc} = \frac{1}{T}\sum_{n\in\Psi_F}\left\{-\left[c\sigma(\zeta_{n-1}) - \mu(\zeta_{n-1})\right]\frac{\partial \zeta_{n-1}}{\partial c} + \sum_{k=0}^{S_i}\sigma_{i,k}b_{i,k}\right\} \tag{13}$$

It is also worth pointing out that although the IPA estimator expressions (10) and (12) seem complicated, their algorithmic implementation is quite simple.

IV. UNBIASEDNESS

In this section we establish the unbiasedness of the IPA estimators (10) and (12). Normally, the unbiasedness of an IPA derivative $d\mathcal{L}(\theta)/d\theta$ for some performance metric $\mathcal{L}(\theta)$ is ensured by the following two conditions (see [22], Lemma A2, p.70): (i) For every $\theta \in \tilde{\Theta}$, the sample derivative exists w.p.1, and (ii) W.p.1, the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout $\tilde{\Theta}$, and the (generally random) Lipschitz constant has a finite first moment. Consequently, establishing unbiasedness reduces to verifying the Lipschitz continuity of $\mathcal{L}(\theta)$ over $\tilde{\Theta}$. In the case of $L_T(c)$, however, the presence of invalid intervals in $\mathcal{C}$ creates a problem that we circumvent in what follows. In order to proceed, we shall need one additional mild technical condition:

Assumption 5: Let $W(c)$ be the total number of jumps of $\sigma(t)$ and $\mu(t)$ in the time interval $[0, T]$. Then, for any $c \in \mathcal{C}$, $E[W(c)] \le W_{\max} < \infty$.

Lemma 11: Under Assumptions 1-5, let c and $c + \Delta c$, $\Delta c > 0$, be in the same valid interval in $\mathcal{C}$. Then,

$$|\Delta L_T| \le r \cdot \Delta c$$

in which r is a random variable with a finite expectation.

Theorem 12: Assume $c \in \mathcal{C}$ is in a valid interval. Then, the IPA estimators (12) and (10) are unbiased, i.e.,

$$\frac{dE[L_T(c)]}{dc} = E\left[\frac{dL_T(c)}{dc}\right], \quad \frac{dE[Q_T(c)]}{dc} = E\left[\frac{dQ_T(c)}{dc}\right]$$

V. Conclusions and Future Directions

SFMs have recently been used to capture the dynamics of complex stochastic discrete event systems and to implement control and optimization methods based on gradient estimates of performance metrics obtained through IPA. In [14] we showed that IPA can be used in SFMs with additive feedback and in this paper we have further explored the effect of feedback by considering a single-node SFM with a controllable inflow rate as a multiplicative function of state (i.e., queue level) feedback parameterized by a feedback gain c and a threshold ϕ (capturing a quantization in the state feedback). We have developed IPA estimators for the loss volume and average workload with respect to the feedback gain parameter c and shown their unbiasedness, despite the complications brought about by the presence of feedback. The multiplicative feedback scheme bypasses the need for continuous state information seen in additive mechanisms and involves only knowledge of a single event representing the queue level crossing the threshold ϕ. Moreover, even if this state information is not instantaneously supplied, the delays involved are naturally built into the IPA estimator, based on which appropriate control parameters can be selected.

The work in this paper opens up a variety of possible extensions. First, looking at the feedback function (1), while c represents the intensity of feedback, ϕ represents its range. Instead of controlling c or ϕ separately (along the lines of previous work in [23]), it may be more effective to control the (c, ϕ) pair jointly. Next, noticing that probabilistic dropping/marking mechanisms are widely adopted in computer networks (e.g., in Random Early Detection or Random Early Marking), it is appealing to apply IPA specifically to these algorithms although the effect of feedback will be more complex. Finally, of obvious interest is the possibility of applying our SFM-based IPA estimators to an actual underlying DES, i.e., to determine the value of c that minimizes a weighted sum of loss volume and average workload, as we have done in [10] and [14]. As mentioned earlier, one advantage of IPA is that the estimators depend only on data directly observable along a sample path of the actual DES (not just the SFM which is an abstraction of the system); see, for example, [10] and [14]. In this paper, however, we have seen that this direct connection to the DES no longer holds because the estimators rely on the identification of "modes" whose definition has no direct correspondence to a DES. As a result, in order to successfully apply the SFM-based IPA estimators to an actual DES, we need to carefully select and interpret an appropriate abstraction of the underlying DES.

References

[1] D. Anick, D. Mitra, and M. Sondhi, "Stochastic theory of a data-handling system with multiple sources," *The Bell System Technical Journal*, vol. Vol. 61, pp. 1871–1894, 1982.

[2] H. Kobayashi and Q. Ren, "A mathematical theory for transient analysis of communications networks," *IEICE Transactions on Communications*, vol. E75-B, pp. 1266–1276, 1992.

[3] R. Cruz, "A calculus for network delay, Part I: Network elements in isolation," *IEEE Transactions on Information Theory*, 1991.

[4] G. Kesidis, A. Singh, D. Cheung, and W. Kwok, "Feasibility of fluid-driven simulation for ATM network," in *Proc. IEEE Globecom*, vol. 3, pp. 2013–2017, 1996.

[5] K. Kumaran and D. Mitra, "Performance and fluid simulations of a novel shared buffer management system," in *Proceedings of IEEE INFOCOM*, March 1998.

[6] B. Liu, Y. Guo, J. Kurose, D. Towsley, and W. Gong, "Fluid simulation of large scale networks: Issues and tradeoffs," in *Proceedings of the International Conference on Parallel and Distributed Processing Techniques and Applications*, June 1999. Las Vegas, Nevada.

[7] R. Suri and B. Fu, "On using continuous flow lines for performance estimation of discrete production lines," in *Proc. 1991 Winter Simulation Conference*, (Piscataway, NJ), pp. 968–977, 1991.

[8] Y. Wardi, B. Melamed, C. G. Cassandras, and C. G. Panayiotou, "IPA gradient estimators in single-node stochastic fluid models," *Journal of Optimization Theory and Applications*, vol. 115, no. 2, pp. 369–406, 2002.

[9] Y. Liu and W. Gong, "Perturbation analysis for stochastic fluid queueing systems," in *Proc. 38th IEEE Conf. Dec. and Ctrl*, pp. 4440–4445, 1999.

[10] C. G. Cassandras, Y. Wardi, B. Melamed, G. Sun, and C. G. Panayiotou, "Perturbation analysis for online control and optimization of stochastic fluid models," *IEEE Trans. Automatic Control*, vol. 47, August 2002.

[11] G. Sun, C. G. Cassandras, and C. G. Panayiotou, "Perturbation analysis of a multiclass stochastic fluid model with finite buffer capacity," *Proc. 41th IEEE Conf. On Decision and Control.*, 2002.

[12] L.Takacs, "A single-server queue with feedback," *Bell System Tech. J.*, vol. 42, pp. 505–519, 1963.

[13] M.A.Wortman, R.L.Disney, and P. Kiessler, "The M/GI/1 bernoulli feedback queue with vacations," *Queueing Systems Theory Appl.*, vol. 9, no. 4, pp. 353–363, 1991.

[14] H. Yu and C. G. Cassandras, "Perturbation analysis of feedback-controlled stochastic flow systems," in *Proceedings of 42th IEEE Conf. On Decision and Control*, pp. 6277–6282, 2003.

[15] C. G. Cassandras, G. Sun, C. G. Panayiotou, and Y. Wardi, "Perturbation analysis of multiclass stochastic fluid models," in *Proceedings of 15th IFAC World Congress*, pp. 848–853, 2002.

[16] R. Akella and P.R.Kumar, "Optimal control of production rate in a failure prone manufacturing system," *IEEE Trans. Automatic Control*, vol. AC-31, pp. 116–126, Feb. 1986.

[17] J. Perkins and R. Srikant, "Failure-prone production systems with uncertain demand," *IEEE Trans. Automatic Control*, vol. 46, pp. 441–449, 2001.

[18] H. Yu and C. G. Cassandras, "Perturbation analysis of stochastic flow systems with multiplicative feedback," Tech. Rep., Boston Univ. Dept. of Manuf. Eng., 2004.

[19] Y. Ho and X. Cao, *Perturbation Analysis of Discrete Event Dynamic Systems*. Boston, Massachusetts: Kluwer Academic Publishers, 1991.

[20] P. Glasserman, *Gradient Estimation via Perturbation Analysis*. Kluwer Academic Pub., 1991.

[21] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*. Kluwer Academic Publishers, 1999.

[22] R. Y. Rubinstein and A. Shapiro, *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*. New York, New York: John Wiley and Sons, 1993.

[23] H. Yu and C. G. Cassandras, "Perturbation analysis for production control and optimization of manufacturing systems," *Automatica*, 2004. To appear.