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The fault detection problem in nonlinear systems

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Abstract— **In this work high gain nonlinear observers are used as residual generators to study the fault detection problem in a differential algebraic framework. We analyze the stability of the residual generator when a fault occurs. We consider two faults types: constant and constant sign time-varying. It is shown that under some soft conditions over the aforementioned faults the residual is different from zero.**

I. Introduction

The high reliability required in industrial processes has created the necessity of detecting abnormal conditions while processes are operating. These conditions are called *faults* and it is important to detect and to isolate them in the early stages. A fault in a process is considered as a not allowable deviation which can be detected by an appropriated signal evaluation. State observers are suitable structures to evaluate this change. The difference between the measured outputs of the process and the observer is the so called *residual value* which is used to detect the fault. In this paper we consider the fault detection problem with a residual generators approach using high gain nonlinear observers in a differential algebraic framework. We study two types of faults: constant faults and time-varying faults with constant sign. The differential algebraic approach allows to define the concept of algebraic observability [3] and supplies state estimation through observers designed for systems described by differential algebraic equations [4], [12], [10], [13], [14], [15], [16], [17]. The paper is organized as follows. Section 2 presents the some differential algebra basic definitions. In Section 3, we present the observation problem, the residual generation problem and the residual generator stability using the *Uniform Ultimate Boundedness* (UUB) theorem [2]. Section 4 presents two fault cases. Finally, we close the paper with some concluding remarks.

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II. Basic Definitions

We introduce some basic definitions and notations, further details on the differential algebraic approach can be found in [3], [6], [11], [12], [10].

Definition 1: A *differential field extension* L/k is given by two differential fields k and L, such that: i) k is a subfield of L, ii) the derivation of k is the restriction to k of the derivation of L.

Definition 2: A dynamics is defined as a finitely generated differentially algebraic field extension $G/k\langle u\rangle$ ($G = k\langle u, \xi\rangle, \xi \in G$). Any element of G satisfies a differential algebraic equation with coefficients which are rational functions over k in the components of u and a finite number of their time derivatives, i.e., a system with input u and output y consists of a differential algebraic extension $k\langle u, y\rangle/k\langle u\rangle$ and the components $y_1, ..., y_n$ are differentially algebraic on $k\langle u\rangle$.

Definition 3: Consider the subset $\{u, y\}$. An element x in G is said to be algebraically observable with respect to $\{u, y\}$ if it is algebraic over $k\langle u, y\rangle$. This means that x can be expressed as an algebraic function of the components of $\{u, y\}$ and a finite number of their time derivatives. A dynamics $G/k\langle u\rangle$ with output y in G is algebraically observable if and only if, any state has this property.

Here the concept of algebraic observability means that the differential field extension $G/k\langle u\rangle$ is algebraic, i.e., the whole differential information is contained in $k\langle u, y\rangle$.

III. Observation problem, the fundamental problem of residual generation and stability

A. Observation problem

Consider the following nonlinear system in the so called *GOCF* (*generalized observability canonical* *form*) [6], [11], [12].

$$\begin{cases} \dot{\xi}_i = \xi_{i+1}, & 1 \le i \le n-1 \\ \dot{\xi}_n = -L_0\left(\xi, u, ..., u^{(\nu)}\right) \\ y = \xi_1 \end{cases} \tag{1}$$

where L_0 is a C^1 real−valued function, $\xi = Col\,(\xi_1, ..., \xi_n) \in R^n$, $u \in R^m$, $y(t) \in R$, and some integer $\nu \ge 0$.

Remark 1: In general, a nonlinear system

$$\begin{cases} \dot{x}(t) = g(x, u) \\ y(t) = h(x, u) \end{cases} \tag{2}$$

where $x \in R^n$, $u \in R^m$, $y \in R$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are polynomial functions of their arguments, may be transformed to the GOCF described by (1) as a consequence of the differential primitive element for nonlinear systems [3], [6], [11], [12], [18].

System (1) may be written in compact form as

$$\begin{cases} \dot{\xi} = A\xi + \varphi\,(\xi, \bar{u}) \\ y = C\xi \end{cases} \tag{3}$$

with $C = \begin{pmatrix} 1 & 0 & ... & 0 \end{pmatrix}$, $\bar{u} = \left(u, ..., u^{(\nu)}\right)$, the elements of A are given by

$$A_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi\,(\xi, \bar{u}) = Col \begin{pmatrix} 0 & ... & 0 & -L_0(\xi, \bar{u}) \end{pmatrix} \tag{4}$$

is continuously differentiable. Hence, an estimate $\hat{\xi}$ of ξ can be given by an exponential nonlinear observer ($\mathcal{O}$) of the form

$$(\mathcal{O}): \quad \dot{\hat{\xi}} = A\hat{\xi} + \varphi\left(\hat{\xi}, \bar{u}\right) - S_\theta^{-1} C^T C \left(\hat{\xi} - \xi\right) \tag{5}$$

where

$$\varphi\left(\hat{\xi}, \bar{u}\right) = Col \begin{pmatrix} 0 & ... & 0 & -L_0(\hat{\xi}, \bar{u}) \end{pmatrix} \tag{6}$$

and S_θ is the positive definite solution of [7]

$$S_\theta \left(A + \frac{\theta}{2} I\right) + \left(A^T + \frac{\theta}{2} I\right) S_\theta = C^T C \tag{7}$$

for some $\theta > 0$. The coefficients of $(S_\theta)_{ij}$ are given by

$$(S_\theta)_{ij} = \frac{\alpha_{ij}}{\theta^{i+j-1}} \tag{8}$$

where (α_{ij}) are the entries of a symmetric positive-definite matrix which does not depend on θ, and $C = (1, 0, ..., 0)$.

Now, from (3) and (5), the estimation error dynamic $\epsilon = \xi - \hat{\xi}$ is given by:

$$\dot{\epsilon} = (A - S_\theta^{-1} C^T C)\epsilon + \Phi(\epsilon, \bar{u}) \tag{9}$$

where

$$\Phi(\epsilon, \bar{u}) = \varphi\left(\hat{\xi} + \epsilon, \bar{u}\right) - \varphi\left(\hat{\xi}, \bar{u}\right) \tag{10}$$

Now, first of all, we introduce the following notations and definitions. Denote $\|x\|_{S_\theta} = (x^T S_\theta x)^{1/2}$, being S_θ the solution to equation (7). Then, if $\Phi(\epsilon, \bar{u})$ is differentiable, we get $\|\Phi(\epsilon, \bar{u})\|_{S_\theta} \le \gamma \|\epsilon\|_{S_\theta}$, for some $\gamma > 0$. In what follows, we present a result which shows some characteristics and structural properties of matrix $(A - S_\theta^{-1} C^T C)$.

Lemma 1: $(A_\theta - S_\theta^{-1} C^T C)$ is a Hurwitz matrix. Furthermore, the characteristic polynomial of A_θ is $P(\lambda) = (\lambda + \theta)^n$.

Proof: (Sketch) Matrix $(A_\theta)_{2\times 2}$ is given by

$$A_\theta = \begin{bmatrix} -2\theta & 1 \\ -\theta^2 & 0 \end{bmatrix} \tag{11}$$

and its eigenvalues are given by: $\lambda_1 = \lambda_2 = -\theta$.
For the case $(A_\theta)_{3\times 3}$ we have

$$A_\theta = \begin{bmatrix} -3\theta & 1 & 0 \\ -3\theta^2 & 0 & 1 \\ -\theta^3 & 0 & 0 \end{bmatrix} \tag{12}$$

with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -\theta$.
Finally, by induction, the matrix A_θ for the case $n \times n$ is given by

$$A_\theta = \begin{bmatrix} -n\theta & 1 & 0 & \cdots & 0 \\ -\frac{n(n-1)}{2!}\theta^2 & 0 & 1 & \cdots & 0 \\ -\frac{n(n-1)(n-2)}{3!}\theta^3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{n(n-1)\cdots(n-(n-(r-1)))!}{(n-(r-1))!(r-1)!}\theta^{r-1} & 0 & 0 & \cdots & 0 \\ -\frac{n(n-1)\cdots(n-(n-r)!}{(n-r)!r!}\theta^{r} & 0 & 0 & \cdots & 0 \\ -\frac{n(n-1)\cdots(n-(n-(r+1)))!}{(n-(r+1))!(r+1)!}\theta^{r+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{n(n-1)}{2!}\theta^{n-2} & 0 & 0 & \cdots & 0 \\ -n\theta^{n-1} & 0 & 0 & \cdots & 1 \\ -\theta^n & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{13}$$

which has the following eigenvalues:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_r = \cdots \lambda_{n-1} = \lambda_n = -\theta,$$

the above means, A_θ is Hurwitz matrix and characteristic polynomial is given by
$P(\lambda) = \det(\lambda I - A_\theta) = (\lambda + \theta)^n$ ■

B. Fundamental problem of residual generation

The fault detection scheme is composed of a residual generator and a *fault mode rule*. The residual generator is a filter whose inputs correspond to the inputs and outputs of the plant. And its outputs called *residual values* are used for fault detection purposes.

Now, we consider a nonlinear system from (2) with an additional fault

$$\begin{cases} \dot{x}(t) = g(x,u) + v(t) \\ y(t) = h(x,u) \end{cases} \tag{14}$$

where $x(t) \in R^n$ is the state vector, $u(t) = (u_1(t), ..., u_m(t))^T \in \mathcal{U} \subset R^m$ is the input vector, $v(t) \in R$, is a scalar function representing a system fault. For the sake of simplicity we consider that only one fault occurs at a given time, $y(t) \in R$ is the measured output vector. $g(x,u)$, and $h(x,u)$ are functions of class C^∞. $u(t)$ and $y(t)$ are the system input and output signals, respectively, and we suppose that they are known. Function $v(t)$ is unknown and arbitrary and belongs to a compact set.

In this work, we define a *good input* to a system input such that the coordinate transformation is not singular and it can carry out the system (14) into the GOCF (3). Here, a residual generator can be defined [8], [9] as a nonlinear dynamic system given by

$$\begin{cases} \dot{z}(t) = G(z,y,u) \\ r(t) = H(y,z) \end{cases} \tag{15}$$

where $z(t) \in R^{\bar{n}}$ is the state vector, $r(t) \in R^{\bar{p}}$ is the output vector, u and y are the inputs system and corresponding to input and output vectors of (14). A residual generator must satisfy the following conditions

C1) If $v(t) = 0$ for each initial condition $(x(0), z(0))$ of the extended system (14)-(15) and for all admissible good input u, $\underset{t\to\infty}{lím}\ r = 0$. Then, in absence of fault, r asymptotically converge to zero.

C2) If $v(t) \neq 0$ for $t \geq t_o$, then $r(t) \neq 0$ for $t \geq t_o$.

If the above conditions are satisfied, we then say that r is a residual value.

C. Residual generator stability

In this part, we consider the stability analysis of the residual generator for the system with fault 14 using the Uniform Ultimate Stability (UUB) Theorem, given in [2].

Lemma 2: Suppose that there exists a differential primitive element such that it is possible to transform system (14) into the GOCF given by

$$\begin{array}{l} \dot{\xi} = A\xi + \varphi(\xi, \bar{u}) + W(t) \\ y(t) = C\xi \end{array} \tag{16}$$

where the term $W(t) = Col\begin{pmatrix} 0 & ... & 0 & w(t) \end{pmatrix}$ is the fault in the transformed system. Furthermore, the following system:

$$(\mathcal{O}') : \begin{cases} \dot{\hat{\xi}} = A\hat{\xi} + \varphi\left(\hat{\xi}, \bar{u}\right) + S_\theta^{-1}C^T\left(y - \hat{y}\right) \\ \hat{y} = C\hat{\xi} \\ r = y - \hat{y} \end{cases} \tag{17}$$

is an observer for (16), with S_θ the gain matrix satisfying (7) and r is the residual value. Then, the estimation error dynamic given by

$$\begin{array}{l} \dot{\epsilon} = A_\theta\epsilon + \Phi(\epsilon, \bar{u}) + W(t) \\ r = C\epsilon \end{array} \tag{18}$$

is Uniform Ultimate Bounded, where $\Phi(\epsilon, \bar{u})$ is given in (10), $A_\theta = (A - S_\theta^{-1}C^TC)$, and ϵ belongs to the compact set $B_b = \{\epsilon \mid \|\epsilon\| \leq b, b > 0\}$.

Proof: **(**Sketch**)** Consider the following Lyapunov function candidate for system (18):

$$V(t) = \epsilon^T S_\theta \epsilon > 0$$

Taking the time derivative we have

$$\begin{aligned} \dot{V}(t) &= \dot{\epsilon}^T S_\theta \epsilon + \epsilon^T S_\theta \dot{\epsilon} \\ &= -\theta\epsilon^T S_\theta\epsilon - \epsilon^T C^T C\epsilon + 2\epsilon^T S_\theta \Phi(\epsilon, \bar{u}) \\ &\quad +2\epsilon^T S_\theta W \end{aligned}$$

For the sake of simplicity we have drop the argument t in $W(t)$. Since

$$A^T S_\theta + S_\theta A - C^T C = -\theta S_\theta \tag{19}$$

and using the fact $\epsilon^T C^T C\epsilon > 0$, $\dot{V}(t)$ is upper bounded as

$$\dot{V}(t) \leq -\theta\epsilon^T S_\theta\epsilon + 2\epsilon^T S_\theta\Phi(\hat{\xi}, \epsilon) + 2\epsilon^T S_\theta W \tag{20}$$

analizing each right hand side term of inequality, using the Cholesky decomposition for a symmetric positive-definite matrix, the Cauchy-Schwarz inequality, and assuming that the fault is bounded, i.e. $\|W\|_{S_\theta} \leq \Gamma$, where $\Gamma > 0$, it readily follows that the time derivative $\dot{V}(t)$ remains bounded, that is to say

$$\dot{V}(t) \leq -\theta \|\epsilon\|_{S_\theta}^2 + 2\gamma \|\epsilon\|_{S_\theta}^2 + 2\Gamma \|\epsilon\|_{S_\theta} \quad (21)$$

Here, it is clear, that if the fault is zero i.e., $W = 0$, this leads us to obtain the particular case of exponential convergence of the high gain nonlinear observer,

$$\dot{V}(t) \leq -\theta \|\epsilon\|_{S_\theta}^2 + 2\gamma \|\epsilon\|_{S_\theta}^2 = -(\theta - 2\gamma)V(t)$$

i.e.

$$\|\epsilon\|_{S_\theta} \leq \|\epsilon(0)\|_{S_\theta} e^{-\left(\frac{\theta}{2}-\gamma\right)t} \quad (22)$$

with $\theta > 2\gamma$. Now, if $W \neq 0$, consider inequality (21), then,

$$\dot{V}(t) \leq -(\theta - 2\gamma) \|\epsilon\|_{S_\theta}^2 + 2\Gamma \|\epsilon\|_{S_\theta}$$

Using the Rayleigh-Ritz inequality

$$\lambda_{\min}(S_\theta) \|\epsilon\|^2 \leq \|\epsilon\|_{S_\theta}^2 \leq \lambda_{\max}(S_\theta) \|\epsilon\|^2$$

we conclude that $\dot{V}(t)$ satisfies

$$\dot{V}(t) \leq -(\theta - 2\gamma)\lambda_{\min}(S_\theta) \|\epsilon\|^2 + 2\Gamma\sqrt{\lambda_{\max}(S_\theta)} \|\epsilon\|$$

By applying the UUB Theorem [2], it directly follows that $\epsilon(t)$ is bounded uniformly for any initial state $\epsilon(0)$, and $\epsilon(t)$ remains in a compact set

$$B_b = \{\epsilon \,|\, \|\epsilon\| \leq b, b > 0\}$$

where

$$b = \sqrt{\frac{\lambda_{\max}(S_\theta)}{\lambda_{\min}(S_\theta)}} \left(\frac{2\Gamma\sqrt{\lambda_{\max}(S_\theta)}}{(\theta - 2\gamma)\lambda_{\min}(S_\theta)} \right) > 0$$

■

In the next section we will study two fault cases: constant fault and constant sign time-varying fault.

IV. Study Cases of Faults

A. Constant fault case

We consider system (14) in the GOCF (16). From Lemma 2, an observer for this system is given by (17) with estimation error dynamics (18). First, we tackle the constant fault case problem with a lemma related with the existence and uniqueness of solutions of the error dynamics (18).

Lemma 3: Consider the estimation error dynamics (18)

$$\begin{aligned} \dot{\epsilon} &= A_\theta \epsilon + \Phi(\epsilon, \bar{u}) + W(t) \\ r &= C\epsilon \end{aligned} \quad (23)$$

when the fault $W(t)$ is constant, i.e. $W(t) = W_c = Col\begin{pmatrix} 0 & 0 & \dots & w_c \end{pmatrix}$, $|w_c| > 0$. Then, there exists a unique constant solution $\epsilon_s \; \forall \; t \in [0, \infty)$.

Proof: The proof is split into two parts.
a) Existence. Since $\Phi(\epsilon, \bar{u})$ is differentiable, then, there exists a solution for (23) $\forall \; t \in [0, \infty)$ [1].
b) Uniqueness. First, let ϵ_s be a solution of the algebraic equation

$$A_\theta \epsilon_s + \Phi(\epsilon_s, \bar{u}) + W_c = 0 \quad (24)$$

with

$$\Phi(\epsilon_s, \bar{u}) = Col\begin{pmatrix} 0 & \dots & 0 & -\bar{L}_0(\epsilon_s, \bar{u}) \end{pmatrix}$$

and defining $z = \epsilon - \epsilon_s$, we have

$$\dot{z} = A_\theta z + A_\theta \epsilon_s + \Phi(z + \epsilon_s, \bar{u}) + W_c$$

From (24), we replace $A_\theta \epsilon_s$

$$\dot{z} = A_\theta z + \{\Phi(z + \epsilon_s, \bar{u}) - \Phi(\epsilon_s, \bar{u})\} \quad (25)$$

Consider the Lyapunov function candidate $V = z^T S_\theta z$, then,

$$\begin{aligned} \dot{V} &= \dot{z}^T S_\theta z + z^T S_\theta \dot{z} \\ &= \left[z^T A_\theta^T + \{\Phi(z + \epsilon_s, \bar{u}) - \Phi(\epsilon_s, \bar{u})\} \right] S_\theta z \\ &\quad + z^T S_\theta \left[A_\theta z + \{\Phi(z + \epsilon_s, \bar{u}) - \Phi(\epsilon_s, \bar{u})\} \right] \end{aligned}$$

now, substituting $A_\theta = (A - S_\theta^{-1} C^T C)$ and after (19), and since $z^T C^T C z > 0$, $z^T S_\theta z = \|z\|_{S_\theta}^2$, as well as $\Phi(\epsilon, \bar{u})$ is differentiable, then we have

$$\dot{V} \leq -\theta \|z\|_{S_\theta}^2 + 2\gamma \|z\|_{S_\theta}^2 = -(\theta - 2\gamma) \|z\|_{S_\theta}^2$$

From which it follows that z converges exponentially to zero for $\theta > 2\gamma$, then, ϵ converges to ϵ_s. ■

Using results from Lemma 1 we establish the following theorem.

Theorem 4: Consider equation (24)

$$A_\theta \epsilon_s + \Phi(\epsilon_s, \bar{u}) + W_c = 0 \quad (26)$$

with solution ϵ_s, $\|W_c\| > 0$, and $\Phi(\epsilon, \bar{u})$ is differentiable, i.e.

$$\|\Phi(\epsilon, \bar{u})\| \leq \gamma \|\epsilon\| \quad (27)$$

then, the residual value $r_s = C\epsilon_s$ satisfies the following inequality

$$|w_c| \leq (\theta^n + \gamma\sqrt{H(\theta)}) \, |r_s| \quad (28)$$

where $\sqrt{H(\theta)} \, |r_s| = \|\epsilon_s\|$, with $H(\theta)$ a positive function for $\theta > 0$.

Proof: From (26) and (13) we have obtain

$$\begin{aligned} \epsilon_{s_2} &= \epsilon_{s_1}(n\theta) \\ \epsilon_{s_3} &= \epsilon_{s_1}\left(\tfrac{n(n-1)}{2!}\theta^2\right) \\ &\vdots \\ \epsilon_{s_{n-(n-r)-1}} &= \epsilon_{s_1}\left(\tfrac{n(n-1)(n-2)\cdots(n-(n-r))!}{(n-r)!r!}\theta^{n-(n-r)}\right) \\ &\vdots \\ \epsilon_{s_{n-1}} &= \epsilon_{s_1}\left(\tfrac{n(n-1)}{2!}\theta^{n-2}\right) \\ \epsilon_{s_n} &= \epsilon_{s_1}\left(n\theta^{n-1}\right) \\ w_c - \bar{L}_0(\epsilon_s,\bar{u}) &= \epsilon_{s_1}\theta^n \end{aligned} \tag{29}$$

where $\epsilon_s = \begin{bmatrix} \epsilon_{s_1} & \epsilon_{s_2} & \cdots & \epsilon_{s_{n-1}} & \epsilon_{s_n} \end{bmatrix}^T$. Then, the norm of ϵ_s is written compactly as

$$\|\epsilon_s\|^2 = H(\theta)\epsilon_{s_1}^2$$

or

$$\|\epsilon_s\| = \sqrt{H(\theta)}\,|r_s| \tag{30}$$

where $r_s = C\epsilon_s = \epsilon_{s_1}$ and

$$\begin{aligned} H(\theta) \;=\; & [1 + (n\theta)^2 + \left(\frac{n(n-1)}{2!}\theta^2\right)^2 + \cdots \\ & + \left(\frac{n(n-1)(n-2)\cdots(n-(n-r))!}{(n-r)!r!}\right. \\ & \left.\theta^{n-(n-r)}\right)^2 + \cdots + \left(\frac{n(n-1)}{2!}\theta^{n-2}\right)^2 \\ & + \left(n\theta^{n-1}\right)^2 \;] \end{aligned}$$

Note that $H(\theta) > 0$ for $\theta > 0$. Now, from (29), we have

$$w_c - \bar{L}_0(\epsilon_s,\bar{u}) = \epsilon_{s_1}\theta^n$$

Note that $\|\bar{L}_0(\epsilon_s,\bar{u})\| = \|\Phi(\epsilon_s,\bar{u})\| \le \gamma\|\epsilon_s\|$. Then

$$|w_c| \le \theta^n\,|\epsilon_{s1}| + \gamma\,\|\epsilon_s\|$$

and using (30) we finally obtain

$$|w_c| \le (\theta^n + \gamma\sqrt{H(\theta)})\,|r_s|$$

■

B. Constant sign time-varying fault case

In this case, the fault term is considered as $W(t) = W_c + F(t)$ where $W_c = Col\begin{pmatrix} 0 & 0 & \ldots & w_c \end{pmatrix}$ and $F(t) = Col\begin{pmatrix} 0 & 0 & \ldots & f(t) \end{pmatrix}$. We assume that $W(t)$ satisfies the following properties:

P1) $\|W_c\| > 0$.

P2) $0 < \|F(t)\| \le F_{\max} < |w_c|$

Theorem 5: Consider the estimation error dynamics (18) with $W(t) = W_c + F(t)$,

$$\dot{\epsilon} = A_\theta\epsilon + \Phi(\epsilon,\bar{u}) + W_c + F(t) \tag{31}$$

Then, r is strictly greater that zero if the following inequality is satisfied

$$\frac{|w_c|}{\theta^n + \gamma\sqrt{H(\theta)}} - \frac{\lambda_{\max}(S_\theta)}{[\lambda_{\min}(S_\theta)]^{3/2}}\left(\frac{2F_{\max}}{(\theta-2\gamma)}\right) > 0 \tag{32}$$

Proof: Let us define $z = \epsilon - \epsilon_s$, then, from (31) and (26) we have

$$\dot{z} = A_\theta z + \Phi(z,\bar{u}) + F(t) \tag{33}$$

where we have used the fact that

$$\begin{aligned} \Phi(\epsilon,\bar{u}) - \Phi(\epsilon_s,\bar{u}) &= \Phi(z+\epsilon_s,\bar{u}) - \Phi(\epsilon_s,\bar{u}) \\ &= \varphi(\hat{\xi}+z+\epsilon_s,\bar{u}) - \varphi(\hat{\xi},\bar{u}) \\ &\quad -\varphi(\hat{\xi}+\epsilon_s,\bar{u}) + \varphi(\hat{\xi},\bar{u}) \\ &= \varphi(z+\hat{\xi}+\epsilon_s,\bar{u}) \\ &\quad -\varphi(\hat{\xi}+\epsilon_s,\bar{u}) \\ &= \Phi(z,\bar{u}) \end{aligned}$$

The above dynamics is analyzed using the following Lyapunov function candidate $V = z^T S_\theta z$, then

$$\begin{aligned} \dot{V} &= \dot{z}^T S_\theta z + z^T S_\theta \dot{z} \\ &= \left[z^T A_\theta^T + \Phi^T(z,\bar{u}) + F^T(t)\right] S_\theta z \\ &\quad + z^T S_\theta\left[A_\theta z + \Phi(z,\bar{u}) + F(t)\right] \end{aligned}$$

Using $A_\theta = (A - S_\theta^{-1}C^TC)$ and $\theta S_\theta + A^TS_\theta + S_\theta A = C^TC$, and since $\Phi(z,\bar{u})$ is differentiable $\left\|\Phi^T(z,\bar{u})S_\theta z\right\| \le \gamma\|z\|_{S_\theta}^2$ and $\|F^T(t)S_\theta z\| \le F_{\max}\|z\|_{S_\theta}$, then, $\dot{V}$ is upper bounded as follows

$$\begin{aligned} \dot{V} &\le -\theta\|z\|_{S_\theta}^2 + 2\gamma\|z\|_{S_\theta}^2 + 2F_{\max}\|z\|_{S_\theta} \\ &\le -(\theta-2\gamma)\|z\|_{S_\theta}^2 + 2F_{\max}\|z\|_{S_\theta} \end{aligned}$$

with $\lambda_{\min}(S_\theta)\|z\|^2 \le \|z\|_{S_\theta}^2 \le \lambda_{\max}(S_\theta)\|z\|^2$ we obtain

$$\dot{V} \le -(\theta-2\gamma)\lambda_{\min}(S_\theta)\|z\|^2 + 2F_{\max}\sqrt{\lambda_{\max}(S_\theta)}\|z\| \tag{34}$$

By applying the UUB Theorem [2] we have that z is bounded uniformly and converges to the compact set $B_R = \{z \;/\; \|z\| \le R\}$ where

$$R = \frac{\lambda_{\max}(S_\theta)}{[\lambda_{\min}(S_\theta)]^{3/2}}\left(\frac{2F_{\max}}{(\theta-2\gamma)}\right) \tag{35}$$

then

$$\|z\| \leq R = \frac{\lambda_{\max}(S_\theta)}{[\lambda_{\min}(S_\theta)]^{3/2}} \left(\frac{2F_{\max}}{(\theta - 2\gamma)} \right)$$

Now, since $z = \epsilon - \epsilon_s$, then, $\|\epsilon - \epsilon_s\| \leq R$ from which we have $|\epsilon_{s_1}| - |\epsilon_1| \leq |\epsilon_1 - \epsilon_{s_1}| \leq R$, i.e.

$$|r_s| - R \leq |\epsilon_1| \tag{36}$$

where we have using the fact that $r = \epsilon_1$ and $r_s = \epsilon_{s1}$. From (28) it follows that

$$|r_s| \geq \frac{|w_c|}{\theta^n + \gamma\sqrt{H(\theta)}} \tag{37}$$

Then, replacing (35) and (37) into (36), we get

$$|r_s| - R > \left[\frac{|w_c|}{\theta^n + \gamma\sqrt{H(\theta)}} - \frac{\lambda_{\max}(S_\theta)}{[\lambda_{\min}(S_\theta)]^{3/2}} \left(\frac{2F_{\max}}{(\theta - 2\gamma)} \right) \right]$$

Finally, if inequality (32) is satisfied we conclude that $|r| \geq |r_s| - R > 0$. ■

From the above result it is clear that the constant part W_c of the fault must dominate over the time-varying part $F(t)$ for a nonzero residual r.

V. Concluding Remarks

In this paper we studied the fault detection problem using a residual generator based upon a high gain nonlinear observer obtained using a differential algebraic approach. Stability of the residual generator under a fault was studied. We have also considered two faults types: constant and constant sign time-varying. It was shown that under some mild conditions over the aforementioned faults the residual is different from zero. For the sake of space we left the incipient fault case outside of this work.

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