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On the Well Posedness of Singularly Perturbed Fault Detection Filters

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***Abstract*— In this paper, we study the well posedness of observer-based fault detection filters using the theory of singular perturbation. By proper scaling of the fault maps, it is shown that the ill-conditioning of the residual fault projector can be alleviated. This allows the construction of an approximate projector in terms of the projectors for the slow and fast subsystems.**

I. Problem Formulation

Consider an LTI singularly perturbed system

$$\left.\begin{array}{rcl} \dot{x}_1(t) & = & A_{11}x_1(t) + A_{12}x_2(t) + F_{11}\mu_1(t) \\ & & +\sum_{j=2}^{q} F_{j1}\mu_j(t), \\ \epsilon\dot{x}_2(t) & = & A_{21}x_1(t) + A_{22}x_2(t) + F_{12}\mu_1(t) \\ & & +\sum_{j=2}^{q} F_{j2}\mu_j(t), \\ y(t) & = & C_1x_1(t) + C_2x_2(t) + v(t), \end{array}\right\} \quad (1)$$

where $x = [x_1^T \; x_2^T]^T \in \mathcal{R}^n$ is the state vector, $y \in \mathcal{R}^p$ is the output, $\mu_1 \in \mathcal{R}^{p_1}$ is the target fault, $\mu_j \in \mathcal{R}^{p_j}$ are the nuisance faults for $j = 2, \ldots, q$, and $0 < \epsilon \ll 1$ is the perturbation parameter. The partitioned states are $x_1 \in \mathcal{R}^{n_1}$ and $x_2 \in \mathcal{R}^{n_2}$. All matrices are assumed to have appropriate dimensions. If a matrix is a function of ϵ, it is assumed to be analytic at $\epsilon = 0$. However, this dependence is not shown for simplicity of notation. The standard assumption that $A_{22}(0)$ is nonsingular is also made. The failure maps F_{ij} are modeled to represent the fault associated with the plant, the actuators, or the sensors as discussed in [3].

By defining, for $i = 1, 2$,

$$\tilde{F}_{2i} = \left[\begin{array}{ccc} F_{2i} & \ldots & F_{qi} \end{array}\right], \; \tilde{\mu}_2(t) = \left[\begin{array}{c} \mu_2(t) \\ \vdots \\ \mu_q(t) \end{array}\right], \quad (2)$$

the entire set of nuisance faults can be represented as a single map $\tilde{\mu}_2$, and the system can be written in the compact form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + F_1\mu_1(t) + \tilde{F}_2\tilde{\mu}_2(t), \\ y(t) & = & Cx(t) + v(t), \end{array}\right\} \quad (3)$$

where

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21}/\epsilon & A_{22}/\epsilon \end{array}\right], \quad F_1 = \left[\begin{array}{c} F_{11} \\ F_{12}/\epsilon \end{array}\right],$$
$$\tilde{F}_2 = \left[\begin{array}{c} \tilde{F}_{21} \\ \tilde{F}_{22}/\epsilon \end{array}\right], \qquad C = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right].$$

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The pair (A, C) is assumed to be observable. It is further assumed that F_1 and each $\tilde{F}_j$, $j = 2, \ldots, q$ is monic, i.e., $F_1\mu_1 \neq 0$ and $\tilde{F}_j\tilde{\mu}_j \neq 0$ for $j = 2, \ldots, q$. Under these conditions, a method to detect the occurences of the failure μ_1, in the presence of the nuisance fault $\tilde{\mu}_2$, is through the construction of an appropriate detection filter

$$\left.\begin{array}{rcl} \dot{\hat{x}}(t) & = & A\hat{x}(t) + F_1\mu_1(t) + \tilde{F}_2\tilde{\mu}_2(t) + Lr(t), \\ r(t) & = & y(t) - C\hat{x}(t), \end{array}\right\} \quad (4)$$

which keeps the reachable subspace of μ_1 and $\tilde{\mu}_2$ in separate and non-intersecting invariant subspaces. A residual projector

$$z(t) = Hr(t) \quad (5)$$

can subsequently constructed subject to the following two constraints:

1) The filter residual $r(t)$ should be projected onto the orthogonal complement of the invariant subspace containing $\tilde{\mu}_2(t)$. This ensures that, for any arbitrary $\tilde{\mu}_2(t)$, the signal $z(t) = 0$ whenever $\mu_1(t) = 0$.
2) To detect the presence of $\mu_1(t)$, it is further desired to have $z(t) \neq 0$ whenever $\mu_1(t) \neq 0$. This requires the transfer function from $\mu_1(s)$ to $z(s)$ to be left-invertible. This assumption can be relaxed by requiring that the mapping from μ_1 to z is input observable.

Our intention, in this paper, is to study the fault detection problem associated with system (1) in terms of two lower order subsystems, namely, the slow subsystem

$$\left.\begin{array}{rl} \dot{x}_S(t) = & A_Sx_S(t) + F_{1S}\mu_1(t) + \tilde{F}_{2S}\tilde{\mu}_2(t), \\ y(t) = & C_Sx_S(t) + E_{1S}\mu_1(t) + \tilde{E}_{2S}\tilde{\mu}_2(t) + v(t), \end{array}\right\} \quad (6)$$

where $A_S = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $F_{1S} = F_{11} - A_{12}A_{22}^{-1}F_{12}$, $\tilde{F}_{2S} = \tilde{F}_{21} - A_{12}A_{22}^{-1}\tilde{F}_{22}$, $C_S = C_1 - C_2A_{22}^{-1}A_{21}$, $E_{1S} = -C_2A_{22}^{-1}F_{12}$, $\tilde{E}_{2S} = -C_2A_{22}^{-1}\tilde{F}_{22}$, and the fast subsystem

$$\left.\begin{array}{rcl} \dot{x}_F(t) & = & A_Fx_F(t) + F_{1F}\mu_1(t) + \tilde{F}_{2F}\tilde{\mu}_2(t), \\ y_F(t) & = & C_Fx_F(t) + v(t), \end{array}\right\} \quad (7)$$

where $A_F = A_{22}$, $F_{1F} = F_{12}$, $\tilde{F}_{2F} = \tilde{F}_{22}$, $C_F = C_2$.

II. Decomposition of the Detection Filter

The principle tool used for the purpose of mode separation is the Chang transformation

$$\left[\begin{array}{c} \zeta(t) \\ \eta(t) \end{array}\right] = \underbrace{\left[\begin{array}{cc} I_{n_1} - \epsilon\Omega(\epsilon)\Gamma(\epsilon) & -\epsilon\Omega(\epsilon) \\ \Gamma(\epsilon) & I_{n_2} \end{array}\right]}_{T(\epsilon)} \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right], \quad (8)$$

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where $T(\epsilon)$ has the inverse

$$T(\epsilon)^{-1} = \begin{bmatrix} I_{n_1} & \epsilon\Omega(\epsilon) \\ -\Gamma(\epsilon) & I_{n_2} - \epsilon\Gamma(\epsilon)\Omega(\epsilon) \end{bmatrix}, \qquad (9)$$

and $\Gamma(\epsilon)$ and $\Omega(\epsilon)$ are analytic at $\epsilon = 0$ with $\Gamma(\epsilon) = A_{22}(0)^{-1}A_{21}(0) + O(\epsilon)$ and $\Xi(\epsilon) = A_{12}(0)A_{22}(0)^{-1} + O(\epsilon)$. Application of this transformation brings system (1) and filter (4) in new coordinate systems with the variables decoupled. Comparison of (6) and (7) with the decoupled system reveals that, under the assumption of stability and by proper choice of initial conditions, the slow and fast subsystems can be obtained from the original system by ignoring the $O(\epsilon)$ terms. The same idea extends to the transformed filter by defining the slow filter as

$$\left.\begin{aligned} \dot{\hat{x}}_S(t) = {} & A_S\hat{x}(t) + F_{1S}\mu_1(t) + \tilde{F}_{2S}\tilde{\mu}_2(t) \\ & + L_S r_S(t), \\ r_S(t) = {} & y(t) - C_S\hat{x}_S(t), \end{aligned}\right\} \qquad (10)$$

and the fast filter as

$$\left.\begin{aligned} \epsilon\dot{\hat{x}}_F(t) = {} & A_F\hat{x}_F(t) + F_{1F}\mu_1(t) + \tilde{F}_{2F}\tilde{\mu}_2(t) \\ & + L_F r(t), \\ r(t) = {} & y_F(t) - C_F\hat{x}_F(t). \end{aligned}\right\} \qquad (11)$$

However, for these filters to be an $O(\epsilon)$ approximation of the original filter in the decoupled form, the fast filter needs to be implemented by taking $y_F(t) = y(t) - C_S\hat{x}_S(t)$, and, as an input to the slow filter, $\hat{x}_F(t)$ should be approximated as $\hat{x}_F(t) = -A_F^{-1}L_F r(t)$; details can be found in [5]. With these choices, the residual for the slow filter, namely $r_S(t)$, can then be obtained from $r(t)$ as $r_S(t) = (I - C_F A_F^{-1} L_F) r(t)$. Consequently, one can represent the slow filter as

$$\dot{\hat{x}}_S(t) = A_S\hat{x}(t) + F_{1S}\mu_1(t) + \tilde{F}_{2S}\tilde{\mu}_2(t) + \tilde{L}_S r(t) \qquad (12)$$

which is now driven by the residual $r(t)$. Thus, under the assumption of stability and by proper choice of initial conditions, the slow and fast filters in (10) and (11) can represent an $O(\epsilon)$ approximation to the original filter by defining

$$L_1 = L_S + (A_{12} - L_S C_F)A_F^{-1}L_F, \qquad (13)$$

$$L_2 = L_F. \qquad (14)$$

Note that the observabilities of (A_S, C_S) and (A_F, C_F) imply the observability of (A, C) for sufficiently small ϵ.

III. Approximation of the Residual Projector

The approximation of the residual projector requires an amplitude scaling of the failure maps as

$$\left.\begin{aligned} F_1 &= \begin{bmatrix} F_{11} \\ F_{12} \end{bmatrix} = \begin{bmatrix} \tilde{f}_{11} & \dots & \tilde{f}_{1p_1} \\ \epsilon^{\theta_1}\tilde{f}_{21} & \dots & \epsilon^{\theta_{p_1}}\tilde{f}_{2p_1} \end{bmatrix}, \\ \tilde{F}_2 &= \begin{bmatrix} \tilde{F}_{21} \\ \tilde{F}_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & \dots & f_{1p_2} \\ \epsilon^{\nu_1} f_{21} & \dots & \epsilon^{\nu_{p_2}} f_{2p_2} \end{bmatrix}, \end{aligned}\right\} \qquad (15)$$

where the ϵ-dependent scalings with non-negative integers θ_j, $j = 1, \dots, p_1$ and ν_i, $i = 1, \dots, p_2$ are included in the definition of the maps for the well-posedness of the problem. Note that the j-th columns of F_1 and the i-th column of $\tilde{F}_2$ can be represented, respectively, as

$$\tilde{f}_j = \begin{bmatrix} \tilde{f}_{1j} \\ \epsilon^{\theta_j}\tilde{f}_{2j} \end{bmatrix}, \ f_i = \begin{bmatrix} f_{1i} \\ \epsilon^{\nu_i} f_{2i} \end{bmatrix}. \qquad (16)$$

The residual fault projector for (1) is given by

$$H = I - C\hat{F}\left[(C\hat{F})^T C\hat{F}\right]^{-1}(C\hat{F})^T \qquad (17)$$

where $\hat{F} = \left[A^{\beta_1} f_1 \ \dots \ A^{\beta_{p_2}} f_{p_2}\right]$. On the other hand, the residual fault projectors for the slow subsystem (6) and the fast subsystem (7) are, respectively, given by

$$\left.\begin{aligned} H_S &= I - C_S\hat{F}_S\left[(C_S\hat{F}_S)^T C_S\hat{F}_S\right]^{-1}(C_S\hat{F}_S)^T, \\ H_F &= I - C_F\hat{F}_F\left[(C_F\hat{F}_F)^T C_F\hat{F}_F\right]^{-1}(C_F\hat{F}_F)^T, \end{aligned}\right\} \qquad (18)$$

where $\hat{F}_S = \left[A_S^{\beta_1} f_{1S} \ \dots \ A_S^{\beta_{p_2}} f_{p_2 S}\right]$ is formed from the columns f_{iS} of F_{2S}, and $\hat{F}_F = \left[A_F^{\beta_1} f_{1F} \ \dots \ A_F^{\beta_{p_2}} f_{p_2 F}\right]$ is formed from the columns f_{iF} of F_{2F} for $i = 1, \dots, p_2$. The index β_i is assumed to be the smallest integer such that either $C_S A_S^{\beta_i} f_{iS} \neq 0$ or $C_F A_F^{\beta_i} f_{iF} \neq 0$. The following theorem gives sufficient conditions under which an $O(\epsilon)$ approximation of the projector H can be constructed based on the knowledge of the projectors H_S and H_F of the slow and fast subsystems.

Theorem Let δ_j be the smallest integer such that either $C_S A_S^{\delta_j}\tilde{f}_{jS} \neq 0$ or $C_F A_F^{\delta_j}\tilde{f}_{jF} \neq 0$ for $j = 1, \dots, p_1$. Further, assume that $\theta_j = \delta_j + 1$ for $j = 1, \dots, p_1$ and $\nu_i = \beta_i + 1$ for $i = 1, \dots, p_2$. Then for sufficiently small ϵ,

1) System (1) is output separable, i.e.,

$$\text{rank}\left[CA^{\delta_1}\tilde{f}_1 \ \dots \ CA^{\delta_{p_1}}\tilde{f}_{p_1} \ CA^{\beta_1} f_1 \ \dots \ CA^{\beta_{p_2}} f_{p_2}\right] = p_1 + p_2,$$

2) $H = H_S + H_F + O(\epsilon)$.

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