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Flatness-based Control of a Flexible Beam in a Gravitational Field

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***Abstract*— This paper considers a flatness-based boundary control of a hub-beam system with tip payload moving in a vertical plane in the presence of gravity. The homogeneous flexible beam is modelled using an Euler-Bernoulli hypothesis which assumes no shear deformation or rotatory inertia. As well, small transverse deformations are assumed. A linearized system model involving a coupled PDE-ODE is derived and a change of coordinates is introduced to simplify this model. The method of control relies on a flatness property of the system; namely, that the system solution can be differentially parameterized in terms of a flat output. This parameterization allows for straightforward motion planning and computation of a control law. The approach is based on power series in the spatial variable, and the convergence of these series is ensured by choosing the flat output to be a nonanalytic, smooth function of appropriate Gevrey class.**

I. INTRODUCTION

Many important control problems involve motion planning. For example, steering a car along a prescribed trajectory or controlling a chemical reactor between operating points. Methods for solving motion control problems often rely on nonlinear models because of significant change in system behaviour for large motions. When a finite-dimensional nonlinear system is differentially flat [1], straightforward means for motion planning exist. In addition, feedback tracking controllers can be readily designed. The flatness property means that trajectories of all system variables (e.g. inputs or states) can be computed from functions of a finite number of time derivatives of a so-called flat output trajectory. This relationship between flat output and system variables, which involves no integration and only function evaluations, means motion planning objectives such as actuator saturation can be accounted for. Having designed a flat output, an expression for an open-loop steering control follows readily. Applications of this flatness-based method can be found in [1], [2], [3] and the references therein.

Recent work on flatness has dealt with the generalization of motion planning to boundary controlled infinite-dimensional distributed parameter systems [4], [3]. For infinite-dimensional systems, the type of PDE determines how the solution is parameterized by the flat output. For example, the flatness-based approach was first applied to distributed parameter systems on a flexible shaft problem which can be modelled using a hyperbolic wave equation. The angular position of the non-drive-end of the shaft acts as a flat output in that the torsion field can be expressed as constant amplitude delays and predictions of this output trajectory and its time derivatives [5], [6]. Other hyperbolic equations modelling heat exchangers, telegraph lines, and heavy chains lead to "distributed delay" operators requiring the integration of the flat output over a finite time interval [7], [8], [9]. Parabolic or biharmonic equations lead to infinite series parameterizations [2], [10], [11], [12], [13], [14]. Work on the biharmonic case includes the Euler-Bernoulli beam moving in a horizontal plane [10], [15] or a cantilever piezoelectric beam [16], [17]. Other work considers the generalization to Rayleigh and Timoshenko beams [3], [18].

There is a vast body of work on the control of flexible beams which does not use a flatness-based approach. These other methods are often based on linear ODE models obtained by spatial discretization (e.g. modal decomposition or finite element analysis) and linearization [19], [20]. Other work treats the PDE-ODE system directly [21], [15], [10], [22]. The approach taken here is in the same spirit as the work in [10], [15] as a nondiscretized model and its solutions are used directly to determine the control law. The main contribution of this paper is to generalize the results in [10], [15] to include the effects of a gravity. A more complete account of the some of the calculations given in this paper can be found in [23].

II. SYSTEM MODELLING

Consider a single-link flexible beam, shown in Fig. 1, which is clamped to a motor and constrained to move in a vertical plane in a gravitational field. A motor with fixed translational position exerts a torque T on the beam and the inertia of the motor-hub assembly is denoted J_h. Let the undeformed length of the beam be L, its constant mass density be ρ, its constant cross-sectional area be S, its constant cross-section area moment of inertia be I, and its Young modulus be E. A payload of mass m and inertia J_p is attached to the non-drive-end of the beam. We assume the Euler-Bernoulli hypothesis is valid: that plane cross sections which are normal to the beam axis before deformation remain plane after deformation and normal to the deformed axis (no shear deformation) and rotatory inertia effects are negligible. We introduce two coordinate systems shown in Fig. 1 and which are related by a rotation of angle θ. The inertial or fixed frame is denoted by XOY and the floating frame is denoted by xOy. The floating-frame is defined such that its x-axis is aligned with the rigid motion of the beam (i.e., the shadow beam). Let $v(x,t)$ be the transverse deformation of any mass point located at $(x,0)$ in xOy. At

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$x = 0$ we have the boundary conditions $v(0,t) = 0$ and $v_x(0,t) = 0$. In order to derive the dynamic model of the beam we make use of the Generalized Hamilton's Principle [24] which states

$$\int_{t_1}^{t_2} \delta L\,dt + \int_{t_1}^{t_2} T\delta\theta\,dt = 0 \tag{1}$$

where $L = K - V$ is the Lagrangian, K the system kinetic energy, V the system potential energy, δ is the variational operator, and t_1, t_2 are any times such that $t_1 \le t_2$. In order to derive expressions for K and V, we introduce a vector r which points to an element of mass on the beam. Represented in XOY, the vector r pointing to an element of mass located at $(x, v(x,t))$ in xOy is $r(x,\theta,v) = \begin{bmatrix} xc_\theta - vs_\theta & xs_\theta + vc_\theta \end{bmatrix}^T$ where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$. The kinetic energy of the system is

$$K = \frac{\rho S}{2}\int_0^L (x^2\dot\theta^2 + 2x\dot v\dot\theta + \dot v^2)\,dx + \frac{m}{2}(L^2\dot\theta^2 + 2L\dot v_L\dot\theta + \dot v_L^2) + \frac{J_h\dot\theta^2}{2} + \frac{J_p}{2}(\dot\theta + \dot v_{Lx})^2$$

where $v(L,t) = v_L(t), v_x(L,t) = v_{Lx}(t)$, and in the last term we have taken the angle of the payload relative to XOY as $\theta + v_{Lx}$. This is because for small v the angle of the payload relative to xOy is $\beta \approx \tan\beta = v_{Lx}$ (see Fig. 1 for a definition of β). Potential energy is due to the energy

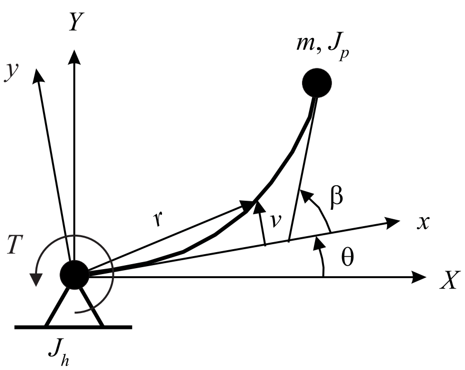

Fig. 1. Flexible beam system

stored internally in the beam and that due to gravity [25]:

$$V = \frac{EI}{2}\int_0^L (v_{xx}^2 + \rho Sg(xs_\theta + vc_\theta))dx + mg(Ls_\theta + v_Lc_\theta)$$

In order to use (1) the variation of K is first computed. This result is integrated from t_1 to t_2 and integration by parts is performed. Using $\delta\theta(t_1) = \delta\theta(t_2) = 0$, $\delta v(x,t_1) = \delta v(x,t_2) = 0$, $x \in [0,L]$, and $\delta v_x(L,t_1) = \delta v_x(L,t_2) = 0$ we have for $\int_{t_1}^{t_2}\delta K\,dt$

$$\int_{t_1}^{t_2}\Big[-\rho S\int_0^L \Big[(x\ddot\theta + \ddot v)\delta v + (x^2\ddot\theta + x\ddot v)\delta\theta\Big]\,dx - m[L^2\ddot\theta + L\ddot v_L]\delta\theta - m(L\ddot\theta + \ddot v_L)\delta v_L - [J_p(\ddot\theta + \ddot v_{Lx}) + J_h\ddot\theta]\delta\theta - J_p(\ddot\theta + \ddot v_{Lx})\delta v_{Lx}\Big]dt \tag{2}$$

The variation of the potential energy is

$$\delta V = EI(v_{Lxx}\delta v_{Lx} - v_{Lxxx}\delta v_L + \int_0^L v_{xxxx}\delta v\,dx) + \rho Sg\int_0^L (xc_\theta\delta\theta + c_\theta\delta v - vs_\theta\delta\theta)\,dx + mg(Lc_\theta\delta\theta + c_\theta\delta v_L - v_Ls_\theta\delta\theta) \tag{3}$$

Substituting (2) and (3) into (1) and equating coefficients of $\delta\theta$, δv, δv_L, and δv_{Lx} to zero gives

$$(\int_0^L \rho Sx^2\,dx + mL^2 + J_p + J_h)\ddot\theta + \int_0^L \rho Sx\ddot v\,dx + J_p\ddot v_{Lx} + \int_0^L \rho Sg(xc_\theta - vs_\theta)\,dx + mg(Lc_\theta - v_Ls_\theta) + mL\ddot v_L = T \tag{4a}$$

$$\rho S(\ddot v + x\ddot\theta + gc_\theta) + EIv_{xxxx} = 0 \tag{4b}$$

$$EIv_{Lxx} = -J_p(\ddot v_{Lx} + \ddot\theta) \tag{4c}$$

$$EIv_{Lxxx} = m(\ddot v_L + L\ddot\theta + gc_\theta) \tag{4d}$$

Substituting (4b) into (4a) and applying boundary conditions (4c) and (4d) gives

$$T = J_h\ddot\theta - EIv_{xx}(0,t) - \int_0^L \rho Sgv\,dxs_\theta - mgv_Ls_\theta$$

The system dynamics can be therefore described by the coupled PDE-ODE system:

$$J_h\ddot\theta = T + EIv_{xx}(0,t) + (\int_0^L \rho Sv\,dx + mv_L)gs_\theta \tag{5a}$$

$$\rho S(\ddot v + x\ddot\theta + gc_\theta) + EIv_{xxxx} = 0 \tag{5b}$$

$$v(0,t) = 0 \tag{5c}$$

$$v_x(0,t) = 0 \tag{5d}$$

$$EIv_{Lxx} = -J_p[\ddot v_{Lx} + \ddot\theta] \tag{5e}$$

$$EIv_{Lxxx} = m[\ddot v_L + L\ddot\theta + gc_\theta] \tag{5f}$$

Linearizing this model about an equilibrium trajectory v_0 and $\theta_0 =$ const. we have

$$J_h\Delta\ddot\theta = \Delta T + \int_0^L Sg\Delta v dxs_{\theta_0} + \int_0^L Sgv_0dxc_{\theta_0}\Delta\theta + mg\Delta v_Ls_{\theta_0} + mgv_{L0}c_{\theta_0}\Delta\theta + EI\Delta v_{xx}(0,t) \tag{6a}$$

$$\rho S(\Delta\ddot v + x\Delta\ddot\theta - gs_{\theta_0}\Delta\theta) + EI\Delta v_{xxxx} = 0 \tag{6b}$$

$$\Delta v(0,t) = 0 \tag{6c}$$

$$\Delta v_x(0,t) = 0 \tag{6d}$$

$$EI\Delta v_{Lxx} = -J_p[\Delta\ddot v_{Lx} + \Delta\ddot\theta] \tag{6e}$$

$$EI\Delta v_{Lxxx} = m[\Delta\ddot v_L + L\Delta\ddot\theta - gs_{\theta_0}\Delta\theta] \tag{6f}$$

where $\Delta\theta = \theta - \theta_0, \Delta v = v - v_0$.

III. FLATNESS-BASED CONTROL

We introduce the new dependent variable

$$w(r,\tau) = \Delta v(x,t) + x\Delta\theta - \int_0^t\int_0^{\xi_1} gs_{\theta_0}\Delta\theta(\xi)\,d\xi\,d\xi_1 \tag{7}$$

and new independent variables $x = Lr$, and $t = \alpha\tau$ where $\alpha = \sqrt{\frac{\rho S}{EI}}L^2$. In order to transform (6) into these new coordinates we first compute

$$\begin{aligned} w_\tau(r,\tau) &= (\Delta\dot{v} + Lr\Delta\dot{\theta} - \int_0^{\alpha\tau} g s_{\theta_0}\Delta\theta(\xi)\,d\xi)\alpha \\ w_{\tau\tau}(r,\tau) &= (\Delta\ddot{v} + Lr\Delta\ddot{\theta} - g s_{\theta_0}\Delta\theta)\alpha^2 \\ w_r(r,\tau) &= (\Delta v_x + \Delta\theta)L, \quad w_{rr}(r,\tau) = \Delta v_{xx}L^2 \\ w_{rrr}(x,t) &= \Delta v_{xxx}L^3, \quad w_{rrrr}(x,t) = \Delta v_{xxxx}L^4 \\ w_{\tau\tau r}(r,\tau) &= (\Delta\ddot{v}_x + \Delta\ddot{\theta})\alpha^2 L \end{aligned}$$

Using the above relations the system (6) transforms to

$$w_{\tau\tau}(r,\tau) + w_{rrrr}(r,\tau) = 0 \tag{8a}$$

$$w(0,\tau) = -\frac{g s_{\theta_0}}{L}\int_0^{\alpha\tau}\int_0^{\xi_1}\Delta u\,d\xi\,d\xi_1 \tag{8b}$$

$$w_r(0,\tau) = L\Delta\theta = \Delta u \tag{8c}$$

$$w_{rr}(1,\tau) = -\frac{J_p}{\rho S L^3}w_{\tau\tau r}(1,\tau) = -\lambda w_{\tau\tau r}(1,\tau) \tag{8d}$$

$$w_{rrr}(1,\tau) = \frac{m}{\rho S L}w_{\tau\tau}(1,\tau) = \mu w_{\tau\tau}(1,\tau) \tag{8e}$$

where $\mu = m/(\rho S L), \lambda = J_p/(\rho S L^3), \Delta u = L\Delta\theta = u - u_0$ and $u_0 = L\theta_0$. We now replace derivatives w.r.t. t by the operator s, in the sense of Mikusiński's operational calculus [26]:

$$s^2\hat{w}(r) + \hat{w}_{rrrr}(r) = 0 \tag{9a}$$

$$\hat{w}(0) = -\frac{\alpha^2 g s_{\theta_0}\Delta\hat{u}}{Ls^2} \tag{9b}$$

$$\hat{w}_r(0) = \Delta\hat{u} \tag{9c}$$

$$\hat{w}_{rr}(1) = -\lambda s^2\hat{w}_r(1) \tag{9d}$$

$$\hat{w}_{rrr}(1) = \mu s^2\hat{w}(1) \tag{9e}$$

where $\hat{w}$ denotes the operational function associated with w. The general solution of (9a) can be expressed as

$$\hat{w}(r) = aC_r^+ + bC_r^- + cS_r^+ + dS_r^- \tag{10}$$

where

$$C_r^+ = \frac{C_r + \overline{C}_r}{2} = \frac{\partial S_r^-}{\partial r}, \qquad C_r^- = \frac{C_r - \overline{C}_r}{2i} = \frac{\partial S_r^+}{\partial r}$$
$$S_r^+ = \frac{iS_r + \overline{S}_r}{2h\sqrt{s}} = -\frac{1}{s}\frac{\partial C_r^+}{\partial r}, \quad S_r^- = \frac{iS_r - \overline{S}_r}{2\bar{h}\sqrt{s}} = \frac{1}{s}\frac{\partial C_r^-}{\partial r}$$

$i = \sqrt{-1}$, $h = e^{i\pi/4}$, $\bar{h} = e^{-i\pi/4}$, $C_r = \cosh[h\sqrt{s}(1-r)]$, $S_r = \sinh[h\sqrt{s}(1-r)]$, $\bar{S}_r = \sinh[\bar{h}\sqrt{s}(1-r)]$, and $\bar{C}_r = \cosh[\bar{h}\sqrt{s}(1-r)]$. The operators $\sqrt{s}, i\sqrt{s}$ are logarithmic [26]. Coefficients a, b, c, d are determined from the boundary relations:

$$\begin{aligned} -\frac{\alpha^2 g s_{\theta_0}}{Ls^2}\Delta\hat{u} &= aC_0^+ + bC_0^- + cS_0^+ + dS_0^-, \quad b = -\lambda s d \\ \Delta\hat{u} &= s(-aS_0^+ + bS_0^-) + cC_0^- + dC_0^+, \qquad c = \mu s a \end{aligned}$$

Solving this linear system gives

$$\begin{aligned} a &= \left(\pi_0 - \frac{(C_0^+ - \lambda s^2 S_0^-)\alpha^2 g s_{\theta_0}}{Ls^2}\right)\frac{\Delta\hat{u}}{Q} \\ b &= -\left(\lambda s\omega_0 + \frac{\lambda s(\mu s C_0^- - sS_0^+)\alpha^2 g s_{\theta_0}}{Ls^2}\right)\frac{\Delta\hat{u}}{Q} \\ c &= \left(\mu s\pi_0 - \frac{\mu s(C_0^+ - \lambda s^2 S_0^-)\alpha^2 g s_{\theta_0}}{Ls^2}\right)\frac{\Delta\hat{u}}{Q} \\ d &= \left(\omega_0 + \frac{(\mu s C_0^- - sS_0^+)\alpha^2 g s_{\theta_0}}{Ls^2}\right)\frac{\Delta\hat{u}}{Q} \end{aligned}$$

where $Q = \omega_0(C_0^+ - \lambda s^2 S_0^-) + \pi_0 s(\mu C_0^- - S_0^+)$ and $\omega_r = C_r^+ + \mu s S_r^+, \pi_r = \lambda s C_r^- - S_r^-$. Using (10) and the expressions for a, b, c, and d, we have $\hat{w}(r) = (P(r)/Q)\Delta\hat{u}$ where

$$\begin{aligned} P(r) = \pi_0\omega_r - \omega_0\pi_r - \frac{\alpha^2 g s_{\theta_0}}{Ls^2}[&\omega_r(C_0^+ - \lambda s^2 S_0^-) \\ &+ \pi_r s(\mu C_0^- - S_0^+)] \end{aligned}$$

We can introduce a Q–basic or flat output $\hat{y}$, such that

$$\hat{w}(r) = P(r)\hat{y}, \quad \Delta\hat{u} = Q\hat{y}$$

Splitting $\hat{w}$ and $P(r)$ into two parts gives

$$\begin{aligned} \hat{w}(r) &= \hat{w}_1(r) + \hat{w}_2(r) = P_1(r)\hat{y} + P_2(r)\hat{y} \\ P_1(r) &= \pi_0\omega_r - \omega_0\pi_r \\ P_2(r) &= \frac{\alpha^2 g s_{\theta_0}}{Ls^2}[(\lambda s^2 S_0^- - C_0^+)\omega_r + (S_0^+ - \mu C_0^-)s\pi_r] \end{aligned} \tag{11}$$

According to [3], $P_1(r)$ can be expressed as

$$\begin{aligned} P_1(r) = &\frac{\lambda\mu s^2 - 1}{2}S^-(r) + \frac{\lambda\mu s^2 + 1}{2}\Big(\Re[S^-(1+i-r)] \\ &- \Im[S^-(1+i-r)]\Big) - \lambda s\Re[C^-(1+i-r)] \\ &+ \mu\Im[C^+(1+i-r)] \end{aligned}$$

where $\Re, \Im$ denote real and imaginary parts respectively. The function P_1 can also be expressed as a series

$$\hat{w}_1(r) = \sum_{n=0}^{\infty} p_n(r)\frac{(-1)^n s^{2n}}{(4n)!}\hat{y} + q_n(r)\frac{(-1)^n s^{2n+2}}{(4n+4)!}\hat{y} \tag{12}$$

where

$$\begin{aligned} p_n(r) = &\frac{r^{4n+1} + \Im[(1-r+i)^{4n+1}] - \Re[(1-r+i)^{4n+1}]}{2(4n+1)} \\ &+ \mu\Im[(1-r+i)^{4n}] \\ q_n(r) = &\frac{\lambda\mu}{2}(4n+4)(4n+3)(4n+2)[\Im[(1-r+i)^{4n+1}] \\ &- \Re[(1-r+i)^{4n+1}] - r^{4n+1}] \\ &- \lambda(4n+4)(4n+3)\Re[(1-r+i)^{4n+2}] \end{aligned}$$

According to (8c),

$$\Delta\hat{u} = \hat{w}_r(0) = \hat{w}_{1r}(0) + \hat{w}_{2r}(0)$$

and

$$\hat{w}_{1r}(0) = \sum_{n=0}^{\infty} \frac{dp_n}{dr}(0)\frac{(-1)^n s^{2n}}{(4n)!}\hat{y} + \frac{dq_n}{dr}(0)\frac{(-1)^n s^{2n+2}}{(4n+4)!}\hat{y}$$

$$\hat{w}_{2r}(0) = \frac{\alpha^2 g s_{\theta_0}\hat{y}}{Ls^2}\Bigg[(\lambda s^2 S_0^- - C_0^+)\frac{d\omega_0}{dr} + (S_0^+ - \mu C_0^-)s\frac{d\pi_0}{dr}\Bigg] = 0$$

Hence Δu expressed as a function of τ is

$$\Delta u = \sum_{n=0}^{\infty} \frac{dp_n}{dr}(0)\frac{(-1)^n}{(4n)!}\frac{d^{2n}y}{d\tau^{2n}} + \frac{dq_n}{dr}(0)\frac{(-1)^n}{(4n+4)!}\frac{d^{2n+2}y}{d\tau^{2n+2}}$$

From (12) we have the relation between w_1 and y:

$$w_1(r,\tau) = \sum_{n=0}^{\infty} p_n(r)\frac{(-1)^n}{(4n)!}\frac{d^{2n}y}{d\tau^{2n}} + q_n(r)\frac{(-1)^n}{(4n+4)!}\frac{d^{2n+2}y}{d\tau^{2n+2}}$$

Next, we obtain the series expression for $w_2(r,\tau)$. From (11), we can rewrite P_2 as

$$P_2(r) = -\frac{\alpha^2 g s_{\theta_0}}{Ls^2}[\lambda\mu s^2(C_0^- C_r^- - sS_r^+ S_0^-) - \lambda s^2(C_r^- S_0^+ + C_r^+ S_0^-) + \mu s(C_0^+ S_r^+ - C_0^- S_r^-) + (C_0^+ C_r^+ + sS_0^+ S_r^-)]$$

In order to express $\hat{w}_2$ in a series we need the following identities [23]:

$$C_0^- C_r^- - sS_r^+ S_0^- = \frac{1}{4}\Big[(1+i)C^+(1-r-i) + (1-i)C^+(1-r+i) - 2C^+(r)\Big]$$

$$C_r^- S_0^+ + C_r^+ S_0^- = \frac{1}{2}\Big[S^+(1-r+i) - S^+(1-r-i)\Big]$$

$$C_0^+ S_r^+ - C_0^- S_r^- = \frac{1}{2}\Big[S^+(1-r+i) + S^+(1-r-i)\Big]$$

$$C_0^+ C_r^+ + sS_0^+ S_r^- = \frac{1}{4}\Big[(1+i)C^+(1-r-i) + (1-i)C^+(1-r+i) + 2C^+(r)\Big]$$

Using $e^r = \sum_{n\geq 0} r^n/(n!)$ we have

$$C_0^- C_r^- - sS_r^+ S_0^- = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n s^{2n}}{(4n)!}\Big[\Re[(1-r+i)^{4n}] + \Im[(1-r+i)^{4n}] - r^{4n}\Big]$$

$$C_r^- S_0^+ + C_r^+ S_0^- = -\sum_{n=0}^{\infty}\frac{(-1)^n s^{2n}}{(4n+1)!}\Im[(1-r+i)^{4n+1}]$$

$$C_0^+ S_r^+ - C_0^- S_r^- = \sum_{n=0}^{\infty}\frac{(-1)^{n+1} s^{2n+1}}{(4n+3)!}\Re[(1-r+i)^{4n+3}]$$

$$C_0^+ C_r^+ + sS_0^+ S_r^- = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n s^{2n}}{(4n)!}\Big[\Re[(1-r+i)^{4n}] + \Im[(1-r+i)^{4n}] + r^{4n}\Big]$$

Thus

$$\hat{w}_2(r) = -\frac{\alpha^2 g s_{\theta_0}}{Ls^2}\sum_{n=0}^{\infty}\Bigg[\frac{\bar{p}_n(r)s^{2n}}{(4n)!} + \frac{\bar{q}_n(r)s^{2n+2}}{(4n+4)!}\Bigg](-1)^n\hat{y}$$

with

$$\bar{p}_n(r) = \frac{1}{2}\Big[\Re[(1-r+i)^{4n}] + \Im[(1-r+i)^{4n}] + r^{4n}\Big] \tag{13}$$

$$\bar{q}_n(r) = \frac{\lambda\mu}{2}\prod_{j=1}^{4}(4n+j)\Big[\Re[(1-r+i)^{4n}] + \Im[(1-r+i)^{4n}] - r^{4n}\Big] + \lambda\prod_{j=2}^{4}(4n+j)\Im[(1-r+i)^{4n+1}] - \mu(4n+4)\Re[(1-r+i)^{4n+3}]$$

Expressed as a function of τ, w_2 is given by

$$w_2(r,\tau) = -\frac{\alpha^2 g s_{\theta_0}\bar{p}_0(r)}{L}\int_0^{\alpha\tau}\int_0^{\xi_1} y(\xi)\,d\xi\,d\xi_1 - \frac{\alpha^2 g s_{\theta_0}}{L}\Bigg[\sum_{n=1}^{\infty}\bar{p}_n(r)\frac{(-1)^n}{(4n)!}\frac{d^{2n-2}y}{d\tau^{2n-2}} + \sum_{n=0}^{\infty}\bar{q}_n(r)\frac{(-1)^n}{(4n+4)!}\frac{d^{2n}y}{d\tau^{2n}}\Bigg] \tag{14}$$

Transforming back to the original dependent coordinate using (7) we have

$$\Delta v(r,\tau) = w_1(r,\tau) - r\Delta u(\tau) + \frac{\alpha^2 g s_{\theta_0}}{L}\Bigg[\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n)!}\left(\frac{dp_n}{dr}(0) - \bar{p}_n(r)\right)\frac{d^{2n-2}y}{d\tau^{2n-2}} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+4)!}\left(\frac{dq_n}{dr}(0) - \bar{q}_n(r)\right)\frac{d^{2n}y}{d\tau^{2n}}\Bigg] \tag{15}$$

IV. Series Convergence

The convergence of w_1 ($g = 0$ case) is treated in [23] and not repeated here. An outline of the convergence proof for the series of w_2 is given here. We begin by choosing y to be smooth and of Gevrey class less than 2. That is, y satisfies

$$\sup_{\tau\in R^+}\left|\frac{d^n y}{d\tau^n}(\tau)\right| \leq M\frac{(n!)^\beta}{\gamma^n}$$

where M and γ are positive reals, and $\beta < 2$ [27]. Defining

$$C_n(r,\tau) = \frac{\alpha^2 g s_{\theta_0}(-1)^n}{L(4n)!}\bar{p}_n(r)\frac{d^{2n-2}y}{d\tau^{2n-2}},$$

$$D_n(r,\tau) = \frac{\alpha^2 g s_{\theta_0}(-1)^n}{L(4n+4)!}\bar{q}_n(r)\frac{d^{2n}y}{d\tau^{2n}}$$

Since $r \in [0,1]$ we take $z = (|r-1|+1)^4 = (2-r)^4$, so $z \geq 1$ and (13) we have

$$\begin{aligned}|C_n(r,\tau)| &\leq \frac{\alpha^2 g|s_{\theta_0}||\bar{p}_n(r)|}{L(4n)!}\left|\frac{d^{2n-2}y}{d\tau^{2n-2}}\right| \\ &\leq \frac{3M\alpha^2|s_{\theta_0}|g}{2L}\frac{((2n-2)!)^\beta}{(4n)!\gamma^{2n-2}}z^n \\ &= \frac{\overline{N}_1((2n-2)!)^\beta}{(4n)!\gamma^{2n-2}}z^n = \overline{C}_n z^n\end{aligned}$$

and

$$\begin{aligned}|D_n(r,\tau)| \leq& \frac{\alpha^2 g|s_{\theta_0}||\bar{q}_n(r)|}{L(4n+4)!}\left|\frac{d^{2n}y}{d\tau^{2n}}\right| \\ \leq& \frac{\alpha^2 g|s_{\theta_0}|}{L}\Bigg[\frac{\lambda\mu}{4}\prod_{j=1}^{4}(4n+j)3(2-r)^{4n} \\ &+ \lambda\prod_{j=2}^{4}(4n+j)(2-r)^{4n+1} \\ &+ \mu(4n+4)(2-r)^{4n+3}\Bigg]\frac{M[(2n)!]^\beta}{\gamma^{2n}(4n+4)!} \\ =& \frac{\alpha^2 g|s_{\theta_0}|M[(2n)!]^\beta}{L(4n)!\gamma^{2n}}\Bigg[\frac{3\lambda\mu z^{-1}}{4} + \frac{\lambda z^{-3/4}}{4n+1} \\ &+ \frac{\mu z^{-1/4}}{\prod_{j=0}^{3}(4n+j)}\Bigg]z^n \\ \leq& \overline{N}_2\frac{((2n)!)^\beta}{(4n)!\gamma^{2n}}z^n = \overline{D}_n z^n\end{aligned}$$

with $\overline{C}_n = \overline{N}_1\frac{((2n-2)!)^\beta}{(4n)!\gamma^{2n-2}}$, $\overline{D}_n = \overline{N}_2\frac{((2n)!)^\beta}{(4n)!\gamma^{2n}}$, $\overline{N}_1 = 3M\alpha^2 g|s_{\theta_0}|/(2L)$, $\overline{N}_2 = (M\alpha^2 g|s_{\theta_0}|/L)(3\lambda\mu/4+\lambda+\mu/6)$, and $n \geq 1$. Using Cauchy-Hadamard's Theorem for radius of convergence [28] and $\sqrt{(2n\pi)}(n/e)^n \leq n! \leq \sqrt{(2n\pi)}(n/e)^n e^{1/12n}$, the radius of convergence for $\sum_n \overline{C}_n z^n$ and $\sum_n \overline{D}_n z^n$ are

$$\begin{aligned}\frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{\overline{C}_n}} &\geq \lim_{n\to\infty}\sqrt[n]{\frac{\prod_{j=0}^{3}(4n-j)(4n-4)!\gamma^{2n-2}}{\overline{N}_1((2n-2)!)^\beta}} \\ &= \lim_{n\to\infty}(\gamma^2 4^{4-\beta}e^{2\beta-4}(n-1)^{4-2\beta}) = \infty \\ \frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{\overline{D}_n}} &\geq \lim_{n\to\infty}\sqrt[n]{\frac{(4n)!\gamma^{2n}}{\overline{N}_2((2n)!)^\beta}} \\ &\geq \lim_{n\to\infty}\sqrt[n]{\frac{(\sqrt{8n\pi})(4n/e)^{4n}\gamma^{2n}}{\overline{N}_2(\sqrt{4n\pi})^\beta(2n/e)^{2n\beta}}} \\ &= \lim_{n\to\infty}(\gamma^2 4^{4-\beta}e^{2\beta-4}n^{4-2\beta}) = \infty\end{aligned}$$

Thus, the radius of convergence for $\sum_{n=0}^{\infty} C_n(r,t)$ and $\sum_{n=0}^{\infty} D_n(r,t)$ are both infinite. Hence, the series for both w_2 and w have an infinite radius of convergence.

V. SIMULATION

The simulation demonstrates a rest to rest motion from $\theta = 0$ to $\pi/4$ on $t \in [0,t^*] = [0, 2.099]$. For the system parameters considered this corresponds to the interval $\tau \in [0,10]$. Equilibrium solutions for (5) are given by $v_0(x,\theta) = \cos(\theta)\mathcal{P}(x)$ where $\mathcal{P}(x) = \big(4(\rho SL+m)x - \rho Sx^2 - 6L(\rho SL+2m)\big)gx^2/(24EI)$. This equilibrium solution corresponds to an equilibrium torque $T_0(\theta) = gLc_\theta(((\rho SL+2m)/2) + [3(\rho SL)^2 + 15\rho SLm + 20m^2]L^2 gs_\theta/(60EI))$. The linearized system (6) has equilibrium solutions Δv_0 given by $\Delta v_0(x,\Delta\theta) = -\Delta\theta\sin\theta_0\mathcal{P}(x)$ where we take $\theta_0 = \pi/8$ as the point of linearization. To ensure initial and final deformations $v = \Delta v + v_0(\pi/8)$ correspond to $v_0(0)$ and $v_0(\pi/4)$ we set

$$\begin{aligned}\Delta v(x,0) &= v(x,0) - v_0(\pi/8) = v_0(0) - v_0(\pi/8) \\ &= (1-\cos(\pi/8))\mathcal{P}(x) = -\sin(\pi/8)\mathcal{P}(x)\Delta\theta(0)\end{aligned}$$

Hence,

$$\Delta\theta(0) = (\cos(\pi/8)-1)/\sin(\pi/8) = y(0)/L \qquad (16)$$

where the last equality follows from the relation between Δu and $y =$ const. Similarly at $t = t^*$ we have

$$\Delta\theta(t^*) = (\cos(\pi/8)-\cos(\pi/4))/\sin(\pi/8) = y(t^*)/L \qquad (17)$$

To ensure the rest to rest motion we incorporate conditions (16) and (17) into y. As well, we choose y so that all its derivatives are zero at $t = 0$ and $t = t^*$. Finally we take y to be of Gevrey class less than 2 for series convergence. We remark that y cannot be analytic as this would prevent the beam from leaving its equilibrium position. We make use of the nonanalytic C^∞ function $\Phi_\sigma : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$\Phi_\sigma(t) = \begin{cases}\frac{\int_0^{t/t_*}\phi_\sigma(\tau)\,d\tau}{\int_0^1\phi_\sigma(\tau)\,d\tau} & \text{for } 0 \leq t \leq t_* \\ 1 & \text{for } t > t_*,\end{cases} \qquad (18)$$

where $\phi_\sigma : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\phi_\sigma(t) = \begin{cases}\exp(-1/(t(1-t))^\sigma) & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1.\end{cases} \qquad (19)$$

The flat output is taken as $y(t) = C_1 + (C_2 - C_1)\Phi_\sigma(t)$ with $C_2 = L(\cos(\pi/8)-\cos(\pi/4))/\sin(\pi/8)$, $C_1 = L(\cos(\pi/8)-1)/\sin(\pi/8)$, and $\sigma = 10/9$. A plot of this function is shown in Fig. 2. We consider the same parameters as in [15]: $m = 5.9$ kg, $L = 1.005$ m, $EI = 47.25$ N m^2, $\rho S = 2.04$ kg/m, $J_h = 0.047$ kg m^2, and $g = 9.81$ m/s^2. Fig. 3 shows the resulting field $v = \Delta v + v_0(\pi/8)$ where Δv is computed using (15) truncated to 4 terms. The open-loop torque $T = \Delta T + T_0(\pi/8)$ is shown in Fig. 4. The function ΔT is computed using (6a).

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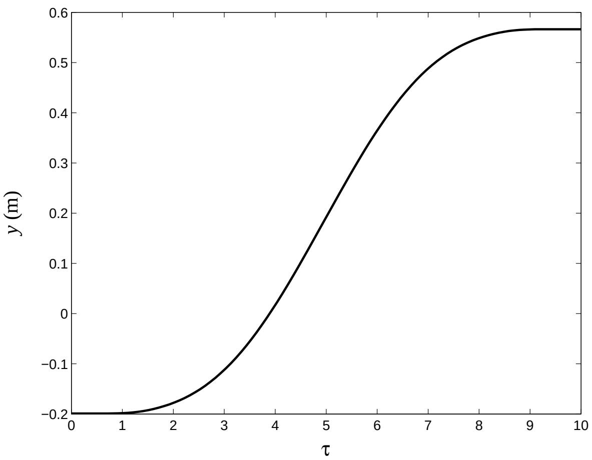

Fig. 2. Flat output trajectory

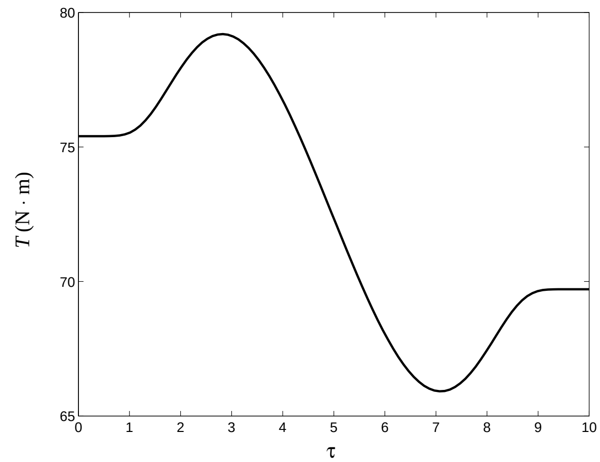

Fig. 4. Open-loop torque $T = \Delta T + T_0(\pi/8)$

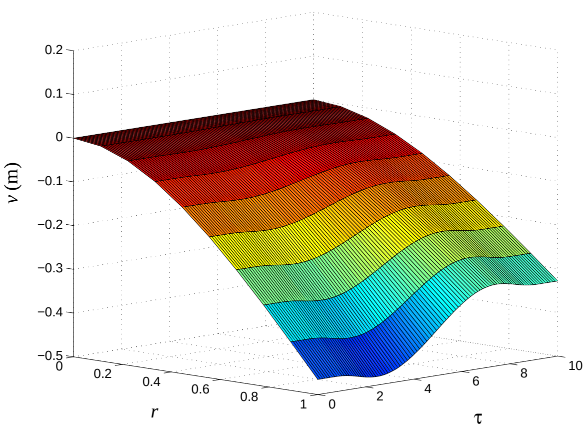

Fig. 3. Deformation field $v = \Delta v + v_0(\pi/8)$

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