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Robust Output Feedback Stabilization for Uncertain Singular Time Delay Systems

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Abstract— **The robust output feedback stabilization problem for singular time delay system with norm-bounded parametric uncertainties is considered in this paper. All the coefficient matrices except the matrix E include uncertainties. The authors derive sufficient conditions about the robust stability of the two closed-loop systems which are obtained by applying an observer-based controller and a compensator to the singular time delay system respectively. Then the strict linear inequality (LMI) design approaches are developed and the desired robust output feedback control laws are given.**

Index Terms — **Singular time delay system, robust stabilization, output dynamic feedback, linear matrix inequality (LMI).**

I. INTRODUCTION

Control of delay systems has been a topic of recurring interest over the past decades since time delays are frequently encountered in physical processes and very often are the causes of instability and poor performance of control systems, see [1]. Recently, increasing attention has been devoted to the problem of robust stability and robust stabilization of linear systems with delayed state and parametric uncertainty, see, e.g., [2]-[4] and the references therein.

On the other hand, control of singular systems has been extensively studied in the past years due to the fact that singular systems can describe practical physical processes more comprehensively than regular ones. A great number of results based on the theory of regular systems have been extended to the area of singular systems. Very recently, much attention has been paid to singular time delay systems. A sufficient condition about the stability of the singular time delay systems is derived in [7] and [8] independently in terms of linear matrix inequality (LMI). Furthermore, the problems of robust stabilization [7] and guaranteed cost control [8] via state feedback for singular time delay systems with norm-bounded uncertainties are discussed respectively. However, when all state variables are not available for feedback, it is necessary to design output feedback controller for system. The purpose of this paper is to design an observer-based output dynamic feedback controller and a compensator such the resultant closed-loop systems are robustly stable. To the best of our knowledge, there are no results on the problems of robust output feedback stabilization for uncertain singular time delay systems in the literature. In addition, control laws are proposed by using strict LMI approaches, which is much more efficient in numerical computation.

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Notation: R denotes the set of all real numbers, R^n denotes the n-dimensional Euclidean space, $R^{n\times m}$ is the set of all $n\times m$ real matrices, $C([-\tau,0],R^n)$ denotes the space of all continuous functions mapping $[-\tau,0]$ into R^n, $diag\{\cdots\}$ is a block-diagonal matrix. For a symmetric matrix $P=P^T$, $P>0$ (<0) means that P is positive (negative) definite. The size of the identity matrix I should to be inferred from the context.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a linear singular system with state delay and parametric uncertainties described by

$$\begin{cases} E\dot{x}(t)=(A+\triangle A)x(t)+(A_\tau+\triangle A_\tau)x(t-\tau) \\ \qquad\quad +(B+\triangle B)u(t) \\ y(t)=(C+\triangle C)x(t) \\ x(t)=\phi(t), \quad t\in[-\tau,0] \end{cases} \tag{1}$$

where $x(t)\in R^n$, $u(t)\in R^p$, $y(t)\in R^m$ are the state, control input and measurement output, respectively. E, A, A_τ, B and C are known real constant matrices with appropriate dimensions and $0<$rank $E=q<n$. $\tau>0$ is a constant time delay, $\phi(t)\in C([-\tau,0],R^n)$ is a compatible vector valued function. $\triangle A,\triangle A_\tau,\triangle B$ and $\triangle C$ are time-invariant matrices representing norm-bounded parametric uncertainties, and are assumed to be of the following form:

$$\begin{bmatrix} \triangle A & \triangle A_\tau & \triangle B \\ \triangle C & * & * \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} F(\sigma) \begin{bmatrix} E_1 & E_\tau & E_2 \end{bmatrix} \tag{2a}$$

$$F^T(\sigma)F(\sigma)\le I_j, \quad F(\sigma)\in R^{i\times j}. \tag{2b}$$

Where, $D_1\in R^{n\times i}$, $D_2\in R^{m\times i}$, $E_1\in R^{j\times n}$, $E_\tau\in R^{j\times n}$, $E_2\in R^{j\times p}$ are known real constant matrices which characterize how the uncertain parameters in $F(\sigma)$ enter the normal matrices A, A_τ, B and C. $\sigma\in\Theta,\Theta$ is a compact set in R. Furthermore, it is assumed that given any matrix $F: F^TF\le I$, there exists a $\sigma\in\Theta$ such that $F=F(\sigma)$. $\triangle A,\triangle A_\tau,\triangle B$ and $\triangle C$ are said to be admissible if (2) is satisfied.

Consider the stability of the nominal unforced singular delay system of (1)

$$\begin{cases} E\dot{x}(t)=Ax(t)+A_\tau x(t-\tau) \\ x(t)=\phi(t), \quad t\in[-\tau,0] \end{cases} \tag{3}$$

A result has been presented in [7] and [8] simultaneously, which is given as a lemma in the following.

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Lemma 1:[7][8] If there exist a matrix P and a matrix $Q > 0$ such that

$$EP^T = PE^T \geq 0 \tag{4}$$

$$\left[\begin{array}{cc} AP^T + PA^T + Q & A_\tau P^T \\ PA_\tau^T & -Q \end{array} \right] < 0 \tag{5}$$

the singular time delay system (3) is asymptotically stable.

Remark 1: From the definition given in [7], the uncertain singular time delay system (1) is said to be robustly stable if the system (1) with $u(t) \equiv 0$ is asymptotically stable for all admissible uncertainties $\triangle A$ and $\triangle A_\tau$.

The objective of this paper is to:

A. Design an observer-based dynamic output feedback controller:

$$\left\{ \begin{array}{l} E\dot{\hat{x}}(t) = A\hat{x}(t) + A_\tau \hat{x}(t-\tau) + Bu(t) \\ \qquad\qquad + L(y(t) - C\hat{x}(t)) \\ u(t) = K\hat{x}(t) \\ \hat{x}(t) = \psi(t), \quad t \in [-\tau, 0] \end{array} \right. \tag{6}$$

such that the resultant closed-loop system is robustly stable, where $\hat{x}(t) \in R^n$ is the observer state vector, $L \in R^{n\times m}$ is the observer gain matrix and $K \in R^{p\times n}$ is the controller gain matrix.

B. Design a compensator

$$\left\{ \begin{array}{l} \dot{\eta}(t) = K_{11}\eta(t) + K_{12}y(t) \\ u(t) = K_{21}\eta(t) \\ \eta(0) = \eta_0 \end{array} \right. \tag{7}$$

such that the resultant closed-loop system is robustly stable, where $\eta(t) \in R^r$ is the controller state vector, r is to be decided and $0 < r \leq n$.

III. MAIN RESULTS

To get the main results of this paper, we first introduce some useful lemmas.

Lemma 2:[5] Given matrices H, E and Q of appropriate dimensions with Q symmetrical, then

$$Q + HF(\sigma)E + (HF(\sigma)E)^T < 0$$

for all $F(\sigma) : F^T(\sigma)F(\sigma) \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$Q + \epsilon HH^T + \epsilon^{-1}E^TE < 0.$$

Lemma 3:[6] Given matrices Ψ, P and Q of appropriate dimensions with Ψ symmetrical, then the inequality

$$\Psi + P^T\Pi^TQ + Q^T\Pi P < 0$$

is solvable for Π if and only if

$$\left\{ \begin{array}{l} W_P^T\Psi W_P < 0 \\ W_Q^T\Psi W_Q < 0 \end{array} \right.$$

where matrices W_P and W_Q are orthogonal complements of P and Q respectively.

A. Design of observer-based controller (6)

When we apply (6) to the system (1), the closed-loop system is given by

$$\left\{ \begin{array}{l} E\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)K\hat{x}(t) \\ \qquad\qquad + (A_\tau + \Delta A_\tau)x(t-\tau) \\ E\dot{\hat{x}}(t) = (LC + L\Delta C)x(t) + (A - LC + BK)\hat{x}(t) \\ \qquad\qquad + A_\tau\hat{x}(t-\tau) \\ \quad x(t) = \phi(t), \quad t \in [-\tau, 0] \\ \quad \hat{x}(t) = \psi(t), \quad t \in [-\tau, 0] \end{array} \right.$$

Define the observer error vector $e(t) = x(t) - \hat{x}(t)$, we get

$$\left\{ \begin{array}{l} E\dot{x}(t) = [(A + \Delta A) + (B + \Delta B)K]x(t) \\ \qquad\qquad - (B + \Delta B)Ke(t) + (A_\tau + \Delta A_\tau)x(t-\tau) \\ E\dot{e}(t) = (\Delta A + \Delta BK - L\Delta C)x(t) + \Delta A_\tau x(t-\tau) \\ \qquad\qquad + (A - LC - \Delta BK)e(t) + A_\tau e(t-\tau) \\ \quad x(t) = \phi(t), \quad t \in [-\tau, 0] \\ \quad e(t) = \phi(t) - \psi(t), \quad t \in [-\tau, 0] \end{array} \right. \tag{8}$$

Introduce an auxiliary variable $x_c(t) = \left[\begin{array}{c} x(t) \\ e(t) \end{array} \right]$, the system (8) can be written as

$$\left\{ \begin{array}{l} E_c\dot{x}_c(t) = (A_c + \Delta A_c)x_c(t) + (A_{c\tau} + \Delta A_{c\tau})x_c(t-\tau) \\ \quad x_c(t) = \left[\begin{array}{c} \phi(t) \\ \phi(t) - \psi(t) \end{array} \right], \quad t \in [-\tau, 0] \end{array} \right. \tag{9}$$

where

$$E_c = \left[\begin{array}{cc} E & 0 \\ 0 & E \end{array} \right], A_c = \left[\begin{array}{cc} A + BK & -BK \\ 0 & A - LC \end{array} \right]$$

$$A_{c\tau} = \left[\begin{array}{cc} A_\tau & 0 \\ 0 & A_\tau \end{array} \right], \Delta A_c = \bar{D}_1F(\sigma)\bar{E} + \bar{D}_2F(\sigma)\bar{E}_1$$

$$\Delta A_{c\tau} = \bar{D}_1F(\sigma)\bar{E}_\tau, \bar{D}_1 = \left[\begin{array}{c} D_1 \\ D_1 \end{array} \right], \bar{D}_2 = \left[\begin{array}{c} 0 \\ -LD_2 \end{array} \right]$$

$$\bar{E} = \left[\begin{array}{cc} E_1 + E_2K & -E_2K \end{array} \right]$$

$$\bar{E}_1 = \left[\begin{array}{cc} E_1 & 0 \end{array} \right], \bar{E}_\tau = \left[\begin{array}{cc} E_\tau & 0 \end{array} \right] \tag{10}$$

Lemma 1 shows that the closed-loop system (9) is robustly stable, if there exist matrices $P_c = \left[\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array} \right]$ and $Q_c = \left[\begin{array}{cc} Q_1 & 0 \\ 0 & Q_2 \end{array} \right] > 0$ satisfying

$$E_cP_c^T = P_cE_c^T \geq 0 \tag{11a}$$

$$\left[\begin{array}{cc} (A_c + \Delta A_c)P_c^T + P_c(A_c + \Delta A_c)^T + Q_c & * \\ P_c(A_{c\tau} + \Delta A_{c\tau})^T & -Q_c \end{array} \right] < 0 \tag{11b}$$

such that for all ΔA_c and $\Delta A_{c\tau}$. Inequalities (11a) and (11b) are just

$$EP_1^T = P_1E^T \geq 0 \tag{12a}$$

$$EP_2^T = P_2E^T \geq 0 \tag{12b}$$

$$\begin{bmatrix} A_cP_c^T+P_cA_c^T+Q_c & A_{c\tau}P_c^T \\ P_cA_{c\tau}^T & -Q_c \end{bmatrix}+ \begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F(\sigma) & 0 \\ 0 & F(\sigma) \end{bmatrix} \begin{bmatrix} \bar{E}P_c^T & \bar{E}_\tau P_c^T \\ \bar{E}_1P_c^T & 0 \end{bmatrix}+ \left(\begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F(\sigma) & 0 \\ 0 & F(\sigma) \end{bmatrix} \begin{bmatrix} \bar{E}P_c^T & \bar{E}_\tau P_c^T \\ \bar{E}_1P_c^T & 0 \end{bmatrix} \right)^T < 0. \tag{12c}$$

From Lemma 2, (12c) holds if there exists a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} A_cP_c^T+P_cA_c^T+Q_c & A_{c\tau}P_c^T \\ P_cA_{c\tau}^T & -Q_c \end{bmatrix}+ \epsilon \begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & 0 \end{bmatrix}^T + \epsilon^{-1} \begin{bmatrix} \bar{E}P_c^T & \bar{E}_\tau P_c^T \\ \bar{E}_1P_c^T & 0 \end{bmatrix}^T \begin{bmatrix} \bar{E}P_c^T & \bar{E}_\tau P_c^T \\ \bar{E}_1P_c^T & 0 \end{bmatrix} < 0. \tag{13}$$

By Schur complements, (13) is equivalent to the following inequality:

$$\begin{bmatrix} A_cP_c^T+P_cA_c^T+Q_c+\epsilon\bar{D}_1\bar{D}_1^T & A_{c\tau}P_c^T & \epsilon\bar{D}_2 & P_c\bar{E}^T & P_c\bar{E}_1^T \\ P_cA_{c\tau}^T & -Q_c & 0 & P_c\bar{E}_\tau^T & 0 \\ \epsilon\bar{D}_2^T & 0 & -\epsilon I & 0 & 0 \\ \bar{E}P_c^T & \bar{E}_\tau P_c^T & 0 & -\epsilon I & 0 \\ \bar{E}_1P_c^T & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{14}$$

Using the expression in (10), inequality (14) can be written as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & A_\tau P_1^T & 0 & 0 & P_1(E_1+E_2K)^T & P_1E_1^T \\ \Gamma_{12}^T & \Gamma_{22} & 0 & A_\tau P_2^T & -\epsilon LD_2 & -P_2(E_2K)^T & 0 \\ P_1A_\tau^T & 0 & -Q_1 & 0 & 0 & P_1E_\tau^T & 0 \\ 0 & P_2A_\tau^T & 0 & -Q_2 & 0 & 0 & 0 \\ 0 & -\epsilon(LD_2)^T & 0 & 0 & -\epsilon I & 0 & 0 \\ (E_1+E_2K)P_1^T & -E_2KP_2^T & E_\tau P_1^T & 0 & 0 & -\epsilon I & 0 \\ E_1P_1^T & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{15}$$

with

$$\Gamma_{11} = (A+BK)P_1^T + P_1(A+BK)^T + Q_1 + \epsilon D_1D_1^T$$

$$\Gamma_{12} = -BKP_2^T + \epsilon D_1D_1^T$$

$$\Gamma_{22} = (A-LC)P_2^T + P_2(A-LC)^T + Q_2 + \epsilon D_1D_1^T.$$

Define matrix $\Lambda = diag\{I, P_2^{-1}, I, P_2^{-1}, I, I, I\}$, (15) is equivalent to $\Lambda\Gamma\Lambda^T < 0$, i. e.,

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & A_\tau P_1^T & 0 & 0 & P_1(E_1+E_2K)^T & P_1E_1^T \\ \Upsilon_{12}^T & \Upsilon_{22} & 0 & P_2A_\tau & -\epsilon P_2LD_2 & -(E_2K)^T & 0 \\ P_1A_\tau^T & 0 & -Q_1 & 0 & 0 & P_1E_\tau^T & 0 \\ 0 & A_\tau^TP_2^T & 0 & -Q_2 & 0 & 0 & 0 \\ 0 & -\epsilon(LD_2)^TP_2^T & 0 & 0 & -\epsilon I & 0 & 0 \\ (E_1+E_2K)P_1^T & -E_2K & E_\tau P_1^T & 0 & 0 & -\epsilon I & 0 \\ E_1P_1^T & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{16}$$

where

$$\Upsilon_{11} = \Gamma_{11}, \Upsilon_{12} = -BK + \epsilon D_1D_1^TP_2^T$$

$$\Upsilon_{22} = P_2(A-LC) + (A-LC)^TP_2^T + Q_2 + \epsilon P_2D_1D_1^TP_2^T$$

and we still denote $P_2 = P_2^{-1}, Q_2 = P_2^{-1}Q_2P_2^{-T}$ for simplicity. Correspondingly, (12b) is equivalent to

$$P_2E = E^TP_2^T \geq 0. \tag{17}$$

Based on above analysis, we know that a sufficient condition guaranteeing the robust stability of the closed-loop system (9) is that there exist matrices $P_1, P_2, Q_1 > 0, Q_2 > 0$ and a scalar $\epsilon > 0$ such that (12a), (17) and (16) are satisfied. Using the method dealing with inequalities (12a) and (17) which was developed in [7], we get a main result in the following theorem.

Theorem 1: The closed-loop system (9) is robustly stable if there exist matrices $X_1 > 0, Y_1, Q_1 > 0, X_2 > 0, Y_2, Q_2 > 0$ and a scalar $\epsilon > 0$, such that the matrix inequality (16) holds, in which P_1 is substituted by $EX_1 + Y_1\Phi^T$ and P_2 is substituted by $E^TX_2 + Y_2\Psi^T$. Where, matrices $\Phi \in R^{n\times(n-q)}, \Psi \in R^{n\times(n-q)}$ satisfy $E\Phi = 0$ and $E^T\Psi = 0$ respectively, and rankΦ =rank$\Psi = n-q$.

According to Theorem 1, an observer-based controller can be obtained by solving the matrix inequality (16). However, it is worth pointing out that (16) is not a linear matrix inequality, so can not be solved using the LMI Toolbox of Matlab. However, note that a necessary condition of (16) is

$$\begin{bmatrix} \Upsilon_{11} & A_\tau P_1^T & P_1(E_1+E_2K)^T & P_1E_1^T \\ P_1A_\tau^T & -Q_1 & P_1E_\tau^T & 0 \\ (E_1+E_2K)P_1^T & E_\tau P_1^T & -\epsilon I & 0 \\ E_1P_1^T & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{18}$$

with P_1 being substituted by $EX_1 + Y_1\Phi^T$. Let $Z_1 = K(EX_1 + Y_1\Phi^T)^T$, then (18) can be written as

$$\begin{bmatrix} \Omega_{11} & A_\tau P_1^T & P_1E_1^T + Z_1^TE_2^T & P_1E_1^T \\ P_1A_\tau^T & -Q_1 & P_1E_\tau^T & 0 \\ E_1P_1^T + E_2Z_1 & E_\tau P_1^T & -\epsilon I & 0 \\ E_1P_1^T & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{19}$$

where

$$\Omega_{11} = AP_1^T + P_1A^T + BZ_1 + Z_1^TB^T + Q_1 + \epsilon D_1D_1^T$$

with P_1 being substituted by $EX_1+Y_1\Phi^T$. Obviously, (19) is a strict LMI about matrices $Q_1>0, X_1>0, Y_1, Z_1$ and a scalar $\epsilon>0$, which can be solved numerically very efficiently by using the LMI Toolbox of Matlab.

Remark 2: Here, without loss of generality, we can assume $EX_1+Y_1\Phi^T$ is nonsingular, then the controller gain matrix is given by $K=Z_1(EX_1+Y_1\Phi^T)^{-T}$. Otherwise, we can choose nonsingular matrices $M\in R^{n\times n}$ and $N\in R^{n\times n}$ such that $\hat{E}=MEN=\begin{bmatrix} I_q & 0\\ 0 & 0\end{bmatrix}$. Denote $\hat{A}=MAN, \hat{A}_\tau=MA_\tau N, \hat{B}=MB, \hat{D}_1=MD_1, \hat{E}_1=E_1N, \hat{E}_\tau=E_\tau N, \hat{X}_1=N^{-1}X_1N^{-T}, \hat{Y}_1=MY_1, \hat{Z}_1=Z_1M^T, \hat{Q}_1=M^TQ_1M$, and let $\hat{\Phi}=N^{-1}\Phi$ (obviously, $\hat{\Phi}$ satisfies $\hat{E}\hat{\Phi}=0$ and rank$\hat{\Phi}=n-q$). Hence, (18) is equivalent to

$$\begin{bmatrix} \hat{\Omega}_{11} & \hat{A}_\tau(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi})^T & (\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)\hat{E}_1^T+\hat{Z}_1^TE_2^T & (\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)\hat{E}_1^T\\ (\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)\hat{A}_\tau^T & -\hat{Q}_1 & (\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)\hat{E}_\tau^T & 0\\ \hat{E}_1(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)^T+E_2\hat{Z}_1 & \hat{E}_\tau(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)^T & -\epsilon I & 0\\ \hat{E}_1(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)^T & 0 & 0 & -\epsilon I \end{bmatrix}<0 \tag{20}$$

with $\hat{\Omega}_{11}=\hat{A}(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)^T+(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T)\hat{A}^T+\hat{B}\hat{Z}_1+\hat{Z}_1^T\hat{B}^T+\hat{Q}_1+\epsilon\hat{D}_1\hat{D}_1^T$. If $EX_1+Y_1\Phi^T$ is singular, then $\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T=M(EX_1+Y_1\Phi^T)N^{-T}$ is also a singular matrix. We can choose a sufficient small $\theta>0$ such that $\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T+\theta I$ is nonsingular and satisfies (20) with $\hat{E}(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T+\theta I)^T=(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T+\theta I)\hat{E}^T$. If (20) holds, the controller gain matrix K is obtained as $K=\hat{Z}_1(\hat{E}\hat{X}_1+\hat{Y}_1\hat{\Phi}^T+\theta I)^{-T}N^{-1}$.

Substituting the matrices Q_1, Z_1, X_1, Y_1, K and the scalar $\epsilon>0$ obtained by solving (19) into (16) and letting $Z_2=(E^TX_2+Y_2\Psi^T)L$, we get the strict LMI about Q_2, Z_2, and Y_2 as follows:

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & A_\tau P_1^T & 0 & 0 & P_1E_1^T+Z_1^TE_2^T & P_1E_1^T & 0\\ \Omega_{12}^T & \Omega_{22} & 0 & P_2A_\tau & -\epsilon Z_2D_2 & -(E_2K)^T & 0 & \epsilon P_2D_1\\ P_1A_\tau^T & 0 & -Q_1 & 0 & 0 & P_1E_\tau^T & 0 & 0\\ 0 & A_\tau^TP_2^T & 0 & -Q_2 & 0 & 0 & 0 & 0\\ 0 & -\epsilon D_2^TZ_2^T & 0 & 0 & -\epsilon I & 0 & 0 & 0\\ E_1P_1^T+E_2Z_1 & -E_2K & E_\tau P_1^T & 0 & 0 & -\epsilon I & 0 & 0\\ E_1P_1^T & 0 & 0 & 0 & 0 & 0 & -\epsilon I & 0\\ 0 & \epsilon D_1^TP_2^T & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix}<0 \tag{21}$$

with $\Omega_{12}=\Upsilon_{12}, \Omega_{22}=P_2A+A^TP_2^T-Z_2C-C^TZ_2^T+Q_2$ and in which P_1 is substituted by $EX_1+Y_1\Phi^T$ and P_2 is substituted by $E^TX_2+Y_2\Psi^T$.

Remark 3: Similar to Remark 1, without loss of generality, $E^TX_2+Y_2\Psi^T$ can be assumed to be nonsingular, so the observer gain matrix can be obtained by solving LMI (21) and $L=(E^TX_2+Y_2\Psi^T)^{-1}Z_2$.

B. Design of compensator (7)

The closed-loop system of (1) under compensator law (7) is

$$\begin{cases} \begin{bmatrix} I_r & 0\\ 0 & E\end{bmatrix}\begin{bmatrix}\dot{\eta}(t)\\ \dot{x}(t)\end{bmatrix} \\ \quad =\begin{bmatrix} K_{11} & K_{12}(C+\triangle C)\\ (B+\triangle B)K_{21} & A+\triangle A\end{bmatrix}\begin{bmatrix}\eta(t)\\ x(t)\end{bmatrix} \\ \quad +\begin{bmatrix} 0 & 0\\ 0 & A_\tau+\triangle A_\tau\end{bmatrix}\begin{bmatrix}\eta(t-\tau)\\ x(t-\tau)\end{bmatrix} \\ \begin{bmatrix}\eta(t)\\ x(t)\end{bmatrix}=\begin{bmatrix}\eta_0\\ \phi(t)\end{bmatrix}, \quad t\in[-\tau,0]\end{cases} \tag{22}$$

Introduce an auxiliary variable $\tilde{x}(t)=\begin{bmatrix}\eta^T(t) & x^T(t)\end{bmatrix}^T$ and gather all controller parameters into the single variable $K_0=\begin{bmatrix} K_{11} & K_{12}\\ K_{21} & 0\end{bmatrix}$, then the system (22) can be written as

$$\begin{cases} \tilde{E}\dot{\tilde{x}}(t) &=& (\tilde{A}+\triangle\tilde{A})\tilde{x}(t)+(\tilde{A}_\tau+\triangle\tilde{A}_\tau)\tilde{x}(t-\tau)\\ \tilde{x}(t) &=& \begin{bmatrix}\eta_0\\ \phi(t)\end{bmatrix}, \quad t\in[-\tau,0]\end{cases} \tag{23}$$

where,

$$\tilde{E}=\begin{bmatrix} I_r & 0\\ 0 & E\end{bmatrix}, \tilde{A}=A_0+B_0K_0C_0, \triangle\tilde{A}=D_{10}F_0(\sigma)E_{10}$$

$$\tilde{A}_\tau=\begin{bmatrix} 0 & 0\\ 0 & A_\tau\end{bmatrix}, \triangle\tilde{A}_\tau=D_{10}F_0(\sigma)\begin{bmatrix}E_{\tau 0}\\ 0\end{bmatrix}$$

$$A_0=\begin{bmatrix} 0 & 0\\ 0 & A\end{bmatrix}, B_0=\begin{bmatrix} I_r & 0\\ 0 & B\end{bmatrix}, C_0=\begin{bmatrix} I_r & 0\\ 0 & C\end{bmatrix}$$

$$D_{10}=\begin{bmatrix}\bar{D}_{10} & B_0K_0\bar{D}_{20}\end{bmatrix}, E_{10}=\begin{bmatrix}\bar{E}_{10}+\bar{E}_{20}K_0C_0\\ \bar{E}_{10}\end{bmatrix}$$

$$F_0(\sigma)=\begin{bmatrix} F(\sigma) & 0\\ 0 & F(\sigma)\end{bmatrix}, \bar{D}_{10}=\begin{bmatrix}0\\ D_1\end{bmatrix}$$

$$\bar{D}_{20}=\begin{bmatrix}0\\ D_2\end{bmatrix}, \bar{E}_{10}=\begin{bmatrix}0 & E_1\end{bmatrix}$$

$$\bar{E}_{20}=\begin{bmatrix}0 & E_2\end{bmatrix}, E_{\tau 0}=\begin{bmatrix}0 & E_\tau\end{bmatrix}. \tag{24}$$

Lemma 1 shows that the closed-loop system (23) is robustly stable if there exist matrices $\tilde{P}$ and $\tilde{Q}>0$ such that the following inequalities

$$\tilde{E}\tilde{P}^T=\tilde{P}\tilde{E}^T\geq 0 \tag{25a}$$

$$\begin{bmatrix} (\tilde{A}+\triangle\tilde{A})\tilde{P}^T+\tilde{P}(\tilde{A}+\triangle\tilde{A})^T+\tilde{Q} & *\\ \tilde{P}(\tilde{A}_\tau+\triangle\tilde{A}_\tau)^T & -\tilde{Q}\end{bmatrix}<0 \tag{25b}$$

hold for all $\triangle\tilde{A}$ and $\triangle\tilde{A}_\tau$. (24b) can be written as for all $F_0(\sigma)$

$$\begin{bmatrix} \tilde{A}\tilde{P}^T+\tilde{P}\tilde{A}^T+\tilde{Q} & \tilde{A}_\tau\tilde{P}^T \\ \tilde{P}\tilde{A}_\tau^T & -\tilde{Q} \end{bmatrix} + \begin{bmatrix} D_{10} \\ 0 \end{bmatrix} F_0(\sigma) \begin{bmatrix} E_{10}\tilde{P}^T & E_{\tau 0}\tilde{P}^T \\ & 0 \end{bmatrix} + \left(\begin{bmatrix} D_{10} \\ 0 \end{bmatrix} F_0(\sigma) \begin{bmatrix} E_{10}\tilde{P}^T & E_{\tau 0}\tilde{P}^T \\ & 0 \end{bmatrix} \right)^T < 0 \tag{26}$$

$F^T(\sigma)F(\sigma) \le I_j$ implies that $F_0^T(\sigma)F_0(\sigma) \le I_{2j}$. From Lemma 2, we can get a sufficient condition guaranteeing that (26) holds for all $F_0(\sigma)$ is that there exists a scalar $\epsilon_0 > 0$ such that

$$\begin{bmatrix} \tilde{A}\tilde{P}^T+\tilde{P}\tilde{A}^T+\tilde{Q} & \tilde{A}_\tau\tilde{P}^T \\ \tilde{P}\tilde{A}_\tau^T & -\tilde{Q} \end{bmatrix} + \epsilon_0 \begin{bmatrix} D_{10} \\ 0 \end{bmatrix} \begin{bmatrix} D_{10} \\ 0 \end{bmatrix}^T + \epsilon_0^{-1} \begin{bmatrix} E_{10}\tilde{P}^T & E_{\tau 0}\tilde{P}^T \\ & 0 \end{bmatrix}^T \begin{bmatrix} E_{10}\tilde{P}^T & E_{\tau 0}\tilde{P}^T \\ & 0 \end{bmatrix} < 0. \tag{27}$$

By Schur complements, (27) is equivalent to

$$\begin{bmatrix} \tilde{A}\tilde{P}^T+\tilde{P}\tilde{A}^T+\tilde{Q} & \tilde{A}_\tau\tilde{P}^T & \epsilon_0 D_{10} & \tilde{P}E_{10}^T \\ \tilde{P}\tilde{A}_\tau^T & -\tilde{Q} & 0 & \tilde{P}E_{\tau 0}^T \quad 0 \\ \epsilon_0 D_{10}^T & 0 & -\epsilon_0 I_{2i} & 0 \\ E_{10}\tilde{P}^T & E_{\tau 0}\tilde{P}^T \ \ 0 & 0 & -\epsilon_0 I_{2j} \end{bmatrix} < 0. \tag{28}$$

Applying the expression in (24) to (28), it follows that

$$\begin{bmatrix} \Xi & \tilde{A}_\tau\tilde{P}^T & \epsilon_0 B_0 K_0 \bar{D}_{20} & \tilde{P}(\bar{E}_{10}+\bar{E}_{20}K_0C_0)^T & \tilde{P}\bar{E}_{10}^T \\ \tilde{P}\tilde{A}_\tau^T & -\tilde{Q} & 0 & \tilde{P}E_{\tau 0}^T & 0 \\ \epsilon_0\bar{D}_{20}^T K_0^T B_0^T & 0 & -\epsilon_0 I_i & 0 & 0 \\ (\bar{E}_{10}+\bar{E}_{20}K_0C_0)\tilde{P}^T & E_{\tau 0}\tilde{P}^T & 0 & -\epsilon_0 I_j & 0 \\ \bar{E}_{10}\tilde{P}^T & 0 & 0 & 0 & -\epsilon_0 I_j \end{bmatrix} < 0 \tag{29}$$

with

$$\Xi = (A_0+B_0K_0C_0)\tilde{P}^T + \tilde{P}(A_0+B_0K_0C_0)^T + \tilde{Q} + \epsilon_0\bar{D}_{10}\bar{D}_{10}^T.$$

From above discussion, we know that if there exist matrices $\tilde{P}, \tilde{Q} > 0$ and $\epsilon_0 > 0$ such that (25a) and (29) hold simultaneously, the closed-loop system (23) is robustly stable. We can solve (25a) and (29) using the method in [7]. Next, we will give another method to solve (25a) and (29). It is well known that restricted system equivalent transformation will not change the stability of a system. We can find nonsingular matrices M and N such that $\breve{E} = M\tilde{E}N = \begin{bmatrix} I_{r+q} & 0 \\ 0 & 0 \end{bmatrix}$ and denote $\breve{A} = MA_0N, \breve{A}_\tau = M\tilde{A}_\tau N, \breve{B}_0 = MB_0, \breve{C}_0 = C_0N, \breve{D}_{10} = M\bar{D}_{10}, \breve{E}_{10} = \bar{E}_{10}N, \breve{E}_{\tau 0} = E_{\tau 0}N, \breve{P} = M\tilde{P}N^{-T}, \breve{Q} = M\tilde{Q}M^T$. Then (25a) and (29) are equivalent to

$$\breve{E}\breve{P}^T = \breve{P}\breve{E}^T \ge 0 \tag{30a}$$

$$\begin{bmatrix} \breve{\Lambda} & \breve{A}_\tau\breve{P}^T & \epsilon_0\breve{B}_0K_0\bar{D}_{20} & \breve{P}(\breve{E}_{10}+\bar{E}_{20}K_0\breve{C}_0)^T & \breve{P}\breve{E}_{10}^T \\ \breve{P}\breve{A}_\tau^T & -\breve{Q} & 0 & \breve{P}\breve{E}_{\tau 0}^T & 0 \\ \epsilon_0\bar{D}_{20}^T K_0^T\breve{B}_0^T & 0 & -\epsilon_0 I_i & 0 & 0 \\ (\breve{E}_{10}+\bar{E}_{20}K_0\breve{C}_0)\breve{P}^T & \breve{E}_{\tau 0}\breve{P}^T & 0 & -\epsilon_0 I_j & 0 \\ \breve{E}_{10}\breve{P}^T & 0 & 0 & 0 & -\epsilon_0 I_j \end{bmatrix} < 0 \tag{30b}$$

with

$$\breve{\Lambda} = (\breve{A}_0+\breve{B}_0K_0\breve{C}_0)\breve{P}^T + \breve{P}(\breve{A}_0+\breve{B}_0K_0\breve{C}_0)^T + \breve{Q} + \epsilon_0\breve{D}_{10}\breve{D}_{10}^T.$$

From $\breve{E} = \begin{bmatrix} I_{r+q} & 0 \\ 0 & 0 \end{bmatrix}$, we know that the matrix $\breve{P}$ satisfying $\breve{E}\breve{P}^T = \breve{P}\breve{E}^T \ge 0$ is of the form $\breve{P} = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ with $P_{11} \in R^{(r+q)\times(r+q)}$. Then we can obtain another main result of this paper in the following theorem.

Theorem 2: The closed-loop system (23) is robustly stable if there exist matrices $\breve{P} = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ with $P_{11} \in R^{(r+q)\times(r+q)}$, $\breve{Q} > 0$ and a scalar $\epsilon_0 > 0$ satisfying the inequality (30b).

Next, we will give the design method for K_0 based on above analysis. (30b) can be represented by

$$\Psi + \Theta^T\Pi\Sigma\Upsilon + \Upsilon^T\Sigma^T\Pi^T\Theta < 0 \tag{31}$$

where,

$$\Psi = \begin{bmatrix} \breve{A}_0\breve{P}^T+\breve{P}\breve{A}_0^T+\breve{Q}+\epsilon_0\breve{D}_{10}\breve{D}_{10}^T & \breve{A}_\tau\breve{P}^T & 0 & \breve{P}\breve{E}_{10}^T & \breve{P}\breve{E}_{10}^T \\ \breve{P}\breve{A}_\tau^T & -\breve{Q} & 0 & \breve{P}\breve{E}_{\tau 0}^T & 0 \\ 0 & 0 & -\epsilon_0 I_i & 0 & 0 \\ \breve{E}_{10}\breve{P}^T & \breve{E}_{\tau 0}\breve{P}^T & 0 & -\epsilon_0 I_j & 0 \\ \breve{E}_{10}\breve{P}^T & 0 & 0 & 0 & -\epsilon_0 I_j \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \breve{B}_0^T & 0 & 0 & \bar{E}_{20}^T & 0 \\ \breve{B}_0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \Pi = \begin{bmatrix} K_0 & 0 \\ 0 & K_0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \breve{C}_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon\bar{D}_{20} & 0 & 0 \end{bmatrix}$$

$$\Upsilon = \begin{bmatrix} \breve{P}^T & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

From Lemma 3, a necessary condition about the existence of Π solving the inequality (31) is

$$W_{\Theta}^{T}\Psi W_{\Theta}<0 \qquad (32a)$$

$$W_{\Sigma}^{T}\Upsilon^{-T}\Psi\Upsilon^{-1}W_{\Sigma}<0 \qquad (32b)$$

In (32a) and (32b), though the controller gain matrix K_0 is eliminated, $\check{P}$ and $\check{P}^{-1}$ are included simultaneously. To get K_0 mathematically efficiently using the LMI Toolbox in Matlab, we can first solve the LMI (32a) for $\check{P} = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}, \check{Q} > 0$ and $\epsilon_0 > 0$. Then substitute the $\check{P}, \check{Q}$ and ϵ_0 into (30b) to get a LMI about K_0. Solve (30b) for $K_0 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & 0 \end{bmatrix}$ and the output dynamic feedback controller (7) is obtained.

IV. CONCLUSIONS

The problems of robust stabilization via observer-based controller and compensator for uncertain singular systems with time delay and parametric uncertainties have been studied. Sufficient conditions about the robust stability of the closed-loop systems are presented. The control laws proposed by using strict LMI approaches can guarantee the resultant closed-loop systems are stable for all admissible uncertainties.

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