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Accounting for uncertainty in anti-windup synthesis

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Abstract— **This paper describes an approach to synthesising anti-windup compensators which can improve the behaviour of systems subject to actuator saturation while also taking into account uncertainty in the system. The class of uncertainty considered is reasonably large and, moreover, is of the type often used in practice and often considered in linear robust control. The development of the ideas makes use of a 'decoupled' representation of an anti-windup scheme which is useful for comparing the results from standard approaches to anti-windup compensation to those compensators obtained using the new robust approach. An interesting, but perhaps not surprising feature of these results is that the often-criticised internal model control (IMC) anti-windup solution emerges as an 'optimally' robust solution.**

I. INTRODUCTION

The problems associated with robustness, or lack thereof, to model uncertainty and the problems associated with actuator saturation have focused the minds of control engineers for over a decade now. Remarkably though, these problems have often been considered in isolation and the authors are only aware of one substantial body of work which attempts to unify some of the results ([1]). On a second look though, perhaps this is not so odd: actuator saturation could be considered, crudely, as model uncertainty and taken into account in the same way as other uncertainty; this could be handled quite routinely in the $\mathscr{H}_\infty$ and μ-synthesis approaches to controller design. Unfortunately, the introduction of another uncertainty into the optimisation problems has the tendency to make the resulting design overly conservative and potentially of low performance.

In contrast, those engineers studying the behaviour of systems subject to actuator saturation have, by and large, chosen to ignore the effect of model uncertainty in their proposals. In anti-windup particularly this has been the case, with the prevailing attitude being to assume that if the nominal linear design is robust, then the anti-windup compensated system will inherit this robustness. This makes some intuitive sense, although it seems more logical to hypthosise that nominal linear robustness is a necessary, but not sufficient, condition for the robustness of the overall anti-windup compensated (nonlinear) system.

The origin of this work has two sources. The first inspiration came from the work of [1] which contains a useful account of pioneering early work on linear systems subject to actuator saturation *and* model uncertainty. In [1] is a collection of papers which address the analysis and synthesis of controllers which, *a priori*, account for actuator saturation and also ensure some degree of robustness for the feedback interconnection. Most of these papers deal with the same type of uncertainty: parametric or state-space uncertainty, generally of the form

$$\dot{x} = Ax + B\text{sat}(u) + \underbrace{\Delta(x,t)}_{\text{uncertainty}} \tag{1}$$

Although this type of parametric uncertainty is certainly useful, in practice it is quite limited in scope and is not very useful for capturing *unmodelled dynamics* which can be more of an obstacle than their modelled, but uncertain, counterparts.

The second point of origin was the paper [2] where some initial ideas were put forward to anti-windup compensation in the presence of model uncertainty. This current paper is a continuation of those ideas, but with more constructive synthesis techniques. It is important to remark that the motivation for this work, and also for that of [2] is practical: in our experience it has really been the unmodelled dynamics which have caused the most difficulty in controller design and, while present, the parametric uncertainty, has played a less prominent role (it is often easily countered by large enough low frequency gain).

The aims of this paper are two-fold. Firstly it aims to bring robustness to the fore in anti-windup compensation, where, except for [3], it has had little prominence. Secondly, this paper aims to promote the use of a type of uncertainty which is closer to that often used in practice (and to that used in linear robust control theory, which has had great success recently).

Notation used in the paper is standard. In particular we define the induced $\mathscr{L}_2$ norm, or finite $\mathscr{L}_2$ gain, of an operator $\mathscr{H}$ as $\|\mathscr{H}\|_{i,2} := \sup_{0\neq x\in\mathscr{L}_2} \frac{\|\mathscr{H}x\|_2}{\|x\|_2}$ where $\|x\|_2 = \sqrt{\int_0^\infty \|x\|^2 dt}$ is the $\mathscr{L}_2$ norm of the vector $x(t)$ and $\|x\|$ is its Euclidean norm. The $\mathscr{H}_\infty$ norm for a linear operator P is defined as $\|P\|_\infty := \sup_\omega \bar{\sigma}[P(j\omega)]$ where $\bar{\sigma}(.)$ denotes the maximum singular value and $P(j\omega)$ is the frequency response matrix associated with the linear operator P. We do not explicitly distinguish between a linear operator and its transfer function. Equivalently, the $\mathscr{H}_\infty$ norm may be defined as $\|P\|_\infty = \|P\|_{i,2}$.

II. A GENERAL ANTI-WINDUP FRAMEWORK

We begin from the scheme introduced in [4], where anti-windup is interpreted as choosing an appropriate transfer function matrix $M(s)$. The scheme is shown in Figure 1 where $G(s) = [G_1(s) \quad G_2(s)]$ is the plant and $K(s) = [K_1(s) \quad K_2(s)]$ is the controller. This can be re-drawn as the decoupled scheme in Figure 2. We have used the fact that the saturation function and the deadzone functions are related by the identity

$$\text{sat}(u) = u - \text{Dz}(u) \tag{2}$$

$$\text{sat}(u) = \begin{bmatrix} \text{sat}_1(u_1) \\ \vdots \\ \text{sat}_m(u_m) \end{bmatrix} \quad \text{Dz}(u) = \begin{bmatrix} \text{Dz}_1(u_1) \\ \vdots \\ \text{Dz}_m(u_m) \end{bmatrix} \tag{3}$$

where $\text{sat}_i(u_i) = \text{sign}(u_i)\min(|u_i|, \bar{u}_i) \quad \forall i$ and $\text{Dz}_i(u_i) = \text{sign}(u_i)\max(0, |u_i| - \bar{u}_i)\forall i$. Also $\bar{u}_i > 0 \quad \forall i \in \{1,\ldots,m\}$.

In [5], it was shown that most anti-windup schemes can be interpreted as certain choices of $M(s)$ and therefore schemes such as the Hanus conditioning scheme ([6]) and the high gain approach ([7], [8]) can be analysed in terms of Figure 2. The advantages of viewing anti-windup in terms of Figure 2 is that the nominal linear performance is separated from the nonlinear part of the scheme and moreover, the stability of the scheme is dependent on the stability of the nonlinear loop. From Figure 2, it can be seen that the performance of the anti-windup compensator is

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intimately related to the mapping $\mathscr{T}_p : u_{lin} \mapsto y_d$: if the norm of this mapping is small, then the anti-windup compensator is successful at keeping performance close to linear (which we assume is the desired performance). In [9] (see also [10], [11]), the $\mathscr{L}_2$ gain of $\mathscr{T}_p$ was minimised using a system of linear matrix inequalities and, furthermore, $M(s)$ was chosen such that it corresponded to static or low order anti-windup compensators. [9] demonstrated, using suitable examples, that direct minimisation of $\mathscr{T}_p$ was central to good anti-windup performance and compensators designed according to the ideas in [9] seemed to perform a least as well, and often better, than most other anti-windup compensators.

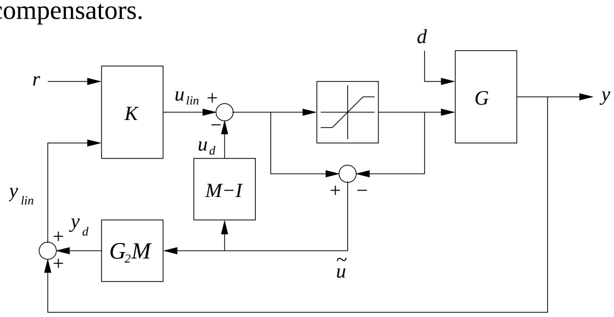

Fig. 1. Conditioning with $M(s)$

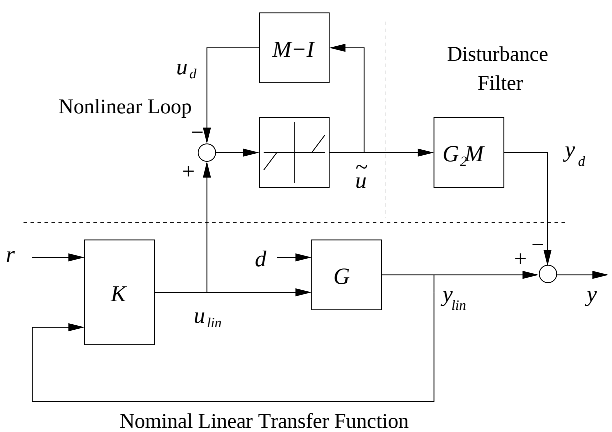

Fig. 2. Equivalent representation of Figure 1

A. With uncertainty

Let us now consider the configuration in Figure 3 where $\tilde{G}$ is the true plant given by $\tilde{G}(s) = [G_1(s) \quad G_2(s) + \Delta_G(s)]$, where $G(s) = [G_1(s) \quad G_2(s)]$ is the model of the plant with which we work and Δ_G is additive uncertainty to the feedback part[1] which we assume is stable and linear. Other types of uncertainty such as output-multiplicative where $\tilde{G}_2(s) = (I + \Delta_o(s))G_2(s)$ and input multiplicative uncertainty $\tilde{G}_2(s) = G_2(s)(I + \Delta_i(s))$ could be used instead. However, it is easy to see that both these uncertainties can be caputured by additive uncertainty ($\Delta_G = \Delta_o G_2$ or $\Delta_G = G_2 \Delta_i$), although the converse is not always true (unless G_2 is invertible), so we prefer to work with additive uncertainty. When uncertainty is present in the system, the appealing decoupled structure of the original scheme is lost. Figure 4 shows an equivalent representation of Figure 3 . Note the term $\Delta_G M : \tilde{u} \mapsto y_\Delta$ destroys the decoupling of the linear system and nonlinear loop.

[1]It is likely that there will also be a perturbation of the disturbance feedforward portion of the plant,G_1, although this will have no bearing on stability, so for simplicity we do not consider it

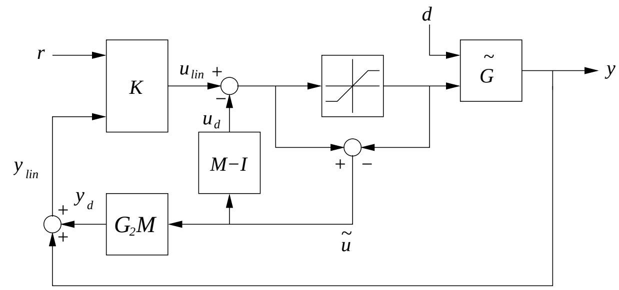

Fig. 3. Anti-windup with uncertainty

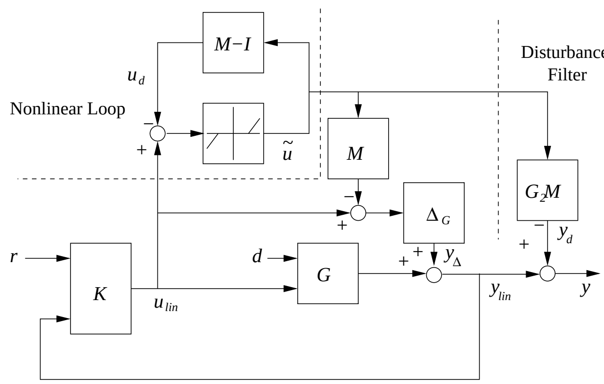

Fig. 4. Equivalent representation of Figure 3

B. Assumptions

1) *The open-loop plant, $G(s)$, is asymptotically stable*. This assumption is, in most cases, necessary for us to obtain global results.
2) *The (linear) uncertainty $\Delta_G(s)$ is asymptotically stable*. This mirrors the case in standard $\mathscr{H}_\infty$ control theory and allows us to use small-gain methods.
3) *The nominal linear closed-loop system is robustly asymptotically stable*. When the saturation nonlinearity is replaced by the identity operator, the closed-loop system is stable and furthermore can tolerate a certain amount of uncertainty ($\|\Delta_G\|_\infty < \gamma$, where $\gamma = \|(I - K_2 G_2)^{-1} K_2\|_\infty$) before becoming unstable. This essentially amounts to assuming that the design of the linear controller $K(s)$ is "good" in the sense that it robustly stabilises the system. We also assume that the nominal linear closed-loop is well-posed.
4) *The nominal linear closed-loop yields desirable performance*. This is a common assumption in the anti-windup literature, and the performance of the anti-windup compensator can be measured against the deterioration of this performance when the control signals saturate. This is related to the foregoing point in the sense that we also assume that the linear closed-loop yields desirable robustness properties and thefore the performance of the anti-windup compensator can also be assessed against its preservation of the linear system's robustness properties.

On the basis of these assumptions three features are evident from Figure 4:

1) If Δ_G is small in some sense, then the robustness of the anti-windup scheme is similar to that of the nominal, unconstrained linear system (via a Small Gain argument).
2) If the mapping from $u_{lin} \mapsto M\tilde{u}$ is small, again, the robustness of the anti-windup system is similar to that

of the nominal linear system. (again using a Small gain argument). So in other words the map $u_{lin} \mapsto M\tilde{u}$ contains important robustness information.

3) The robustness of the system with anti-windup compensation can never be better than the robustness of the linear system. Denoting the 'modified' uncertainty $\tilde{\Delta}_G : u_{lin} \mapsto y_\Delta$, this follows by noting that $\|\Delta_G\|_\infty = \|\Delta_G\|_{i,2} \leq \|\tilde{\Delta}_G\|_{i,2}$ (by using a contradiction argument). So, in a sense, the retention of the linear system's robustness can be considered as an optimal property (discussed in more detail later).

III. SPECIAL CASE: IMC ANTI-WINDUP

Before we explore the consequences of uncertainty in anti-windup further, it is interesting to consider a special case: the much-maligned IMC anti-windup scheme. IMC anti-windup was introduced in [12] as an anti-windup methodology but many examples have shown it to be a poorly performing anti-windup scheme (e.g. [13]). This can be easily seen by viewing IMC anti-windup in Figure 2: to obtain IMC-anti-windup we simply choose $M = I$. The nonlinear 'loop' becomes simply the deadzone operator, and the disturbance filter becomes the open-loop plant. Hence the IMC performance will be poor if the open-loop plant has lightly damped modes or nonminimum phase zeros.

As is often the case in linear control theory, there is a trade-off between performance and robustness and this seems to extend to anti-windup compensation. Consider uncertain anti-windup in Figure 4 and choose $M = I$, then again the nonlinear loop degenerates to the deadzone function and the troublesome term, which destroys the decoupling of the linear and nonlinear parts of the system, simply becomes the uncertainty, Δ_G. This scenario is redrawn in Figure 5. For consistency we have retained the notation y_{lin} and u_{lin}, although it should be understood that these signals are no longer generated by a purely linear system. Assuming no saturation, simple small gain analysis

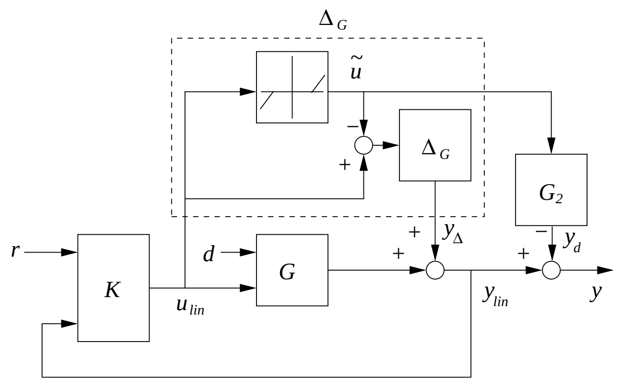

Fig. 5. IMC anti-windup with uncertainty

shows that we have stability robustness against all input additive uncertainty

$$\|\Delta_G\|_\infty < \frac{1}{\gamma} \tag{4}$$

where $\|(I - K_2G_2)^{-1}K_2\|_\infty := \gamma$. Carrying out a small gain analysis on Figure 5 we see we have stability providing that the 'modified' nonlinear uncertainty $\tilde{\Delta}_G$ satisfies

$$\|\tilde{\Delta}_G\|_{i,2} < \frac{1}{\gamma} \tag{5}$$

But as

$$\begin{aligned} \|\tilde{\Delta}_G\|_{i,2} &\leq \|\Delta_G\|_\infty \|I - Dz(.)\|_{i,2} = \|\Delta_G\|_\infty \|\text{sat}(.)\|_{i,2} && (6) \\ &= \|\Delta_G\|_\infty && (7) \end{aligned}$$

we see that we have stability robustness to all uncertainty satisfying inequality (4). However, as we also have $\|\Delta_G\|_\infty \leq \|\tilde{\Delta}_G\|_{i,2}$ we must have that $\|\tilde{\Delta}_G\|_{i,2} = \|\Delta_G\|_\infty$. In other words, the IMC anti-windup scheme is *guaranteed to be robustly stable for the same class of additive uncertainties as the nominal linear system*. Recall, that it is not possible for an anti-windup scheme to be more robust than the nominal linear system because much of the anti-windup scheme's time is spent operating as a linear system. So, in a sense, the retention of the linear system's robustness properties is optimal. Hence, although IMC schemes can be criticised for their performance, they are in fact optimally robustly stable!

IV. GENERAL CASE

A. A stability robustness criterion

From Figure 4, we have that

$$y_{lin} = G_1d + G_2u_{lin} + \Delta_G[u_{lin} - M\mathscr{F}(u_{lin})] = G_1d + G_2u_{lin} + \tilde{\Delta}_G(u_{lin}) \tag{8}$$

where $\mathscr{F}(u_{lin})$ is the map from u_{lin} to $\tilde{u}$. Carrying out a small gain analysis we see that the system is robust against all additive perturbations such that

$$\|\tilde{\Delta}_G\|_{i,2} = \|\Delta_G[I - M\mathscr{F}(u_{lin})]\|_{i,2} < \frac{1}{\gamma} \tag{9}$$

So nominal robustness is retained if $\|[I - M\mathscr{F}(u_{lin})]\|_{i,2} \leq 1$. However as $\mathscr{F}(u_{lin}) = 0$ around $u_{lin} = 0$ (as it contains the deadzone), $\|[I - M\mathscr{F}(u_{lin})]\|_{i,2}$ can never be strictly less than unity. Again this conclusion coincides with our prior discussion as we could not expect an anti-windup scheme to yield greater robustness margins than the linear system upon which it is constructed. This also serves as justification for the IMC scheme, although it is unlikely to be the *unique* compensator which achieves this optimality.

B. Stability robustness optimisation

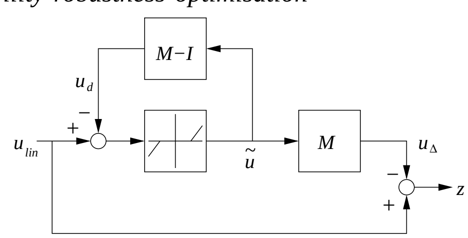

Fig. 6. Robustness optimisation for general anti-windup schemes: graphical representation of $\mathscr{T}_r$

Here, we consider robustness optimisation using full-order anti-windup compensators. As in [5] we choose $M(s)$ to be part of a right-coprime factorisation of $G_2(s) = N(s)M^{-1}(s)$ (this is a dual result to [14], where anti-windup is described as a left coprime factorisation of the controller) and attempt to choose a particular factorisation such that robustness is optimised.

To preserve as much robustness as possible we would like to minimise $\|\mathscr{T}_r\|_{i,2} := \|[I - M\mathscr{F}(u_{lin})]\|_{i,2}$. This can be shown as Figure 6 where we want to minimise the $\mathscr{L}_2$ gain from u_{lin} to z. As mentioned in [9], this optimisation is typically a difficult problem to solve, so instead we seek to ensure a certain $\mathscr{L}_2$ gain bound holds for the map $\mathscr{T}_r$.

Given a nominal plant realisation

$$G_2 \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] \tag{10}$$

a full-order right coprime factorisation can be described as

$$\begin{bmatrix} M \\ N \end{bmatrix} \sim \left[\begin{array}{c|c} A_p+B_pF & B_p \\ \hline F & I \\ C_p+D_pF & D_p \end{array}\right] \tag{11}$$

where F is chosen such that A_p+B_pF is a Hurwitz matrix.

To ensure robustness, that is to ensure that $\|\mathscr{T}_r\|_{i,2} < \gamma$, it suffices for the following inequality to hold for sufficiently small γ:

$$J = \frac{d}{dt}x'Px + \|z\|^2 - \gamma^2\|u_{lin}\|^2 < 0, \quad x,z,u_{lin} \neq 0 \tag{12}$$

where x is the state vector associated with the realisation of $[M-I, M]$. As shown in, for example [15], [9] this ensures that the $\mathscr{L}_2$ gain from u_{lin} to z is less than γ and that the system in Figure 6 is asymptotically stable.

The deadzone nonlinearity belongs to the Sector $[0,I]$ (see [16], chapter 10), so we make use of

$$2\tilde{u}'W(u_{lin} - u_d - \tilde{u}) \geq 0 \tag{13}$$

where $W > 0$ is a diagonal matrix, to form

$$\tilde{J} = \frac{d}{dt}x'Px + \|z\|^2 - \gamma^2\|u_{lin}\|^2 + 2\tilde{u}'W(u_{lin} - u_d - \tilde{u}) \tag{14}$$

If $\tilde{J} < 0$, this implies that $J < 0$. Evaluating $\tilde{J}$ in a similar manner to [9] (see also [10], [11] for the discrete-time cases) yields the following LMI

$$\begin{bmatrix} QA_p' + A_pQ + L'B_p' + B_pL & B_pU - L' & 0 & L' \\ \star & -2U & I & U \\ \star & \star & -\mu I & -I \\ \star & \star & \star & -I \end{bmatrix} < 0 \tag{15}$$

in $Q > 0, U = W^{-1} = \mathrm{diag}(\nu_1, \ldots, \nu_m) > 0, L, \mu > 0$.

Satisfaction of this LMI means that inequality (14) is satisfied and hence that the $\mathscr{L}_2$ gain from u_{lin} to z is less than $\gamma = \sqrt{\mu}$ and a suitable choice of F is given by $F = LQ^{-1}$. From the $\begin{bmatrix} -\mu I & -I \\ \star & -I \end{bmatrix}$ term of this LMI we can see that, as anticipated earlier, the $\mathscr{L}_2$ gain can be no less than unity, which is achieved for the IMC scheme.

C. Optimisation for robustness and performance

The primary goal of anti-windup compensation is to provide performance improvement during saturation, but optimising the LMI (15) alone does not guarantee this. Indeed, there is little point in optimising (15) when an optimal solution can be found by inspection as the IMC anti-windup solution. The real use of (15) and the arguments of the previous subsection is to use them in conjuction with performance optimisation, the goal being to optimise performance and robustness together, although there will often be a trade-off.

In [9] it was argued that $\mathscr{T}_p$, the map from u_{lin} to y_d was central to the "true goal" of anti-windup compensation: if the induced norm of this operator was minimised, the deviation of the system's nonlinear behaviour during and after saturation would be minimised. The paper [9] solved this problem with $\Delta_G = 0$, in the $\mathscr{L}_2$ sense, for static and low order compensators (see also [10], [11]).

Realistically, we would really like to optimise some weighted combination of $\mathscr{T}_p$ and $\mathscr{T}_r$. This can be accomplished by solving the LMI.

$$\begin{bmatrix} QA_p' + A_pQ + L'B_p' + B_pL & B_pU - L' & 0 & QC_p' + L'D_p' & L' \\ \star & -2U & I & UD_p' & U \\ \star & \star & -\mu I & 0 & -I \\ \star & \star & \star & -W_p^{-1} & 0 \\ \star & \star & \star & \star & -W_r^{-1} \end{bmatrix} < 0 \tag{16}$$

in the variables $Q > 0, U = \mathrm{diag}(\nu_1, \ldots, \nu_m) > 0, L, \mu > 0$. W_p and W_r are positive definite weighting matrices which reflect the relative importance of performance and robustness respectively and are chosen by the designer. The derivation of this LMI is carried out in a similar way to that of the previous section in the spririt of that done in [9].

Remark 1: Throughout this paper, we have only discussed full-order anti-windup compensation for two reasons: (i) A full-order anti-windup compensator always exists, and (ii) the expressions and derivations of formulae for static and low-order anti-windup compensators are more complex, although the same ideas are certainly applicable to these types of compensator. □□

Remark 2: Another advantage of using LMI (16) to synthesise full-order compensators is that it tends to prevent fast poles appearing in the compensator dynamics. If a robustness weight (W_r) was not included in the optimisation - or if W_r was only chosen small - the poles of the anti-windup compensator tend to be rather fast, lying far to the left of the imaginary axis. Obviously this would require a very high sampling frequency for implementation, which is not always possible in practice. However, when simultaneously optimising performance and robustness using (16), the poles are placed in regions more comparable to that of the controller. This feature is reminiscent of solving 'singular' $\mathscr{H}_\infty$ problems with LMI's, where poles tend to get placed far from the imaginary axis. □□

D. Stability robustness of the work in [10]

The work in [10], [9] advocates only the optimisation of anti-windup performance, that is the minimisation of the $\mathscr{L}_2$ gain of the operator $\mathscr{T}_p$. By setting $W_r = 0$ and solving the LMI (16) we obtain a full-order compensator which only optimises this performance. As argued in [9], [10], this operator is central to obtaining desirable anti-windup behaviour. It is interesting to examine whether this approach has any intrinsic robustness properties.

Suppose that we consider output multiplicative uncertainty instead of additive uncertainty, that is $\tilde{G}(s) = (I + \Delta_o(s))G_2(s)$, or equivalently that $\Delta_G = \Delta_o G_2$. Now, our expression for y_{lin} becomes

$$y_{lin} = G_1 d + G_2 u_{lin} + \Delta_o G_2 (I - M\mathscr{F}(u_{lin}))u_{lin} \tag{17}$$

We are sure that the system is robustly stable when $\mathscr{F}(u_{lin}) = 0$ as this is a property of the nominal linear system. Therefore the smaller we can make the extra term $-\Delta_o G_2 M\mathscr{F}(u_{lin})$ the closer to nominal robustness we shall be. We can do nothing about Δ_o, so the logical approach is to make

$$\|G_2 M\mathscr{F}(u_{lin})\|_{i,2} = \|N\mathscr{F}(u_{lin})\|_{i,2} \tag{18}$$

as small as possible. As in [10], because $G_2(s) = N(s)M^{-1}(s)$ is a right coprime factorisation of the plant $G_2(s)$, the quantity in equation (18) is exactly our performance operator norm $\|\mathscr{T}_p\|_{i,2}$. Therefore the minimisation of $\|\mathscr{T}_p\|_{i,2}$ leads not only to desirable anti-windup performance, but it also endows the saturated system with some indirect robustness when the the uncertainty is of the output multiplicative type. However, the robustness is not *guaranteed* to approach that of the linear system (that is, the robustness of the system with 'robust' anti-windup is not guaranteed to be as great as the robustness of the unconstrained linear system). Nevertheless, it does appear to explain some of the results of [17] where a discrete-time version of the results of [9] were implemented on a hard-disk system. In that work, few robustness problems were encountered and the above analysis goes some way to explaining this.

V. EXAMPLE

To demonstrate the implications of our results we use an example introduced in [18]. The example consists of a plant with a large resonant peak and the controller used is a two-degree-of-freedom controller with large feedback gain. However, we shall take this plant to be the *perturbed* plant, $\tilde{G}(s)$ rather than our nominal plant. We shall also use a controller with a slightly lower gain, for reasons which shall become clear later.

A. The unperturbed system

For our nominal plant we take the example of [18] without the resonant peak (i.e the system is critically damped). Thus, $G_2(s) \sim (A_p, B_p, C_p, D_p)$ is described by the state-space matrices:

$$A_p = \begin{bmatrix} 0 & 1 \\ -10 & -10 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_p = [0] \quad (19)$$

The linear controller $K(s) = [K_1(s) \quad K_2(s)] \sim (A_c, [B_{cr} \quad B_c], C_c, [D_{cr} \quad D_c])$, which was designed for the plant $G(s)$, is described by the state-space matrices

$$A_c = \begin{bmatrix} -80 & 0 & 2.5 \\ 1 & 0 & 0 \\ 0 & 0 & -2.5 \end{bmatrix}, B_{cr} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_c = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -9450 & 3375 & 337.5 \end{bmatrix}, D_{cr} = [0], D_{cy} = [-135]$$

This is the same controller as in [18] but with a lower gain in the feedback loop. The dashed line in Figure 7 shows the response of the linear system to a pulse input of magnitude 1.2; the system is well behaved with no overshoot and a fast settling time. When the control input is saturated at ± 1 however, the system degrades to that shown by the dash-dotted line in Figure 7; the system still exhibits no overshoot, but actuator saturation has impaired the system's ability to track it's reference signal accurately, the reference and response being out of phase.

To improve the behaviour of the system, static anti-windup compensation as suggested in [9] is introduced. This anti-windup compensation minimises the difference between the nominal linear system's response and the saturated system's response and was synthesised as

$$\Theta = \begin{bmatrix} -0.1909 \\ 0.1402 \end{bmatrix}, \quad \|\mathscr{T}_p\|_{i,2} < \gamma \approx 24 \quad (20)$$

A diagram of how this compensator is implemented is given in Figure 8. Note that the unperturbed system is quadratically stabilisable by static anti-windup compensation, but we cannot be sure that the perturbed system also has this desirable property (in fact it does not). The solid line in Figure 7 shows the response of the system with static anti-windup: the response has improved and the system output is now in-phase with the reference demand, although the infeasibility of the reference means it is not possible for the output to track the input with the correct magnitude. The robust and full order anti-windup compensators introduced in the following sections both yield a similar response to that in Figure 7 when used on the nominal system $G(s)$.

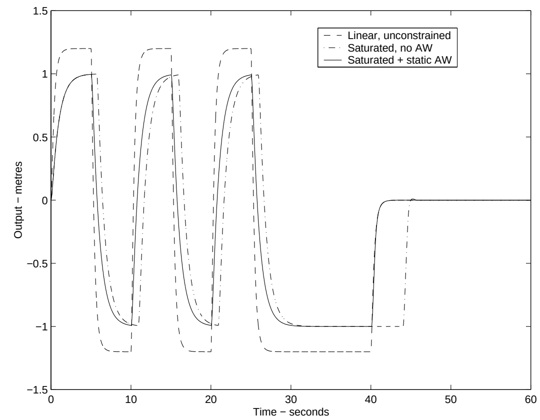

Fig. 7. Response of unperturbed system

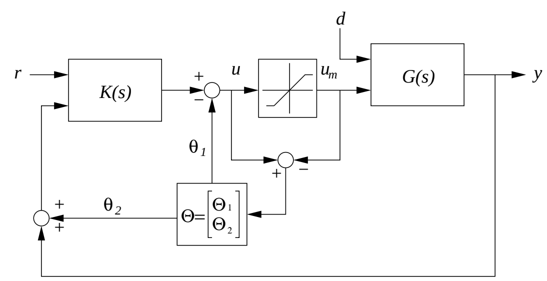

Fig. 8. Structure of static anti-windup scheme

B. The perturbed system

The true, or perturbed, plant, $\tilde{G}(s) = G(s) + \Delta_G(s)$ is the plant given in [18]. This has a large resonnant peak and is described by the following state-space matrices:

$$\tilde{A}_p = \begin{bmatrix} 0 & 1 \\ -10 & -0.01 \end{bmatrix}, \tilde{B}_p = B_p, \tilde{C}_p = C_p, \tilde{D}_p = D_p \quad (21)$$

The dashed line in Figure 9 shows the response of this perturbed plant using the same controller as before; the controller yields a similar type of performance to before and hence can be considered satisfactory. However, when input saturation is introduced, the static anti-windup compensator actually drives the system unstable as depicted by the solid line in Figure 9. In fact, this static anti-windup is worse than *no anti-windup at all*, which at least remains stable. Note that for this perturbed plant and controller, static anti-windup is not feasible, so we cannot expect it to stabilise the system in question. To overcome this problem, we choose $W_p = 0.001$ and $W_r = 1$ and we synthesise a robust dynamic compensator according to the LMI (16) . This yields the matrix F as

$$F = [0.2242 \quad 0.0446] \times 10^{-4}, \quad \gamma \approx 1 \quad (22)$$

In this case we have essentially the IMC solution, as F is almost zero. As $\gamma \approx 1$ we can expect to recover the robustness results of the linear system. Figure 10 shows the system's response; the system is stable and although the response is not as good as the unperturbed system, it is substantially better than that of the static anti-windup compensation. Note that this robust anti-windup compensation also performed as well as the optimal static anti-windup compensation when applied to the unperturbed plant.

C. Other anti-windup compenators

As discussed in Section IV-D the full-order anti-windup compensation method obtained by setting $W_r = 0$ and solving the LMI (16) can, in a certain sense, provide a robust

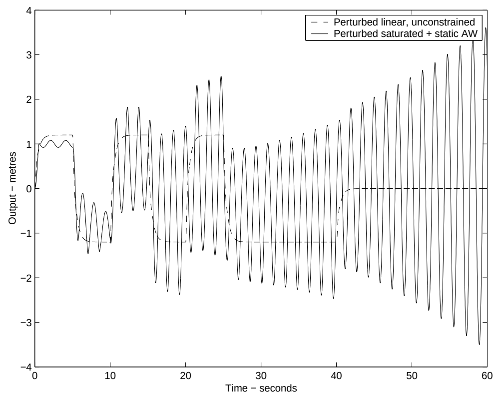

Fig. 9. Response of perturbed system with optimal static anti-windup

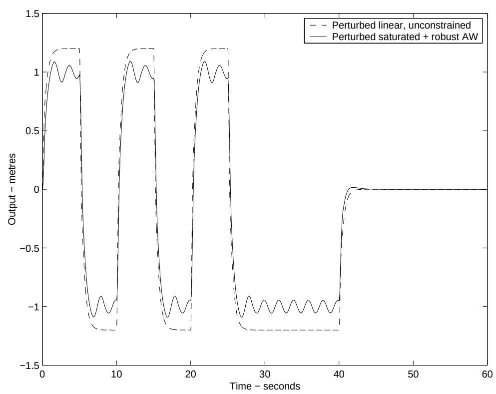

Fig. 10. Response of perturbed system with robust anti-windup

anti-windup solution. This type of anti-windup solution is the continuous time counterpart of the discrete-time full-order compensator described in [10]. We designed a full-order anti-windup compensator according to this approach, choosing $W_p = 1$ and $W_r = 0.0001 \approx 0$ and then solving the LMI (16). The optimal gain matrix F was given by

$$F = [-1.3138 \quad -0.1424] \times 10^4, \quad \gamma \approx 1 \qquad (23)$$

It is important to realise that in this case the *guaranteed* robustness margin of the system is now given by $\sqrt{\mu/W_r} \approx 100$, although this appears to be a conservative estimate. Again note that this places the poles of the anti-windup far into the left half plane and, therefore, this would require a fast sampling frequency for correct implementation. Figure 11 shows the response of the saturated system using this compensator; the stable response is indicative of the scheme's intrinsic robustness properties, although it does appear to be more oscillatory than that of the robust anti-windup compensator.

VI. CONCLUSION

This paper has introduced a framework for synthesising robust anti-windup compensators for open-loop systems subject to additive uncertainties. The problem was posed in a similar way to that of linear $\mathscr{H}_\infty$ control theory and the solution which was proposed appears as a set of LMI's of a similar type to those proposed in [9]. The attractive feature of the proposed solution is that the class of uncertainties considered are those which are routinely considered by control practitioners. As an important aside, we have also demonstrated the optimal robustness of the much denigrated IMC anti-windup strategy.

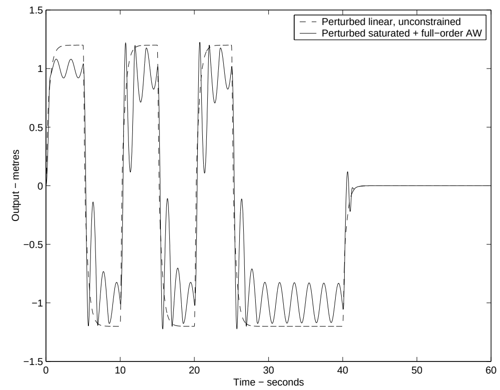

Fig. 11. Response of perturbed system with full-order anti-windup

We note that many simple anti-windup schemes also seem to be quite robust in practice. For example the Hanus scheme ([6]) has been the practitioners method of choice for some time (see [8], [19]) although it is not so easy to prove this theoretically as the Hanus scheme is only globally *stable* for a small class of systems, so proving that it is globally *robustly stable*, generally, is impossible. However, it seems likely that the Hanus scheme could be examined with respect to something which might be described as *local robust stability*.

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