Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A Feasible Two-Stage LQ Reliable Control Via Partial Actuator Failures Estimation

Chien-Shu Hsieh

Department of Electrical Engineering, Ta Hwa Institute of Technology, Chiunglin, Hsinchu 30740, Taiwan, R.O.C.

Abstract

This paper introduces a feasible version of the previously proposed two-stage LQ reliable control (TSLQRC) in order to overcome the infeasible problem encountered in the control structure of the TSLQRC in which the controller has exactly known the faults in the preselected set of actuators. A modified unified gain margin constraint and a parameter estimating technique are proposed to facilitate the design. Analytical and simulated results show that the proposed feasible TSLQRC serves as a practical implementation of the TSLQRC.

1 Introduction

Reliable controller designs guaranteeing stability while permitting control component failures (e.g., actuator failures and/or sensor failures) have received great attentions in the literature [1]-[9]. Among these, three approaches have been used to facilitate the design. One is the redundant control in which the reliability of control structures is guaranteed by using multiple controllers [1, 6]. The second is the reliable control where the designs will tolerate failures within a preselected set of actuators or sensors, while maintaining stability and a known performance bound [2, 3, 4, 8]. All of the abovementioned works are focused on an H_∞ framework. On the other hand, Veillette [5] presented a reliable LQ design approach, and Yang *et al.* [7] also presented a discrete-time LQ reliable guaranteed cost controller. The last is the adaptive control which is capable of dealing with systems for unknown actuator failures (see [9] and the references therein). It should be stressed that most of the aforementioned works, except [7], are developed for continuous-time systems. However, in this paper we focus on the extension of Veillette's reliable LQ design results [5] to discrete-time systems.

To extend the stability/performance gain margin properties of Veillette's reliable LQ regulator [5] to discrete-time systems, the author has successfully proposed the two-stage LQ reliable control (TSLQRC) [11]. It was further shown in [12] that the TSLQRC also serves as a reliable guaranteed cost control which guarantees the performance cost to be within a certain bound. However, the gain margin results of the TSLQRC may be too restricted to be applied for a certain system, and the upper and lower bounds of the considered failure model of actuators are assumed to be in the same distance with the nominal condition. In [13], a modified version of the TSLQRC, i.e., MTSLQRC, was proposed to relax the above restrictions. Furthermore, the gain margin issues of the MTSLQRC were addressed in [15]. Nevertheless, it should be stressed that the control structure of the TSLQRC may encounter the following infeasible problem: the controller presumes a need of feedback of the actuator faults which are associated with the selected subset of unreliable actuators. This may disturb the TSLQRC in further development.

The main aim of this paper is to propose a feasible version of the TSLQRC in order to overcome the aforementioned infeasible problem encountered in the original control structure of the TSLQRC. The paper is organized as follows. The problem of interest and a review of the TSLQRC are given in Section 2. In Section 3, the derivation of the feasible TSLQRC is presented. A modified unified gain margin constraint is proposed to facilitate the design. The gain margin issues of the feasible TSLQRC are then addressed in Section 4. The issue of determining the design parameters of the proposed feasible TSLQRC is given in Section 5. A numerical example is given in Section 6 to illustrate the usefulness of the proposed results. Section 7 has some concluding remarks.

2 Problem Statement and Preliminaries

Consider the following discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

where $x_k \in R^n$ is the system state, $u_k \in R^m$ is the control input whose components may fail during system operation, and matrices A and B are known constant matrices. The quadratic performance index associated with the system is given by

$$J = \sum_{k=0}^{\infty}(x_k' Q x_k + u_k' R u_k) \tag{2}$$

where $Q \geq 0$ and $R > 0$ are given weighting matrices and $'$ denotes transpose.

In a fault-tolerant control system design, the standard optimal control may not be the most suitable one in the sense that it does not generally tolerate the complete outage of any actuators. On the other hand, the reliable LQ regulator of Veillette [5] seems to be an attractive mean to guarantee system's stability and performance. However, Veillette's results are derived mainly for continuous-time systems. In [11], the author has proposed the TSLQRC to extend the Veillette's reliable LQ regulator to discrete-time systems where the failed control input u_k^F is expressed as follows:

$$u_k^F = -\begin{bmatrix} N_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}(I - B_\Omega N_\Omega \bar{K}_\Omega) \\ N_\Omega \bar{K}_\Omega \end{bmatrix} A x_k. \quad (3)$$

In (3), $\bar{K}_{\bar{\Omega}}$ and $\bar{K}_\Omega$ are the designed controller gain matrices which are given as follows:

$$\begin{aligned} \bar{K}_{\bar{\Omega}} &= S_{\bar{\Omega}}^{-1} B_{\bar{\Omega}}' P, \quad S_{\bar{\Omega}} = B_{\bar{\Omega}}' P B_{\bar{\Omega}} + R_{\bar{\Omega}} && (4) \\ \bar{K}_\Omega &= S_\Omega^{-1} B_\Omega' \bar{P}, \quad S_\Omega = B_\Omega' \bar{P} B_\Omega + R_\Omega && (5) \\ \bar{P} &= P(I - B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}) && (6) \\ P &= A' P (I - B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}) A + Q && (7) \end{aligned}$$

$N_{\bar{\Omega}}$ and N_Ω are their corresponding gain perturbation matrices, and B and R are partitioned as follows: $B = \begin{bmatrix} B_{\bar{\Omega}} & B_\Omega \end{bmatrix}$ and $R = diag\{R_{\bar{\Omega}}, R_\Omega\}$. Note that in the above formulations, Ω denotes the selected subset of unreliable actuators within which outages must be tolerated while $\bar{\Omega}$ denotes the complementary subset of actuators within which actuator outages are not taken into account by the design. Also note that the actuator failure considered in this paper is given as follows [5]:

$$\bar{K}_{\bar{\Omega}} \to N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}, \qquad \bar{K}_\Omega \to N_\Omega \bar{K}_\Omega. \quad (8)$$

The key task of the considered LQ reliable control design problem is to determine the admissible gain perturbation matrices, i.e., $N_{\bar{\Omega}}$ and N_Ω, such that in the presence of actuator failures in (8), the performance index (2) is still bounded. As given in [12], the considered problem is solved if the gain perturbation matrices satisfy the following unified gain margin constraint:

$$\bar{K}_\Omega' D_\Omega \bar{K}_\Omega - \bar{U}' \bar{K}_{\bar{\Omega}}' D_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \bar{U} + a\tilde{Q} \geq 0 \quad (9)$$

where $a \in R$,

$$\begin{aligned} D_\Omega &= S_\Omega - (I - N_\Omega)' S_\Omega (I - N_\Omega) && (10) \\ D_{\bar{\Omega}} &= (I - N_{\bar{\Omega}})' S_{\bar{\Omega}} (I - N_{\bar{\Omega}}) && (11) \\ \tilde{Q} &= \bar{U}' \bar{K}_{\bar{\Omega}}' N_{\bar{\Omega}}' R_{\bar{\Omega}} N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \bar{U} + \bar{K}_\Omega' N_\Omega' R_\Omega N_\Omega \bar{K}_\Omega && (12) \\ \bar{U} &= I - B_\Omega N_\Omega \bar{K}_\Omega. && (13) \end{aligned}$$

Through (9), the stability and performance gain margin results of Veillette [5] may be seen as limiting cases of the presented results in case that the sampling time is considered to be sufficiently small.

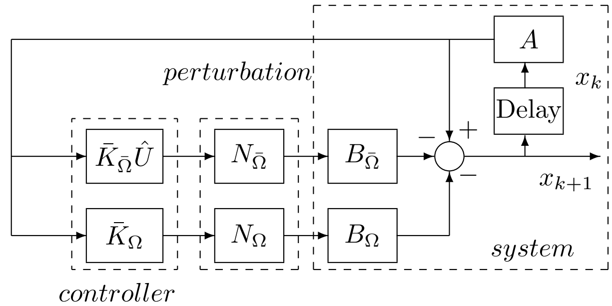

Figure 1: Block diagram of the feasible TSLQRC

Unfortunately, the control structure of the TSLQRC may encounter the following infeasible problem: the controller presumes a need of feedback of the actuator faults associated with the set Ω. The main aim of this paper is to propose a feasible version of the TSLQRC in order to overcome the aforementioned restriction.

3 The Feasible TSLQRC

In the following discussions, the perturbation gain matrices $N_{\bar{\Omega}}$ and N_Ω are assumed as follows:

$$\begin{aligned} N_{\bar{\Omega}} &= diag\{n_{\bar{\Omega}}^1, n_{\bar{\Omega}}^2, \cdots, n_{\bar{\Omega}}^p\}, \ \underline{n}_{\bar{\Omega}}^i \leq n_{\bar{\Omega}}^i \leq \bar{n}_{\bar{\Omega}}^i && (14) \\ N_\Omega &= diag\{n_\Omega^1, n_\Omega^2, \cdots, n_\Omega^{m-p}\}, \ \underline{n}_\Omega^j \leq n_\Omega^j \leq \bar{n}_\Omega^j && (15) \end{aligned}$$

where $0 < \underline{n}_{\bar{\Omega}}^i \leq 1$, $\bar{n}_{\bar{\Omega}}^i \geq 1$, $0 \leq \underline{n}_\Omega^j \leq 1$, and $\bar{n}_\Omega^j \geq 1$. In order to achieve the aim of the paper, the author proposes the following feasible TSLQRC:

$$u_k = -\begin{bmatrix} \bar{K}_{\bar{\Omega}} \hat{U} \\ \bar{K}_\Omega \end{bmatrix} A x_k \quad (16)$$

where $\hat{U}$, $\bar{K}_{\bar{\Omega}}$, and $\bar{K}_\Omega$ are calculated as follows:

$$\begin{aligned} \hat{U} &= I - B_\Omega \hat{N}_\Omega \bar{K}_\Omega && (17) \\ \bar{K}_{\bar{\Omega}} &= (S_{\bar{\Omega}} \Phi_{\bar{\Omega}})^{-1} B_{\bar{\Omega}}' P && (18) \\ \bar{K}_\Omega &= (S_\Omega \Phi_\Omega)^{-1} B_\Omega' \bar{P} && (19) \\ \bar{P} &= P(I - B_{\bar{\Omega}} \Phi_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}) = P - P B_{\bar{\Omega}} S_{\bar{\Omega}}^{-1} B_{\bar{\Omega}}' P && (20) \\ P &= A' P (I - B_{\bar{\Omega}} \Gamma_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}) A + Q && (21) \end{aligned}$$

in which $\hat{N}_\Omega$, $\Phi_{\bar{\Omega}}$, Φ_Ω, and $\Gamma_{\bar{\Omega}}$ are design parameters, which will be determined in the next section. Note that the controller gain matrices of the above feasible TSLQRC adopt the forms given in [13]. The structure of the proposed feasible TSLQRC in a practical faulty system is depicted in Fig. 1. Thus, the failed control input u_k^F (3) becomes

$$u_k^F = -\begin{bmatrix} N_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}(I - B_\Omega \hat{N}_\Omega \bar{K}_\Omega) \\ N_\Omega \bar{K}_\Omega \end{bmatrix} A x_k. \quad (22)$$

Using (22) in (1) yields the following system dynamics matrix:

$$F = \bar{F} + B_\Omega(\hat{N}_\Omega - N_\Omega)\bar{K}_\Omega A \quad (23)$$

where

$$\bar{F} \;=\; (I - B_{\bar{\Omega}} N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}})\hat{U} A. \quad (24)$$

Using (23) and (24), the main result of this paper, which shows the general performance gain margin property of the feasible TSLQRC, is given in the following theorem.

Theorem 1. *The state-feedback system obtained by applying the feasible TSLQRC (16)-(21) satisfies the performance bound*

$$J \leq x_0'(P + a\tilde{P})x_0 \quad (25)$$

where x_0 is the initial state, P is the stabilizing solution of (21), $a \in R$, and $\tilde{P}$ is the stabilizing solution of the following equations:

$$\begin{aligned} \tilde{P} &= F'\tilde{P}F + A'\bar{Q}A & (26)\\ \bar{Q} &= \hat{U}'\bar{K}_{\bar{\Omega}}' N_{\bar{\Omega}}' R_{\bar{\Omega}} N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \hat{U} + \bar{K}_\Omega' N_\Omega' R_\Omega N_\Omega \bar{K}_\Omega & (27)\end{aligned}$$

if the gain matrices $N_{\bar{\Omega}}$ and N_Ω satisfy the following modified unified gain margin constraint:

$$\begin{aligned} &\bar{K}_\Omega' D_\Omega^\Phi \bar{K}_\Omega + \bar{K}_{\bar{\Omega}}' \Phi_{\bar{\Omega}}' S_{\bar{\Omega}}(\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}})\bar{K}_{\bar{\Omega}} + a\bar{Q}\\ \geq\;& \hat{U}'\bar{K}_{\bar{\Omega}}' D_{\bar{\Omega}}^\Phi \bar{K}_{\bar{\Omega}} \hat{U} + \bar{K}_\Omega' E_1 \bar{K}_\Omega + E_2 & (28)\end{aligned}$$

where

$$\begin{aligned} D_\Omega^\Phi &= \Phi_\Omega' S_\Omega \Phi_\Omega - (\Phi_\Omega - N_\Omega)' S_\Omega (\Phi_\Omega - N_\Omega) & (29)\\ D_{\bar{\Omega}}^\Phi &= (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}})' S_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}}) & (30)\\ E_1 &= (\hat{N}_\Omega - N_\Omega)' B_\Omega' \bar{K}_{\bar{\Omega}}' (N_{\bar{\Omega}}' S_{\bar{\Omega}} N_{\bar{\Omega}} - D_{\bar{\Omega}}^\Phi)\\ &\quad \times \bar{K}_{\bar{\Omega}} B_\Omega (\hat{N}_\Omega - N_\Omega) & (31)\\ E_2 &= \bar{K}_\Omega' (\hat{N}_\Omega - N_\Omega)' B_\Omega' P B_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}}) \bar{K}_{\bar{\Omega}} \bar{U}\\ &\quad + \bar{U}' \bar{K}_{\bar{\Omega}}' (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}})' B_{\bar{\Omega}}' P B_\Omega (\hat{N}_\Omega - N_\Omega) \bar{K}_\Omega. & (32)\end{aligned}$$

Proof. First, we note that the performance index (2), in the presence of actuator failures, can be represented as follows:

$$J = \sum_{k=0}^{\infty}\{x_k'(Q + A'\bar{Q}A)x_k\}. \quad (33)$$

Then, the key point to prove the theorem is to show the following equation:

$$\begin{aligned} &F'(P + a\tilde{P})F - (P + a\tilde{P}) + Q + A'\bar{Q}A\\ =\;& -A'(\bar{K}_{\bar{\Omega}}' \Phi_{\bar{\Omega}}' S_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}}) \bar{K}_{\bar{\Omega}} + \{\bullet\} + a\bar{Q})A & (34)\end{aligned}$$

where

$$\{\bullet\} = \bar{K}_\Omega'(D_\Omega^\Phi - E_1)\bar{K}_\Omega - \hat{U}'\bar{K}_{\bar{\Omega}}' D_{\bar{\Omega}}^\Phi \bar{K}_{\bar{\Omega}} \hat{U} - E_2. \quad (35)$$

Using the same procedures as given in deriving (34) of [13], one obtains

$$\bar{F}'P\bar{F} + A'\bar{Q}A = P - Q - A'\Psi A \quad (36)$$

where

$$\begin{aligned} \Psi &= \bar{K}_\Omega' D_\Omega^\Phi \bar{K}_\Omega - \hat{U}'\bar{K}_{\bar{\Omega}}' D_{\bar{\Omega}}^\Phi \bar{K}_{\bar{\Omega}} \hat{U} + \bar{U}'\bar{P}\bar{U}\\ &\quad + \bar{K}_{\bar{\Omega}}' \Phi_{\bar{\Omega}}' S_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}}) \bar{K}_{\bar{\Omega}} - \hat{U}'\bar{P}\hat{U}. & (37)\end{aligned}$$

Next, using (13) and (17), we obtain

$$\begin{aligned} &\bar{U}'\bar{P}\bar{U} - \hat{U}'\bar{P}\hat{U}\\ =\;& \bar{K}_\Omega'(\hat{N}_\Omega - N_\Omega)' B_\Omega' \bar{P}\bar{U} + \bar{U}'\bar{P}B_\Omega(\hat{N}_\Omega - N_\Omega)\bar{K}_\Omega\\ &- \bar{K}_\Omega'(\hat{N}_\Omega - N_\Omega)' B_\Omega' \bar{P} B_\Omega (\hat{N}_\Omega - N_\Omega)\bar{K}_\Omega. & (38)\end{aligned}$$

Then, using (13), (17)-(18), (20), (23)-(24), (30)-(32), and (38), we obtain

$$\bar{F}'P\bar{F} = F'PF - A'\Upsilon A \quad (39)$$

where

$$\begin{aligned} \Upsilon &= \bar{K}_\Omega'(\hat{N}_\Omega - N_\Omega)' B_\Omega' P (I - B_{\bar{\Omega}} N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}})\hat{U}\\ &\quad + \hat{U}'(I - B_{\bar{\Omega}} N_{\bar{\Omega}} \bar{K}_{\bar{\Omega}})' P B_\Omega (\hat{N}_\Omega - N_\Omega)\bar{K}_\Omega\\ &\quad + \bar{K}_\Omega'(\hat{N}_\Omega - N_\Omega)' B_\Omega' P B_\Omega (\hat{N}_\Omega - N_\Omega)\bar{K}_\Omega\\ &= \bar{K}_\Omega' E_1 \bar{K}_\Omega + E_2 + \bar{U}'\bar{P}\bar{U} - \hat{U}'\bar{P}\hat{U}. & (40)\end{aligned}$$

Using (26), (35)-(37), and (39)-(40), (34) is verified. Finally, using (34) one can easily show that the performance index (33) can be represented alternatively as follows:

$$J = x_0'(P + a\tilde{P})x_0 - x_0' \sum_{k=0}^{\infty}(AF^k)'\{\bullet\}_1 AF^k x_0 \quad (41)$$

where

$$\{\bullet\}_1 = \bar{K}_{\bar{\Omega}}' \Phi_{\bar{\Omega}}' S_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}}) \bar{K}_{\bar{\Omega}} + \{\bullet\} + a\bar{Q}. \quad (42)$$

From (41), the performance bound (25) holds if matrix $\{\bullet\}_1$ is positive semidefinite, which establishes the constraint (28). □

4 Gain Margin Issues of the Feasible TSLQRC

In this section, we shall evaluate the stability gain margins (SGM) and the performance gain margins (PGM) of the proposed feasible TSLQRC. First, we give the SGM of the feasible TSLQRC, which is closely related to the reliable control design problem, in the following corollary.

Corollary 1. *The state-feedback system obtained by applying the feasible TSLQRC (16)-(21) remains stable provided that the independent gains $N_{\bar{\Omega}}$ associated* *with $\bar{\Omega}$ and the gains N_Ω associated with Ω satisfy the following constraints:*

$$\begin{aligned}\{\bullet\}^{\Phi}_{\bar{\Omega}} &= N'_{\bar{\Omega}}R_{\bar{\Omega}}N_{\bar{\Omega}} - (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}})'S_{\bar{\Omega}}(\Phi_{\bar{\Omega}} - N_{\bar{\Omega}}) \\ &\geq 0 \qquad (43)\\ \{\bullet\}^{\Phi}_{\Omega} &= N'_{\Omega}R_{\Omega}N_{\Omega} + \Phi'_{\Omega}S_{\Omega}\Phi_{\Omega} \\ &\quad -(\Phi_{\Omega} - N_{\Omega})'S_{\Omega}(\Phi_{\Omega} - N_{\Omega}) \geq 0 \qquad (44)\\ \{\bullet\}^{\Gamma}_{\bar{\Omega}} &= \bar{K}'_{\bar{\Omega}}\Phi'_{\bar{\Omega}}S_{\bar{\Omega}}(\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}})\bar{K}_{\bar{\Omega}} - \bar{K}'_{\Omega}E_1\bar{K}_{\Omega} - E_2 \\ &\geq 0. \qquad (45)\end{aligned}$$

Proof. Using (26)-(27), (29)-(30), (34)-(35), and (43)-(45), we obtain

$$\begin{aligned}P - F'PF &= A'(\bar{K}'_{\Omega}\{\bullet\}^{\Phi}_{\Omega}\bar{K}_{\Omega} + (\bar{K}_{\bar{\Omega}}\hat{U})'\{\bullet\}^{\Phi}_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}\hat{U} \\ &\quad +\{\bullet\}^{\Gamma}_{\bar{\Omega}})A + Q. \qquad (46)\end{aligned}$$

From [16] and (46), it is clear that the state-feedback system remains stable provided that the matrices $\{\bullet\}^{\Phi}_{\bar{\Omega}}$, $\{\bullet\}^{\Phi}_{\Omega}$, and $\{\bullet\}^{\Gamma}_{\bar{\Omega}}$ are all positive semidefinite, which establishes the constraints (43), (44), and (45), respectively. This completes the proof. □

Next, we shall determine the PGM of the feasible TSLQRC, which is related to the guaranteed-cost control design problem. The obtained result is a special case of that given in Theorem 1 and is list in the following corollary without proof.

Corollary 2. *The state-feedback system obtained by applying the feasible TSLQRC (16)-(21) satisfies the performance bound*

$$J \leq x'_0 P x_0 \qquad (47)$$

if $N_{\bar{\Omega}}$ and N_Ω satisfy the following new modified gain margin constraint:

$$\begin{aligned}&\bar{K}'_{\Omega}D^{\Phi}_{\Omega}\bar{K}_{\Omega} + \bar{K}'_{\bar{\Omega}}\Phi'_{\bar{\Omega}}S_{\bar{\Omega}}(\Phi_{\bar{\Omega}} - \Gamma_{\bar{\Omega}})\bar{K}_{\bar{\Omega}} \\ \geq\ &\hat{U}'\bar{K}'_{\bar{\Omega}}D^{\Phi}_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}\hat{U} + \bar{K}'_{\Omega}E_1\bar{K}_{\Omega} + E_2 \qquad (48)\end{aligned}$$

where D^{Φ}_{Ω}, $D^{\Phi}_{\bar{\Omega}}$, E_1, and E_2 are given by (29), (30), (31), and (32), respectively.

Remark 1: If one chooses $\hat{N}_\Omega = N_\Omega$, then one has $E_1 = 0$, $E_2 = 0$, and $\hat{U} = \bar{U}$, and hence (48) becomes to the modified gain margin constraint given in [13, Thm. 2].

5 Parameters Determination for the Feasible TSLQRC

In this section, the issue of determining the design parameters of the proposed feasible TSLQRC, i.e., $\hat{N}_\Omega$, $\Phi_{\bar{\Omega}}$, Φ_Ω, and $\Gamma_{\bar{\Omega}}$, is addressed. It is known from [14] that the two-stage structure of the TSLQRC has no obvious advantage over the single-stage LQ reliable control (SSLQRC) in the PGM point of view. Hence, in the following discussions we only focus on the SGM problem. Furthermore, instead of treating (14) and (15) directly, the perturbation gain matrices: $N_{\bar{\Omega}}$ and N_Ω are handled as follows:

$$N_{\bar{\Omega}} = n_{\bar{\Omega}} I, \qquad N_\Omega = n_\Omega I \qquad (49)$$

where

$$min\{\underline{n}^i_{\bar{\Omega}}\}^p_{i=1} = \underline{n}_{\bar{\Omega}} \leq n_{\bar{\Omega}} \leq \bar{n}_{\bar{\Omega}} = max\{\bar{n}^i_{\bar{\Omega}}\}^p_{i=1} \quad (50)$$
$$min\{\underline{n}^i_{\Omega}\}^{m-p}_{i=1} = \underline{n}_{\Omega} \leq n_{\Omega} \leq \bar{n}_{\Omega} = max\{\bar{n}^i_{\Omega}\}^{m-p}_{i=1}. \quad (51)$$

The design parameters: $\hat{N}_\Omega$, $\Phi_{\bar{\Omega}}$, Φ_Ω, and $\Gamma_{\bar{\Omega}}$ are then taken as the following specific forms:

$$\hat{N}_\Omega = \hat{n}_\Omega I,\ \ \Phi_{\bar{\Omega}} = \phi_{\bar{\Omega}} I,\ \ \Phi_\Omega = \phi_\Omega I,\ \ \Gamma_{\bar{\Omega}} = \gamma_{\bar{\Omega}}\Phi_{\bar{\Omega}}. \quad (52)$$

First of all, we shall determine parameter $\gamma_{\bar{\Omega}}$. It is chosen as a suitable value that will yield a satisfactory worst-fault performance, i.e., $x'_0 P x_0$, where P is the stabilizing solution of the following discrete-time algebraic Riccati equation (DARE):

$$P = A'P(I - \gamma_{\bar{\Omega}} B_{\bar{\Omega}} S^{-1}_{\bar{\Omega}} B'_{\bar{\Omega}} P)A + Q. \qquad (53)$$

Next, we consider parameters $\phi_{\bar{\Omega}}$ and ϕ_Ω. Using (49) and (52), the stability constraints (43) and (44) become, respectively,

$$\begin{aligned}n^2_{\bar{\Omega}} R_{\bar{\Omega}} - (\phi_{\bar{\Omega}} - n_{\bar{\Omega}})^2 S_{\bar{\Omega}} &\geq 0 \qquad (54)\\ n^2_{\Omega} R_{\Omega} + n_{\Omega}(2\phi_{\Omega} - n_{\Omega}) S_{\Omega} &\geq 0. \qquad (55)\end{aligned}$$

Solving (54) and (55), one obtains the following SGM:

$$\frac{\phi_{\bar{\Omega}}}{1+\alpha} \leq n_{\bar{\Omega}} \leq \frac{\phi_{\bar{\Omega}}}{1-\alpha}, \qquad 0 \leq n_\Omega \leq \frac{2\phi_\Omega}{1-\beta} \qquad (56)$$

where $\alpha = [\lambda_{min}\{R_{\bar{\Omega}} S^{-1}_{\bar{\Omega}}\}]^{1/2}$ and $\beta = \lambda_{min}\{R_\Omega S^{-1}_\Omega\}$, in which $\lambda_{min}\{\bullet\}$ represents the minimum eigenvalue of a matrix. Assuming that a dedicated actuator failure model is given by (50) and (51), then the design parameters $\phi_{\bar{\Omega}}$ and ϕ_Ω are determined by satisfying the following constraints:

$$(1-\alpha)\bar{n}_{\bar{\Omega}} \leq \phi_{\bar{\Omega}} \leq (1+\alpha)\underline{n}_{\bar{\Omega}},\ \ \phi_\Omega \geq 0.5(1-\beta)\bar{n}_\Omega \quad (57)$$

where the former has a solution if the following condition holds

$$\lambda_{min}\{R_{\bar{\Omega}} S^{-1}_{\bar{\Omega}}\} \geq [(\bar{n}_{\bar{\Omega}} - \underline{n}_{\bar{\Omega}})/(\bar{n}_{\bar{\Omega}} + \underline{n}_{\bar{\Omega}})]^2. \qquad (58)$$

Then, the proposed feasible TSLQRC design is completed by verifying the stability constraint (45). To achieve this end, we reformulate (32) by using (18) as follows:

$$\begin{aligned} E_2 &= \bar{K}'_{\Omega}(\hat{N}_{\Omega} - N_{\Omega})' B'_{\Omega} \bar{K}'_{\bar{\Omega}} \Phi'_{\bar{\Omega}} S_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}}) \bar{K}_{\bar{\Omega}} \bar{U} \\ &\quad + \bar{U}' \bar{K}'_{\bar{\Omega}} (\Phi_{\bar{\Omega}} - N_{\bar{\Omega}})' S_{\bar{\Omega}} \Phi_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} B_{\Omega} (\hat{N}_{\Omega} - N_{\Omega}) \\ &\quad \times \bar{K}_{\Omega}. \end{aligned} \tag{59}$$

Next, using (31), (52), and (59) in (45) yields the following inequality:

$$\tilde{n}_{\Omega}^2 M + \tilde{n}_{\Omega} H + N \geq 0 \tag{60}$$

where $\tilde{n}_{\Omega} = \hat{n}_{\Omega} - n_{\Omega}$,

$$M = \phi_{\bar{\Omega}}(\phi_{\bar{\Omega}} - 2n_{\bar{\Omega}}) X' Y X \tag{61}$$

$$H = -\phi_{\bar{\Omega}}(\phi_{\bar{\Omega}} - n_{\bar{\Omega}})(X' Y \bar{U} + \bar{U}' Y X) \tag{62}$$

$$N = \phi_{\bar{\Omega}}^2 (1 - \gamma_{\bar{\Omega}}) Y \tag{63}$$

in which $X = B_{\Omega} \bar{K}_{\Omega}$ and $Y = \bar{K}'_{\bar{\Omega}} S_{\bar{\Omega}} \bar{K}_{\bar{\Omega}}$. Since $N \geq 0$, it is clear from (60) that if the estimate $\hat{n}_{\Omega}$ is more accurate, i.e., $\tilde{n}_{\Omega} \to 0$, then the performance of the feasible TSLQRC will be more approximate to that of the MTSLQRC [14].

Finally, we suggest a possible solution to determine $\hat{n}_{\Omega}$. This problem can be solved if one recasts the original control design problem into a parameter filtering problem [10]. Using (22) and (52) in (1) yields

$$\begin{bmatrix} x_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} A & -\bar{B}_k \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix} + \begin{bmatrix} 0 \\ w_k^z \end{bmatrix} \tag{64}$$

$$y_k = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix} \tag{65}$$

where $z = \begin{bmatrix} n_{\bar{\Omega}} & n_{\Omega} \end{bmatrix}'$ and

$$\bar{B}_k = \begin{bmatrix} B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \hat{U}_k A x_k & B_{\Omega} \bar{K}_{\Omega} A x_k \end{bmatrix} \tag{66}$$

$$\hat{U}_k = I - \hat{n}_{\Omega,k} B_{\Omega} \bar{K}_{\Omega}. \tag{67}$$

The parameter filtering problem at hand is then to estimate $n_{\Omega,k}$, which can be solved by using the following reduced-order filter:

$$\hat{n}_{\Omega,k} = \begin{bmatrix} 0 & 1 \end{bmatrix} z_{k|k} \tag{68}$$

where $z_{k|k}$ is obtained by using the following OMOLSE (optimal minimum-order least-squares estimator) [15]:

$$z_{k|k-1} = z_{k-1|k-1} \tag{69}$$

$$z_{k|k} = z_{k|k-1} + K_k^z (x_k - \bar{x}_{k|k-1} - S_k z_{k|k-1}) \tag{70}$$

$$P_{k|k-1}^z = P_{k-1|k-1}^z + Q_{k-1}^z \tag{71}$$

$$K_k^z = P_{k|k-1}^z S'_k \{ S_k P_{k|k-1}^z S'_k + \bar{P}_{k|k-1}^x \}^{-1} \tag{72}$$

$$P_{k|k}^z = (I - K_k^z S_k) P_{k|k-1}^z \tag{73}$$

where

$$\bar{x}_{k|k-1} = A x_{k-1} - \bar{B}_{k-1} z_{k-1|k-1} - S_k z_{k|k-1} \tag{74}$$

$$\bar{P}_{k|k-1}^x = \bar{B}_{k-1} P_{k-1|k-1}^z \bar{B}'_{k-1} - S_k P_{k|k-1}^z S'_k \tag{75}$$

$$S_k = -\bar{B}_{k-1} P_{k-1|k-1}^z (P_{k|k-1}^z)^{-1}. \tag{76}$$

Now, we are in the position to simplify the above $z_{k|k}$ filter. Using (69) and (74)-(76) in (70) and (72) yields, respectively,

$$z_{k|k} = (I + K_k^z \bar{B}_{k-1}) z_{k-1|k-1} + K_k^z (x_k - A x_{k-1}) \tag{77}$$

$$K_k^z = -P_{k-1|k-1}^z \bar{B}'_{k-1} \{ \bar{B}_{k-1} P_{k-1|k-1}^z \bar{B}'_{k-1} \}^{-1} \tag{78}$$

Note that if the dimension of state x_k is greater than that of state z_k, i.e., $dim(x_k) > dim(z_k)$, then the inverse in (78) does not exist. If this is the case, (78) is replaced by the following

$$K_k^z = -P_{k-1|k-1}^z \bar{B}'_{k-1} \{ \bar{B}_{k-1} P_{k-1|k-1}^z \bar{B}'_{k-1} \}^{+} \tag{79}$$

where M^+ is an arbitrary generalized inverse of M satisfying $MM^+M = M$. Without loss of generality, in the following derivations, we only consider the case: $dim(x_k) > dim(z_k)$. Assuming $P_{k|k}^z > 0$ and letting

$$P_{k|k}^z = \Lambda_k \Lambda'_k, \qquad \bar{B}_k \Lambda_k = U_k \begin{bmatrix} \Sigma_k \\ 0 \end{bmatrix} V'_k \tag{80}$$

then one has

$$\begin{aligned} &\Lambda_k^{-1} K_{k+1}^z \bar{B}_k \Lambda_k \\ &= -\Lambda'_k \bar{B}'_k \{ \bar{B}_k \Lambda_k (\bar{B}_k \Lambda_k)' \}^{+} \bar{B}_k \Lambda_k = -I \end{aligned}$$

which yields

$$K_{k+1}^z \bar{B}_k = -I. \tag{81}$$

Using (68), (71), (73), (76)-(78), and (81), one obtains the following more compact reduced-order filter $\hat{n}_{\Omega,k}$:

$$\hat{n}_{\Omega,k} = \begin{bmatrix} 0 & 1 \end{bmatrix} K_k^z (x_k - A x_{k-1}) \tag{82}$$

$$K_k^z = -Q_{k-2}^z \bar{B}'_{k-1} \{ \bar{B}_{k-1} Q_{k-2}^z \bar{B}'_{k-1} \}^{+}. \tag{83}$$

Remark 2: Using (64) and (81) in (82) yields

$$\hat{n}_{\Omega,k} = -\begin{bmatrix} 0 & 1 \end{bmatrix} K_k^z \bar{B}_{k-1} z_{k-1} = n_{\Omega,k-1}. \tag{84}$$

From (84), it is clear that the failure estimate (82) can achieve the goal, i.e., satisfies (60), if the rate of change of the actuator failure is small enough.

6 An Illustrative Example

To illustrate the proposed results, the author considered the system given in [13] and the following actuator failure model:

$$0.6 \leq n_{\bar{\Omega}} \leq 8, \qquad 0 \leq n_{\Omega} \leq 30. \tag{85}$$

The design parameters of the feasible TSLQRC are chosen as follows:

$$\phi_{\bar{\Omega}} = 1.12, \quad \phi_{\Omega} = 1.31, \quad \gamma_{\bar{\Omega}} = 0.85. \tag{86}$$

Table 1: Analytical stable estimation error margins of the feasible TSLQRC

$n_{\bar{\Omega}}$	n_Ω	$\tilde{n}_\Omega$ bounds
0.6	0	-0.6 $\leq \tilde{n}_\Omega \leq$ 0.495
0.6	30.05	-0.016 $\leq \tilde{n}_\Omega \leq$ 18.584
8.44	0	-0.035 $\leq \tilde{n}_\Omega \leq$ 0.042
8.44	30.05	-1.32 $\leq \tilde{n}_\Omega \leq$ 0.001

Table 2: Simulated stable estimation error margins of the feasible TSLQRC

$n_{\bar{\Omega}}$	n_Ω	$\tilde{n}_\Omega$ bounds
0.6	0	-245.295 $\leq \tilde{n}_\Omega \leq$ 6.68
0.6	30.05	-2.395 $\leq \tilde{n}_\Omega \leq$ 155.115
8.44	0	-0.212 $\leq \tilde{n}_\Omega \leq$ 14.855
8.44	30.05	-14.115 $\leq \tilde{n}_\Omega \leq$ 0.221

Using (86) in (56) yields the following SGMs:

$$0.6 \leq n_{\bar{\Omega}} \leq 8.44, \quad 0 \leq n_\Omega \leq 30.05 \tag{87}$$

which meet the specifications of the desired actuator failure model (85). The analytical stable estimation error margins associated with the above SGMs, which guarantee the stability constraint (60), are depicted in Table 1. From Table 1, it is clear that if $|\tilde{n}_\Omega| \leq 0.001$ then the SGMs of the proposed feasible TSLQRC will be equivalent to those of the MTSLQRC [14]. Note that this stability requirement is guaranteed by using the actuator failure estimator (82)-(83). Also shown in Table 2 are the corresponding simulated stable estimation error margins, which clearly indicate that the stability constraints (43)-(45) are conservative ones.

Based on the above discussions, we conclude that the proposed feasible TSLQRC serves as a practical implementation of the previously proposed TSLQRC and its performance may not be degraded.

7 Conclusion

In this paper, we have introduced a feasible version of the previously proposed TSLQRC in order to overcome the inherently infeasible problem. A modified unified gain margin constraint is proposed to facilitate the discussions of the SGM and the PGM of the proposed feasible TSLQRC. Analytical and simulated results showed that the SGM of the feasible TSLQRC is almost the same as that of the MTSLQRC [14].

Acknowledgments

The author would like to thank the National Science Council of the Republic of China for financially supporting this research through its grant NSC92-2213-E-233-003.

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