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The Maximal Robust Controlled Invariant Set of Uncertain Switched Systems

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Abstract— **This paper presents a computational method for the maximal robust controlled invariant set(MRCIS) of a class of uncertain switched systems, whose system matrices are unknown but are convex combinations of known matrices. A recursive algorithm is given to compute the MRCIS. It is also shown that the MRCIS can be obtained from the MRCIS of all subsystems.**

I. INTRODUCTION

Invariant sets play a key role in the state constraints problem because an invariant set provides a set of feasible initial states such that state constraints violations can be avoided. Robust invariant sets have been discussed in [1] for uncertain discrete-time switched systems. The backward reachability algorithm in [1] is not feasible for continuous-time cases. The computational method of maximal robust controlled invariant sets(MRCIS) for uncertain continuous-time switched systems is still an open question.

This paper extends the construction method of the maximal controlled invariant set(MCIS) for deterministic continuous-time switched systems in [2] to uncertain continuous-time switched systems. A recursive algorithm is given to compute the MRCIS and it terminates to a fixed point in a finite number of iterations. The MRCIS can be computed by taking the union of the MRCIS of all subsystems, which are computed by taking the intersection of the MCIS of the linear systems with the vertex matrices. This paper only discusses regular switches because the recursive algorithm can also be used for fast switches.

The remainder of this paper is organized as follows. Section II introduces the notations and defines an uncertain switched system. Section III presents the computational method of the MRCIS. Section IV is the conclusion. Because of the space limitations, all the proofs are omitted in this paper.

II. PRELIMINARIES

This section introduces some definitions and notations used in this paper.

A. (A,B)-invariant Subspaces of Linear Systems

We consider a linear system described by the following differential equation for linear spaces $\mathscr{X}$ and $\mathscr{U}$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x \in \mathscr{X}, u &\in \mathscr{U} \end{aligned} \tag{1}$$

We say a subspace $\mathscr{V}(A,B,\mathscr{K})$ is a (A,B)-invariant subspace of a linear space $\mathscr{K}$ if there exists a map $F: \mathscr{X} \to \mathscr{U}$ such that $(A+BF)\mathscr{V} \subset \mathscr{V}$. The sufficient and necessary condition for $\mathscr{V}$ to be (A,B)-invariant is $A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$, where $\mathscr{B} = \mathrm{Im}B$.

If a subspace $\mathscr{V}(A,B,\mathscr{K})$ is (A,B)-invariant, then, for any initial state $x(0) = x_0 \in \mathscr{V}(A,B,\mathscr{K})$, there exists a control $u = Fx$ such that any trajectory $x(t) = e^{(A+BF)t}x_0$ is in $\mathscr{V}(A,B,\mathscr{K})$, for all $t \geq 0$. The maximal invariant subspace $\mathscr{V}^*(A,B,\mathscr{K})$ of the system in (1) is computed using the following algorithm [4]:

$$\begin{aligned} \mathscr{V}_0 &= \mathscr{K} \\ \mathscr{V}_{i+1} &= \mathscr{V}_i \bigcap A^{-1}(\mathscr{V}_i + \mathscr{B}) \end{aligned} \tag{2}$$

This recursion converges to a fixed point, which is the maximal controlled invariant subspace $\mathscr{V}^*(A,B,\mathscr{K})$. This algorithm always terminates after a finite number, d, of iterations, where d is less than or equal to the dimension of the linear space $\mathscr{K}$.

B. Uncertain Switched Systems

The uncertain switched system in this paper is described by

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_\sigma x(t) + B_\sigma u_\sigma(t) \\ \widetilde{A}_\sigma &= \sum_{k=1}^{m_\sigma} \lambda_k A_\sigma^k, \ \sum_{k=1}^{m_\sigma} \lambda_k = 1, \ \lambda_k \geq 0 \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$ and $u_l \in \mathbb{R}^{r_l}, l = 1, \cdots, n$ are piecewise continuous input functions, and $\sigma : [t_0, \infty) \to M = \{1, 2, \ldots, n\}$ is a switching path. Suppose the switching time instants are $t_0 < t_1 < \cdots < t_s \cdots$, then the sequence $\sigma(t_0), \sigma(t_1), \cdots, \sigma(t_s), \cdots$ refers to the switching index sequence. The switching sequence is controlled by a controller. The matrix $\widetilde{A}_{\sigma(t)}$ is uncertain but is the convex combination of known matrices. We assume $t_{s+1} - t_s > \varepsilon > 0$, i.e. we only consider regular switches.

III. MAXIMAL ROBUST INVARIANT SETS OF UNCERTAIN SWITCHED SYSTEMS

Let $\mathscr{K}$ be a linear space in $\mathbb{R}^n$. Given an uncertain switched system (3), the MRCIS in $\mathscr{K}$, V(not necessarily a linear space), satisfies the following condition: for any $x_0 \in V$, there exists a continuous control input u_σ and a switching path σ such that the trajectory of the system remains in V for time $t > 0$. To study the MRCIS of an uncertain switched system, we will first present some results on uncertain linear systems.

A. $(\widetilde{A},B)$-invariant Subspaces of Uncertain Linear Systems

We consider an uncertain linear system described by the following differential equation, where any system matrix $\widetilde{A}$ can be written as $\sum_{k=1}^{m} \lambda_k A_k$, $\sum_{k=1}^{m} \lambda_k = 1$, $\lambda_k \geq 0$, for $k \in 1,2,\cdots,m$:

$$\begin{aligned} \dot{x} &= \widetilde{A}x + Bu, \qquad \widetilde{A} \in co\{A_1,\cdots,A_m\} \\ x &\in X, u \in U \end{aligned} \tag{4}$$

For this class of uncertain linear systems, we have the following results.

Lemma 1: A subspace $\mathscr{V}^*$ is $(\widetilde{A},B)$-invariant if and only if $\mathscr{V}^*$ is (A_k,B)-invariant for any $k \in \{1,2,\cdots,m\}$.

Lemma 2: Given a subspace $\mathscr{V}$ of linear space $\mathscr{K}$, then $\widetilde{A}^{-1}\mathscr{V} = \bigcap_{k=1}^{m} A_k^{-1}\mathscr{V}$.

The operator $\widetilde{A}^{-1}$ can be understood as equivalent to the predecessor operator in [3]. This Lemma is a generalization of Proposition 2 in [3]. Using Lemma 2 and the algorithm 2, we can obtain the following Lemma.

Lemma 3: If a subspace $\mathscr{V}^*$ is the unique MRCIS of the system (4) and $\mathscr{V}_k^*$ is the unique MCIS of the linear system with (A_k,B) matrices, $k \in \{1,\cdots,m\}$, then $\mathscr{V}^* = \bigcap_{k=1}^{m} \mathscr{V}_k^*$.

Lemma 3 shows that the computation of the invariant subspace of the uncertain system 4 with matrix $\widetilde{A}$ can be obtained by taking the intersection of invariant subspaces of the vertex matrix A_k, $k = 1,2,\cdots,m$.

B. Robust Controlled Invariant Sets of Uncertain Switched Systems

This subsection states the relation between the MRCIS of an uncertain switched system and the MRCIS of all subsystems.

There are some properties that we need to notice for controlled invariant sets. These properties have already been mentioned in [2].

- The invariant sets are closed under the union operator: if V_1 and V_2 are both controlled invariant sets, then $V_1 \cup V_2$ is also a controlled invariant set.
- Let V be a controlled invariant set of the system in (1). $ext(V)$ is defined as the scalar extension of V, i.e. $ext(V) = \{x | \exists \lambda \neq 0, \lambda \in \mathbb{R}, s.t. \lambda x \in V\}$. $ext(V)$ is also a controlled invariant set in $\mathscr{K}$.

These two properties imply that the MCIS V^* is a union of linear spaces. We adopted the concept of *space arrangement* from [2].

Definition 1: A space arrangement V is defined as the union of finitely many linear spaces.

The invariant set, V^*, therefore, is a space arrangement i.e. $V^* = \bigcup_{j=1}^{n_l}(\mathscr{V}_j)$. Therefore, it is possible to have switchings between the component subspaces.

The MRCIS can be constructed using the following algorithm, where $\mathscr{B}_i = Im(B_i)$, and $\mathscr{V}_{lj}$, $j = 1,\cdots,n_l$ are the linear spaces that build up V_l:

$$\begin{aligned} V_0 &= \mathscr{K} \\ V_{l+1} &= V_l \cap \left(\bigcup_{j=1}^{n_l} \bigcup_{i=1}^{n} \bigcap_{k=1}^{m_i} \mathscr{V}_{lj} \cap (A_i^k)^{-1}(\mathscr{V}_{lj} + \mathscr{B}_i) \right). \end{aligned} \tag{5}$$

For the deterministic hybrid systems, this algorithm has been proved to be terminated in a finite number of iterations and the limit set is the MCIS in [2]. The uncertainty of the matrix $\widetilde{A}_i$ makes the computation infeasible, but we can use Lemma 2 to compute $\widetilde{A}_i$. We also can compute the MRCIS of uncertain switched systems using the following Proposition:

Proposition 1: The MRCIS V^* in $\mathscr{K}$ of an uncertain switched system (3) can be computed using the following equality:

$$V^* = \bigcup_{i=1}^{n} \bigcap_{k=1}^{m_i} V^*(A_i^k, B_i, \mathscr{K}). \tag{6}$$

Proposition 1 implies that we can compute the MRCIS for each individual subsystem, and then take the union to obtain the MRCIS for the original switched system. During the computation of the individual MRCIS for the subsystems, we can take the intersection of the MCIS for linear systems described by the vertex matrices.

IV. Conclusion

This paper presents the computational method for the MRCIS of a class of uncertain switched systems, whose system matrices are unknown but are convex combinations of known matrices. A recursive algorithm is given to compute the invariant sets and it converges to a fixed point which is the MRCIS. It is also computed by knowing the MRCIS of all subsystems. The MRCIS for the subsystems are computed by taking the intersection of the MCIS for the linear systems described by the vertex matrices.

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