Canonical Forms of Switched Linear Control Systems

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***Abstract*— This paper addresses the problem of finding normal forms for switched linear systems. Based on the recent controllability criterion, we decompose a switched linear control systems into the controllable mode and the uncontrollable mode. For single-input controllable switched systems, we propose a canonical form via coordinate and feedback transformations. A controllable multi-input switched system can be transformed to a controllable single-input one by (nonregular) feedback reduction.**

I. INTRODUCTION

A switched linear control system is a hybrid control system which switches at different time instants among a finite set of linear time-invariant subsystems. Such systems may arise from several situations. First, systems of this type can be used to model systems subject to known or unknown abrupt parameter variations such as synchronously switched linear systems [10], networks with periodically varying switchings [1], and sudden change of system structure due to the failure of a component [20]. Second, when we try to control a single process by means of multi-controller switching, the overall system can be described by a switched control system. Indeed, the multi-controller scheme provides an effective mechanism to cope with highly complex systems and/or systems with large uncertainties [11], [7]. Even for simple linear time-invariant (LTI) systems, the performance (e.g., transient response) can be improved through multiple controllers/compensators switching [13]. Third, such systems arise naturally in the study of multi-rate sampled-data systems [15], [2].

Switched linear systems have attracted increasingly more attention in the past few years. The literature grow exponentially and a lot new ideas and powerful tools have been developed from various disciplines. The reader is referred to [3] and [12] for surveys of recent development.

In this paper, we address switched linear control systems where both the control inputs and the switching signals are design variables. While much work has been devoted to the specification of the switching mechanisms by assuming that the control inputs are given *a priori*, there are only a few literature addressed the controller design issues for switched control systems. Among these, complete controllability and observability criteria have been presented in [17], [23] together with the (controllability) path-planing algorithms; stabilizing feedback controllers were presented in [19], [16] for special classes of switched linear systems; and optimal control issues were addressed in [24].

The objective of this paper is to determine the normal forms of switched linear systems under state and feedback input transformations. Based on the controllability criterion, we decompose a switched linear control systems into submodes with clear structural information: the controllable mode and the uncontrollable mode. For controllable single-input switched systems, we propose normal forms via coordinate and feedback transformations. A controllable multi-input switched system can be transformed into a controllable single-input one by (nonregular) feedback reduction. These results pave the way for further investigation of synthesis problems such as stabilization and regulation.

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II. PRELIMINARIES

Let $M = \{1, \cdots, m\}$ be an index set. If k is a natural number, let $\bar{k} = \{1, \cdots, k\}$ and $\underline{k} = \{0, 1, \cdots, k-1\}$. Let $I_n = [e_1, \cdots, e_n]$ denote the nth order identity matrix.

Consider a switched linear control system given by

$$\sum(A_i, B_i)_M: \qquad \dot{x}(t) = A_\sigma x(t) + B_\sigma u_\sigma(t), \tag{1}$$

where $x \in \Re^n$ are the states, $u_k \in \Re^{p_k}, k = 1, \cdots, m$ are piecewise continuous inputs, $\sigma \to M$ is the switching signal to be designed, and A_k and B_k are real constant matrices with compatible dimensions.

In the sequel, we briefly review some existing results which will be used in the later derivations.

Let $\phi(t; t_0, x_0, u, \sigma)$ denote the state trajectory at time t of switched system (1) starting from $x(t_0) = x_0$ with inputs $u = [u_1, \cdots, u_m]$ and switching signal σ.

The controllable set of system (1) is the set of states which can be transferred to the origin in a finite time by appropriate choices of input and switching path.

Definition 1: State $x \in \Re^n$ is controllable (at t_0), if there exist a time instant $t_f > t_0$, a switching signal $\sigma : [t_0, t_f] \to M$, and inputs $u_k : [t_0, t_f] \to \Re^{p_k}$, $k \in M$, such that $\phi(t_f; t_0, x, u, \sigma) = 0$. The controllable set of system (1) is the set of states which are controllable. System (1) is said to be (completely) controllable, if its controllable set is $\Re^n$.

Recall that the controllable set of matrix pair (A, B) is the minimal A-invariant subspace that contains image space of B. For switched system $\sum(A_i, B_i)_M$, we have similar criterion. To this end, denote by $\mathcal{V}(A_i, B_i)_M$ the minimum subspace of $\Re^n$ which is invariant under all A_i, $i \in M$ and contains all image spaces of B_i, $i \in M$. The expression of this subspace in the system matrices is

$$\mathcal{V}(A_i, B_i)_M = \sum_{i_0, \cdots, i_{n-1} \in M}^{j_1, \cdots, j_{n-1} = 0, \cdots, n-1} A_{i_{n-1}}^{j_{n-1}} \cdots A_{i_1}^{j_1} \Im B_{i_0}. \tag{2}$$

where $\Im B$ denotes the image space of B.

Lemma 1: [17] For switched linear system (1), the controllable set is precisely the subspace $\mathcal{V}(A_i, B_i)_M$.

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III. CANONICAL DECOMPOSITIONS

In this section, we present canonical forms of switched linear systems based on the controllability criterion presented in the previous section.

Suppose T is a nonsingular $n \times n$ real matrix. By letting $\bar{x} = Tx$, it follows from (1) that

$$\dot{\bar{x}}(t) = TA_\sigma T^{-1}\bar{x}(t) + TB_\sigma u_\sigma(t). \quad (3)$$

This equation describes the same system dynamics in different bases of the state space. Thus the two systems are equivalent under the coordinate transformation $\bar{x} = Tx$.

For systems which are equivalent, their controllable sets are also connected by the equivalence transformation in a clear manner.

Proposition 1: Denote $\bar{\mathcal{V}}$ the controllable set of systems (1) and (3), respectively. Then, we have

$$\bar{\mathcal{V}} = T\mathcal{V}.$$

Proof. Simple calculation gives

$$\begin{aligned}\bar{\mathcal{V}} &= \sum_{i_0,\cdots,i_{n-1}\in M}^{j_1,\cdots,j_{n-1}\in \underline{n}} (TA_{i_{n-1}}T^{-1})^{j_{n-1}}\cdots \\ &\quad (TA_{i_1}T^{-1})^{j_1}\Im(TB_{i_0}) \\ &= T\sum_{i_0,\cdots,i_{n-1}\in M}^{j_1,\cdots,j_{n-1}\in \underline{n}} A_{i_{n-1}}^{j_{n-1}}\cdots A_{i_1}^{j_1}\Im B_{i_0} = T\mathcal{V}.\diamondsuit\end{aligned}$$

As a simple implication, we have the following result.

Corollary 1: The property of complete controllability is invariant under any equivalence transformation.

By Lemma 1, the controllable set of any switched linear system is a subspace of $\Re^n$. Denote the set by $\mathcal{C}$. Let $\mathcal{S}$ be a subspace of $\Re^n$ such that

$$\Re^n = \mathcal{C} \oplus \mathcal{S}, \quad (4)$$

where the symbol '$\oplus$' denotes direct sum. It is clear that

$$\mathcal{C} \approx \frac{\Re^n}{\mathcal{S}} \approx \Re^l, \quad (5)$$

where '$\approx$' stands for isomorphism, $\frac{\Re^n}{\mathcal{S}}$ is the quotient space, and l =dim $\mathcal{C}$. Let $P : \Re^n \to \frac{\Re^n}{\mathcal{S}}$ be the canonical projection and $\hat{A}_k,\ k \in M$ the map induced in $\frac{\Re^n}{\mathcal{S}}$ by A_k:

$$\hat{A}_k P = PA_k,\ k \in M. \quad (6)$$

Let $\hat{B}_k = PB_k,\ k \in M$. Then we have

Theorem 1: System $\sum(\hat{A}_k, \hat{B}_k)_M$ is completely controllable.

Proof. The controllable subspace of system $\sum(\hat{A}_k, \hat{B}_k)_M$ is

$$\begin{aligned}\mathcal{V}(\hat{A}_k, \hat{B}_k)_M &= \sum_{i_0,\cdots,i_{n-1}\in M}^{j_1,\cdots,j_{n-1}=0,\cdots,n-1} \hat{A}_{i_{n-1}}^{j_{n-1}}\cdots \hat{A}_{i_1}^{j_1}\Im\hat{B}_{i_0} \\ &= \sum_{i_0,\cdots,i_{n-1}\in M}^{j_1,\cdots,j_{n-1}=0,\cdots,n-1} \hat{A}_{i_{n-1}}^{j_{n-1}}\cdots \hat{A}_{i_1}^{j_1}P\Im B_{i_0} \\ &= \sum_{i_0,\cdots,i_{n-1}\in M}^{j_1,\cdots,j_{n-1}=0,\cdots,n-1} PA_{i_{n-1}}^{j_{n-1}}\cdots \hat{A}_{i_1}^{j_1}\Im B_{i_0} \\ &= P\mathcal{V}(A_k, B_k)_M = \frac{\mathcal{C}+\mathcal{S}}{\mathcal{S}} = \Re^l.\end{aligned}$$

Hence the theorem follows. $\diamondsuit$

Theorem 1 can be rewritten in state space representation. For this sake, let $Q : \Re^n \to \mathcal{S}$ be the canonical projection. Define a matrix $T \in \Re^{n\times n}$ by

$$Tx = Px \oplus Qx, \quad \forall\ x \in \Re^n.$$

It is not hard to see that T is nonsingular. Let

$$\bar{A}_k = TA_kT^{-1},\ \bar{B}_k = TB_k, \quad k \in M. \quad (7)$$

It can be verified that

$$\bar{A}_k = \begin{bmatrix} \bar{A}_{k,1} & \bar{A}_{k,2} \\ 0 & \bar{A}_{k,3}\end{bmatrix},\ \bar{B}_k = \begin{bmatrix} \bar{B}_{k,1} \\ 0 \end{bmatrix},\ k \in M, \quad (8)$$

where $\bar{A}_{k,1} = \hat{A}_k$ and $\bar{B}_{k,1} = \hat{B}_k$. It follows from Theorem 1 that

Theorem 2: Switched system $\sum(A_k, B_k)_M$ is equivalent to $\sum(\bar{A}_k, \bar{B}_k)_M$. Moreover, system $\sum(\bar{A}_{k,1}, \bar{B}_{k,1})_M$ is completely controllable.

System $\sum(\bar{A}_k, \bar{B}_k)_M$ in triangular form (8) is said to be in controllability canonical form.

IV. FEEDBACK EQUIVALENCE AND FEEDBACK REDUCTION

By introducing regular state feedback

$$u_i(t) = F_ix(t) + G_iv_i(t),\ i \in M, v_i \in \Re^{p_i}, \quad (9)$$

where G_i is nonsingular for $i \in M$ and $v_i,\ i \in M$ are the new inputs, the switched system $\sum(A_i, B_i)_M$ is turned into $\sum(A_i + B_iF_i, B_iG_i)_M$.

Proposition 2: The controllable subspace is invariant under any regular state feedback.

Proof. Let $\mathcal{V}$ and $\bar{\mathcal{V}}$ denote the controllable subspaces of $\sum(A_i, B_i)_M$ and $\sum(A_i + B_iF_i, B_iG_i)_M$, respectively. As

$$(A_i + B_iF_i)\mathcal{V} \subseteq A_i\mathcal{V} + \Im B_i = \mathcal{V},$$

$\mathcal{V}$ is $(A_i + B_iF_i)$-invariant. This means that

$$\bar{\mathcal{V}} \subseteq \mathcal{V}.$$

On the other hand, system $\sum(A_i, B_i)_M$ can be seen as the transformed system from $\sum(A_i + B_iF_i, B_iG_i)_M$ via state feedback

$$v_i(t) = -G_i^{-1}F_ix(t) + G_i^{-1}u_i(t),\ i \in M.$$

Hence $\mathcal{V} \subseteq \bar{\mathcal{V}}$ and the proposition follows. $\diamondsuit$

As a corollary, controllability is an invariant property under regular state feedback.

If we implement both coordinate and feedback transformations, the structure of the controllable part can be made simpler than the canonical form presented in the previous subsection. To see this, we first focus on the single-input systems and then on the multi-input systems.

A. FEEDBACK EQUIVALENCE FOR SINGLE-INPUT CASE

Definition 2: Switched system $\sum(A_i, B_i)_M$ is said to be single-input, if rank $B_k = 1$ for some $k \in M$ and $B_j = 0$ for $j \neq k$.

By re-indexing the subsystems, we can always assume that $B_1 \neq 0$ while $B_j = 0$ for $j \geq 2$.

Suppose single-input system $\sum(A_i, B_i)_M$ is completely controllable. Let $\mathcal{S}_0 = \Gamma_{A_1}\Im B_1$. Define recursively that

$$\mathcal{S}_j = \mathcal{S}_{j-1} + \sum_{i\in M} A_i\mathcal{S}_{j-1},\ j = 1, 2, \cdots.$$

Denote $n_k = \dim \mathcal{S}_k$ for $k = 1, 2, \cdots$, and let $\rho = \min\{k : \mathcal{S}_k = \Re^n\} \leq n - n_0$. We can find a basis of $\Re^n$ by the following procedure.

Firstly, let $\gamma_i = A_1^{i-1}B_1$ for $i = 1, \cdots, n_0$.

Secondly, we can find a basis $\gamma_1, \cdots, \gamma_{n_1}$ of $\mathcal{S}_1$ by searching the set

$$\{\gamma_1, \cdots, \gamma_{n_0}, A_2\gamma_1, \cdots, A_2\gamma_{n_0}, \cdots, A_m\gamma_1, \cdots, A_m\gamma_{n_0}\}$$

from left to right.

Continuing the process, suppose we have found a basis $\gamma_1, \cdots, \gamma_{n_0}, \cdots \gamma_{n_{i-1}+1}, \cdots, \gamma_{n_i}$ for $\mathcal{S}_i$. Then, by searching the set

$$\{\gamma_1, \cdots, \gamma_{n_i}, A_1\gamma_{n_{i-1}}, \cdots, A_1\gamma_{n_i}, \cdots, A_m\gamma_{n_{i-1}}, A_m\gamma_{n_i}\}$$

from left to right for linearly independent column vectors, we can find a basis

$$\gamma_1, \cdots, \gamma_{n_0}, \cdots, \gamma_{n_{i-1}+1}, \cdots, \gamma_{n_i}, \gamma_{n_i+1}, \cdots, \gamma_{n_{i+1}}$$

for $\mathcal{S}_{i+1}$.

Finally, we can find a basis

$$\{\gamma_1, \cdots, \gamma_{n_0}, \cdots, \gamma_{n_{\rho-1}+1}, \cdots, \gamma_n\}$$

for $\Re^n$.

By the procedure, for all $j \geq 2$, we can always express γ_j by $A_{i_j}\gamma_{k_j}$ with unique i_j and k_j. For $l \in \bar{n}$, let $\mathcal{E}_l = \text{span}\{e_1, \cdots, e_l\}$. Denote $Q_1 = [\gamma_1, \cdots, \gamma_n]$.

As $\mathcal{S}_0$ is A_1-invariant, we denote by A_{11} the restriction of A_1 in $\mathcal{S}_0$. Suppose the characteristic polynomial of A_{11} is

$$\det(sI - A_{11}) = s^{n_0} + + \cdots + \alpha_{n_0-1}s + \alpha_{n_0}. \qquad (10)$$

Denote

$$Q_2 = \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_{n_0-2} & \alpha_{n_0-1} \\ 0 & 1 & \cdots & \alpha_{n_0-3} & \alpha_{n_0-2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & \alpha_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

and

$$T = Q_1 \begin{bmatrix} Q_2 & 0 \\ 0 & I_{n-n_0} \end{bmatrix}.$$

Let F_1 be the first row of $T^{-1}A_1T$. Introduce coordinate transformation $\bar{x} = T^{-1}x$ and state feedback $u_1 = -F_1T^{-1}x + v_1$, and denote by $\sum(\bar{A}_i, \bar{B}_i)_M$ the transformed system. It is clear that $\bar{B}_1 = e_1$ and the first row of $\bar{A}_1$ is zero. It follows from $\gamma_j = A_{i_j}\gamma_{k_j}$ that

$$T^{-1}A_{i_j}TT^{-1}\gamma_{k_j} = T^{-1}\gamma_j.$$

As $T^{-1}\gamma_j = e_j$, the above equation exactly states that the k_jth column of $\bar{A}_{i_j}$ is e_j. Similarly, from the fact that

$$A_i\mathcal{S}_l \subseteq \mathcal{S}_{l+1},\ \forall\ \ i \in M, l \in \underline{n_0}$$

we know that the jth column of $\bar{A}_i$ is in $\mathcal{E}_{n_{l+1}}$ for $j \geq n_l$. Hence, we arrive at the following conclusion.

Theorem 3: The controllable single-input system $\sum(A_i, B_i)_M$ is equivalent, via suitable coordinate and feedback transformations, to normal system $\sum(\bar{A}_i, \bar{B}_i)_M$ with

(i) $\bar{B}_1 = e_1$ and both the first row and the n_0th column of $\bar{A}_1$ are zero;

(ii) For all $j \geq n_l$ and $i \in M$, the jth column of $\bar{A}_i$ is in $\mathcal{E}_{n_{l+1}}$; and

(iii) For all $j \geq 2$, the k_jth column of $\bar{A}_{i_j}$ is e_j.

In particular, when the system degenerates to the linear time-invariant case, the normal form becomes

$$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \right)$$

which is the standard normal form for controllable single-input systems.

For a general switched system, the normal form is not unique. In fact, only n columns in $\bar{A}_1, \cdots, \bar{A}_m$ are fixed, they are the k_jth column of $\bar{A}_{i_j}$ for $j = 2, \cdots, n$, together with the n_0th column of $\bar{A}_1$. Other $(n-1)m$ columns, though may submit to certain constraints, have free parameters other than zero. To see this, we examine a controllable second-order single-input system with two subsystems. By the above searching procedure, the system must follows into one of the two cases:

(a) rank $[B_1, A_1B_1] = 2$;

(b) rank $[B_1, A_1B_1] = 1$ but rank $[B_1, A_2B_1] = 2$.

In the former case, $(\bar{A}_1, \bar{B}_1)$ is in the standard normal form for controllable single-input systems, and $\bar{A}_2$ is not necessarily in any specific form. In the latter case, the first column $\bar{A}_2$ is fixed, $\bar{A}_1$ is constrained but the other column of $\bar{A}_2$ are totally unspecified. Hence the matrices are in form

$$\begin{aligned} \bar{A}_1 &= \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \\ \bar{A}_2 &= \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}, \end{aligned}$$

where '*' stands for a real-valued number whose value cannot be determined by the controllability property.

Note that for any controllable single-input switched system, it is always possible to classify in the same fashion and then write out the corresponding normal forms. However, as the system order and the number of subsystems increase, a detailed classification becomes more and more tedious.

B. FEEDBACK REDUCTION FOR MULTI-INPUT CASE

For controllable multi-input switched linear systems, normal forms under coordinate and feedback transformations can be obtained using the same method as in the previous subsection. In particular, Theorem 3 could be extended to the multi-input case. However, comparing to the single-input case, the normal forms are more complex and the system structures are less clear. Since the system decomposition mainly serves for addressing synthesis problems such as feedback stabilization and regulation, a better way to this end is to change the multi-input problem into a single-input problem, just as in the standard linear system theory.

To change a multi-input system into a single-input one, we need a nonregular linear state feedback in form

$$u_i(t) = F_i x + G_i v, \ i \in M,$$

where the gain matrix $G = [G_1, \cdots, G_m]$ only has a non-zero column vector. The idea of using nonregular state feedbacks in control system design could be traced back to the work of [8] which showed that a multi–input controllable linear system can always be brought to a single–input controllable linear system via a nonregular static state feedback, thus enabling an easy proof of the pole assignment theorem for the multi–input case. This idea was generalized to nonlinear case in [21], [22]. Other nonregular state feedback scheme could be found in the decoupling problem [14], model matching [9], and feedback linearization [18], [5].

The following lemma plays a central role in finding a nonregular state feedback to change a controllable multi-input switched system into a controllable single-input system.

Lemma 2: Suppose switched linear system $\sum(A_i, B_i)_M$ is controllable. Then, for any non-zero vector $b \in \bigcup_{k\in M} \Im B_k$, there exist index sequences $i_1, \cdots, i_{n-1}$ and $l_1, \cdots, l_{n-1}$ with $l_j \le j$, and input sequence $u_{i_1}, \cdots, u_{i_{n-1}}$, such that the sequence defined by

$$\begin{aligned} \eta_1 &= b, \\ \eta_{k+1} &= A_{i_k}\eta_{l_k} + B_{i_k}u_{i_k} \end{aligned} \tag{11}$$

for $k = 1, \cdots, n-1$ is independent.

Proof. We proceed by induction. As $\eta_1 \ne 0$ hence it is independent. Suppose that $k < n-1$ and $\eta_1, \cdots, \eta_k$ have been constructed according to (11) and are independent. Denote by $\mathcal{L}_k$ the linear subspace generated by $\eta_1, \cdots, \eta_k$. We have to choose i_k, l_k and u_{i_k} such that

$$\eta_{k+1} = A_{i_k}\eta_{l_k} + B_{i_k}u_{i_k} \notin \mathcal{L}_k.$$

If this is not possible, then

$$A_{i_k}\eta_{l_k} + B_{i_k}u_{i_k} \in \mathcal{L}_k, \ \forall\ i_k \in M, \ l_k \in \bar{k}, \ u_{i_k} \in \Re^{p_{i_k}}.$$

Let $u_{i_k} = 0$, we have

$$A_{i_k}\eta_{l_k} \in \mathcal{L}_k, \forall\ i_k \in M, \ l_k \in \bar{k}.$$

In other words, $\mathcal{L}_k$ is A_i-invariant for all $i \in M$. At the same time, $B_{i_k}u_{i_k} \in \mathcal{L}_k$ for all u_{i_k}. This means that $\sum_{i\in M} \Im B_i \subseteq \mathcal{L}_k$. Now that $\mathcal{L}_k$ is A_i-invariant and contains $\sum_{i\in M} \Im B_i$, it must contain the controllable subspace of system $\sum(A_i, B_i)_M$ as a subset. This is a contradiction because the system is controllable. $\diamondsuit$

Remark 1: The lemma is an extension of [6, Lemma 2] from linear systems to switched linear systems. Note that a key difference between this lemma and [6, Lemma 2] is that we does not impose that $l_k = k$, as did in [6, Lemma 2]. This relaxation is necessary as exhibited in the following example:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \\ A_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

For this system, we have $l_2 = l_3 = 1$.

Although Lemma 2 only asserts the existence of the independent vectors $\eta_1, \cdots, \eta_n$, a constructive procedure can be formulated from the proof. Indeed, suppose $\eta_1, \cdots, \eta_k$ are known for some $k < n$. Denote

$$\mathcal{L}_k = \text{span}\{\eta_1, \cdots, \eta_k\},$$

and

$$\mathcal{V}_k^j = A_j\mathcal{L}_k + \Im B_j, \ j \in M.$$

Comparing $\mathcal{V}_k^j$ with $\mathcal{L}_k$ for $j = 1, \cdots, m$, we can always find an $i_k \in M$, such that

$$\mathcal{V}_k^{i_k} \not\subseteq \mathcal{L}_k.$$

If

$$A_{i_k}\mathcal{L}_k \not\subseteq \mathcal{L}_k,$$

then we can find a l_k, such that

$$A_{i_k}\eta_{l_k} \not\subseteq \mathcal{L}_k.$$

In this case, let $u_{i_k} = 0$. Otherwise, we have

$$\Im B_{i_k} \not\subseteq \mathcal{L}_k,$$

hence there is a unit column vector e_j, such that

$$B_{i_k}e_j \not\subseteq \mathcal{L}_k.$$

In this case, let $l_k = k$ and $u_{i_k} = e_j$. In either case, we have

$$A_{i_k}\eta_{l_k} + B_{i_k}u_{i_k} \not\subseteq \mathcal{L}_k. \tag{12}$$

Note that the process is constructive and in this way we can find an independent sequence $\eta_1, \cdots, \eta_n$. Moreover, if (12) holds for a u_{i_k}, it must also hold for almost all $u_{i_k} \in \Re^{p_{i_k}}$. This is because the relation $A_{i_k}\eta_{l_k} + B_{i_k}u_{i_k} \subseteq \mathcal{L}_k$ is an algebraic constraint and hence the solutions (in u_{i_k}) form an algebraic set in $\Re^{p_{i_k}}$.

By means of Lemma 2, we obtain the main result in this subsection.

Theorem 4: Any controllable multi-input system can be reduced to a controllable single-input system via a nonregular state feedback.

Proof. Choose a non-zero vector b from $\cup_{i\in M}\Im B_i$. By Lemma 2, we can construct a basis $\{\eta_1, \cdots, \eta_n\}$ of $\Re^n$ satisfying formula (11). Let gain matrices F_i, $i \in M$ satisfy, otherwise arbitrary, that $F_{i_k}\eta_{l_k} = u_{i_k}$ for $k = 1, \cdots, n-1$. Note that the choice of such F_i is always possible since η_k are independent. Each η_k can be expressed by

$$\eta_k = (A_{\kappa_j} + B_{\kappa_j}F_{\kappa_j}) \cdots (A_{\kappa_1} + B_{\kappa_1}F_{\kappa_1})b$$

for some j and $\kappa_l \in M$. This implies that each η_k is in the controllable subspace of system $\sum(A_i + B_iF_i, b)_M$. As a consequence, system $\sum(A_i + B_iF_i, b)_M$ is controllable.

Suppose $b \in \Im B_j$. Let $b_j = b$ and $b_l = 0$, $l \neq j$. It can be seen that the single-input system $\sum(A_i + B_iF_i, b_i)_M$ is also controllable. By introducing nonregular state feedback

$$\begin{aligned} u_j &= F_jx + G_jv_j, \\ u_k &= F_kx, \ k \neq j, \end{aligned}$$

where G_j satisfies $B_jG_j = b$, the original multi-input system $\sum(A_i, B_i)_M$ is changed into the single-input system $\sum(A_i + B_iF_i, b_i)_M$ which is controllable. $\diamondsuit$

From the proof, and the discussion before the theorem, we in fact can draw a stronger conclusion as follows.

Corollary 2: Suppose multi-input system $\sum(A_i, B_i)_M$ is controllable. Let matrices $G_i \in \Re^{p_i}$ be such that $[B_1G_1, \cdots, B_mG_m]$ has only one non-zero column. Then, for almost all $F_i \in \Re^{p_i \times n}$, $i \in M$, the single-input system $\sum(A_i + B_iF_i, B_iG_i)_M$ is also controllable.

Another corollary can be obtained by combining Theorems 3 and 4.

Corollary 3: Any controllable multi-input switched system can be reduced to a controllable single-input normal form via proper coordinate and (nonregular) feedback input transformations.

Example 1: Suppose we have a multi-input system $\sum(A_i, B_i)_{\bar{2}}$ with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ -2 & 0 & 0 & 0 & 2 \\ 2 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & -2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}; \\ A_2 &= 0, \\ B_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Let

$$\begin{aligned} G_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ G_2 &= 0, \end{aligned}$$

and

$$\begin{aligned} F_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ F_2 &= [1\ 0\ 0\ 0\ 0]. \end{aligned}$$

It can be verified that the single-input system $\sum(A_i + B_iF_i, B_iG_i)_{\bar{2}}$ is controllable. By applying the searching procedure as in the previous subsection, we have

$$Q_1 = \begin{bmatrix} 1 & 1 & -4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Q_2 = I_3.$$

Let $T = Q_1 \text{diag}\ [Q_2, I_2] = Q_1$. Let F_1' be the first line of matrix $T^{-1}(A_1 + B_1F_1)T$. Denote

$$\begin{aligned} \bar{A}_1 &= T^{-1}(A_1 + B_1F_1)T - T^{-1}B_1G_1F_1' \\ &= T^{-1}A_1T + T^{-1}B_1(F_1T - G_1F_1'), \\ \bar{B}_1 &= T^{-1}B_1G_1, \end{aligned}$$

and and

$$\begin{aligned} \bar{A}_2 &= T^{-1}(A_2 + B_2F_2)T, \\ \bar{B}_2 &= T^{-1}B_2G_2. \end{aligned}$$

The system $\sum(\bar{A}_i, \bar{B}_i)_{\bar{2}}$ is in the normal form (Cf. Theorem 3) with

$$\begin{aligned} \bar{A}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \\ \bar{B}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \bar{A}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{B}_2 &= 0. \end{aligned}$$

This normal form is the reduced system from the original system via coordinate change

$$\bar{x} = T^{-1}x,$$

and input transformation

$$\begin{aligned} u_1 &= (F_1 - G_1F_1'T^{-1})x + G_1v_1, \\ u_2 &= F_2x + G_2v_2. \end{aligned}$$

V. CONCLUSION

In this paper, a framework for finding normal forms has been developed for switched linear systems where both the switching signal and the control input are design variables. We proved that, each switched linear system can be changed via an equivalent coordinate transformation into the canonical controllability form with clear structural information. For single-input controllable systems, we presented normal forms which extend the standard controllability normal form of linear time-invariant systems. For multi-input controllable systems, we showed that they could be reduced to single-input controllable systems via nonregular state feedbacks. These results are generalizations of the well-known results from standard linear system theory.

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