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An L_2-Gain Analysis of Piecewise Affine Systems by Piecewise Quadratic Storage Functions

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***Abstract*— Piecewise affine (PWA) systems, which belong to a class of hybrid systems receiving a lot of attention, are useful for describing dynamics of real-world systems. A PWA system, whose dynamic is composed of a finite number of affine dynamics and switching laws, is at an advantage because existing analysis and control methods for linear systems may be applied to the system with a little modification. Based on this concept, in existing literature, some analysis conditions via piecewise quadratic functions were suggested for PWA systems whose switchings of dynamics depend only on their states. The conditions, however, are not satisfactory enough, because real-world systems like control systems with input saturations and dead-zone nonlinearities, which can be frequently described as PWA systems, are usually dominated by switching laws which depend both on their states and on inputs. This paper discusses the L_2-gain analysis problem via piecewise quadratic storage functions for PWA systems whose switching laws of dynamics depend not only on their states but also on exogenous inputs.**

I. Introduction

Hybrid systems, which are dominated by continuous dynamics and discrete event dynamics, have recently attracted interests of many researchers in the field of control engineering[1], [2]. Piecewise affine (PWA) systems, which are driven by dynamics composed of a finite number of affine dynamics and a set of switching rules, are one of the most typical classes of hybrid systems. Although the dynamics of PWA systems appear to be simple, in fact, they are very complicated. Analysis and control of hybrid systems are generally difficult[5], [6].

PWA systems, however, are at an advantage because the structure of their dynamics may allow applications of existing methods for linear systems to themselves with a little modification. For example, a (globally) quadratic Lyapunov function, which makes it possible to analyze exponential stability of a linear system with two linear matrix inequalities (LMIs), enables exponential stability analysis for a PWA system with a finite number of LMIs.

Romanchuk and Smith[3] adopted this methodology and proposed an L_2 incremental gain analysis condition for control systems with input saturations which are equivalently described as PWA systems. Some numerical examples showed that the condition achieves acceptable results with comparatively tractable computations. The use of a quadratic storage function to all affine dynamics of a PWA system, however, may prevent taking the maximum advantage of the structure of its dynamic, in other words, results via the proposed condition may still involve a lot of conservatism.

For this problem, Rantzer and Johansson[4] suggested a stability analysis condition, an L_2-gain analysis condition, etc. for PWA systems via piecewise quadratic functions. By numerical evaluations, the conditions were proved to be more effective than conditions via globally quadratic functions. The conditions, however, can be applied to just PWA systems whose switching laws of dynamics depend on only their states. PWA systems considered in [3], which are derived from control systems with input saturation, are generally driven by switching laws which depend not only on their states but also on exogenous inputs. Since input saturations exist in most real-world systems, the conditions in [4] do not seem to be satisfactory enough for practical use.

The present paper generalizes the technique of [4], and suggests an L_2-gain analysis condition via piecewise quadratic storage functions to the PWA systems whose switching laws depend on both their states and exogenous inputs. Another condition, which is based on globally quadratic storage functions, is also provided for comparison. Unfortunately, from the viewpoint of practical use, the condition via piecewise quadratic storage functions may frequently fail to achieve less conservatism than the condition via globally quadratic storage functions. Therefore, this paper discusses an additional operation to boost possibility of reducing conservatism of the condition via piecewise quadratic storage functions.

Since PWA systems are a class of nonlinear systems, local analysis with limitation on amplitude of inputs is very significant as well as global analysis, in which no limitation is imposed on the amplitude. The conditions given in this paper are also able to achieve the local analysis, though it was discussed neither in [4] nor in [3].

This paper is organized as follows. Formulation of intended PWA systems, whose switching laws of dynamics depend both on their states and inputs, will be given with some assumptions in the next section. In section 3, we will derive two analysis conditions for this class of PWA systems: One is based on globally quadratic storage functions, and the other is based on piecewise quadratic storage functions. An operation to enhance effectiveness of the latter condition will be also discussed in this section. In section 4, the superiority of the latter condition to the former condition will be shown by a numerical example.

II. Piecewise Affine Systems

This paper considers the following PWA system Υ:

$$\begin{aligned}\dot{x} &= f(x,w)\\ &= A_i x + B_i w + a_i,\ [x^T\ \ w^T]^T \in \Omega_i \qquad (1)\\ z &= h(x,w)\\ &= C_i x + D_i w + c_i,\ [x^T\ \ w^T]^T \in \Omega_i \qquad (2)\\ & f(x,w),\ h(x,w) : \text{Continuous about } x \text{ and } w,\end{aligned}$$

where $x \in \boldsymbol{R}^n$, $w \in \boldsymbol{R}^l$ and $z \in \boldsymbol{R}^p$ stand for states, inputs and outputs, respectively. Ω_i, which is described as

$$\Omega_i = \{[x^T\ \ w^T]^T |\ G_i^x x + G_i^w w + g_i \geq 0\}, \qquad (3)$$

is a closed convex polyhedral set obtained by dividing the space of states and inputs $\boldsymbol{R}^{n+l}$, and satisfies

$$\begin{aligned}\cup_{i=1}^{N_O} \Omega_i = \boldsymbol{R}^{n+l},\quad \mathrm{int}\,\Omega_i \neq \phi,\ \forall i\\ \mathrm{int}\,\Omega_i \cap \mathrm{int}\,\Omega_j = \phi,\ i \neq j,\ \forall i,\ j\end{aligned}$$

where N_O stands for the total number of the affine dynamics of Υ. We define index sets I_0 and I_1 as

$$I_0 \equiv \{\tilde{i}|\ 0 \in \Omega_{\tilde{i}}\},\ I_1 \equiv \{\tilde{i}|\ 0 \notin \Omega_{\tilde{i}}\}$$

and assume that

$$a_{i_0} = 0,\ c_{i_0} = 0,\ \forall i_0 \in I_0.$$

Many systems with saturation nonlinearity can be equivalently described as this class of PWA systems[3], [9], [10]. Therefore, development of systematic analysis and controller design approach for this class of PWA systems is practically of great significance.

Through this paper, we assume that the initial state $x(0)$ is 0 and that input signals w satisfy

$$w \in \mathcal{L}_{2+} \equiv \left\{ \tilde{w} \;\middle|\; \int_0^\infty \tilde{w}(\tau)^T \tilde{w}(\tau) d\tau < \infty \right\},$$

and describe $\left(\int_0^\infty w(\tau)^T w(\tau) d\tau\right)^{1/2}$ as $||w||_2$.

Remark 1: A differential equation which describes a dynamic of a hybrid system like a PWA system, may not have the well-posed solution, in other words, the globally unique solution. Furthermore, it is quite difficult to determine whether or not the solution exists for the differential equation[5], [6]. To the equation (1), however, the well-posed solution always exists, since $f(x,w)$ is continuous and w belongs to $\mathcal{L}_{2+}$.

III. L_2-Gain Analysis

In this section, we consider the L_2-gain as a performance criterion of control systems, and derive two analysis conditions of the performance. First, an analysis condition via globally quadratic storage functions is derived, and next, analysis via piecewise quadratic storage functions is discussed.

A. Analysis via Globally Quadratic Storage Functions

Unlike a linear system, input-output gains of a nonlinear system, in which PWA systems are included, are not independent of the amplitude of input signals. Therefore, it is important as well to evaluate input-output gains under limitations on the amplitude of inputs. For this reason, we define the local L_2-gain in which the Euclidean norm of input signals are bounded, as well as the global L_2-gain.

Definition 1: The global L_2-gain $||\Upsilon||_{L_2}$ and the local L_2-gain $||\Upsilon||_{L_2,\xi}$ for the system Υ are defined as follows:

$$\begin{aligned}||\Upsilon||_{L_2} &\equiv \sup_{w \in \mathcal{L}_{2+}\setminus\{0\}} \frac{||z||_2}{||w||_2}\\ ||\Upsilon||_{L_2,\xi} &\equiv \sup_{w \in \mathcal{L}_{2+}\setminus\{0\}, ||w(\cdot)|| \leq \xi} \frac{||z||_2}{||w||_2},\end{aligned}$$

where $||w(\cdot)||$ stands for the Euclidean norm of $w(\cdot)$.

For analysis of the global L_2-gain, the concept of dissipativity of a system is useful, that is, if there exists a positive semidefinite function $V(x)$ which satisfies

$$V(x(t)) \leq V(x(0)) + \int_0^t s(w(\tau), z(\tau)) d\tau,\ \forall t \geq 0 \qquad (4)$$

for a supply rate $s(w,z)$ defined as

$$s(w,z) \equiv \gamma^2 w^T w - z^T z,\ \gamma > 0,$$

then $||\Upsilon||_{L_2}$ is bounded by γ[7].

The function $V(x)$ is called a storage function, and we first consider a globally quadratic storage function

$$V(x) = V_g(x) \equiv x^T X x. \qquad (5)$$

Because x is a function of class C^1, V is differentiable on t and the inequality

$$\frac{dV}{dt}(t) - \gamma^2 w(t)^T w(t) + z(t)^T z(t) \leq 0,\ \forall t \geq 0 \qquad (6)$$

is obtained by differentiation of (4). This is a necessary and sufficient condition of (4), and a sufficient condition of $||\Upsilon||_{L_2} \leq \gamma$ if $V(x(t)) \geq V(x(0))$ holds for any $t \geq 0$. This argument leads to the following theorem.

Theorem 1: If, for all $i \in \{1, \ldots, N_O\}$, there exist a positive semidefinite matrix X, nonnegative numbers α_i and square matrices W_i with nonnegative entries which satisfy

$$\begin{bmatrix} \underline{A}_i^T X \overline{I} + \overline{I}^T X \underline{A}_i + \overline{C}_i^T \overline{C}_i + \overline{E}_i^T W_i \overline{E}_i + \alpha_i \Xi^T \Xi & \overline{I}^T X B_i + \overline{C}_i^T D_i + \overline{E}_i^T W_i E_i^w \\ B_i^T X \overline{I} + D_i^T \overline{C}_i + E_i^{wT} W_i \overline{E}_i & D_i^T D_i - (\gamma^2 + \alpha_i) I + E_i^{wT} W_i E_i^w \end{bmatrix} \leq 0, \quad (7)$$

then $||\Upsilon||_{L_2,\xi} \leq \gamma$. Furthermore, if $\alpha_i = 0,\ \forall i$ holds as well, then $||\Upsilon||_{L_2} \leq \gamma$ holds. Matrices in (7) are defined as

$$\begin{aligned}\overline{I} \equiv [I\ \ 0],\ \underline{A}_i \equiv [A_i\ \ a_i],\ \overline{C}_i \equiv [C_i\ \ c_i]\\ \Xi \equiv [0\ \ \xi],\ \overline{E}_i \equiv [E_i^x\ \ e_i],\end{aligned}$$

and E_i^x, E_i^w and e_i are defined as an appropriate choice of matrices which satisfy

$$[x^T\ w^T]^T \in \Omega_i \Rightarrow E_i^x x + E_i^w w + e_i \geq 0. \qquad (8)$$

Proof: Substitution of (1), (2) and (5) into (6) gives

$$\begin{bmatrix} x(t) \\ 1 \\ w(t) \end{bmatrix}^T \Theta(i) \begin{bmatrix} x(t) \\ 1 \\ w(t) \end{bmatrix} \leq 0, \quad \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \Omega_i \qquad (9)$$

as a necessary and sufficient condition of (6), where $\Theta(i)$ is defined as

$$\Theta(i) \equiv \begin{bmatrix} \underline{A}_i^T X \overline{I} + \overline{I}^T X \underline{A}_i + \overline{C}_i^T \overline{C}_i & \overline{I}^T X B_i + \overline{C}_i^T D_i \\ B_i^T X \overline{I} + D_i^T \overline{C}_i & D_i^T D_i - \gamma^2 I \end{bmatrix}.$$

Although $\Theta(i) \leq 0$, which is a sufficient condition of (9), can be solved with efficient computation, it would be quite conservative because it means that the inequality of (9) holds for all $[x(t)^T\ w(t)^T]^T$, that is, the inequality must hold for all $[x(t)^T\ w(t)^T]^T \notin \Omega_i$ as well. For this problem, the S-procedure is useful[8]. The S-procedure with matrices E_i^x, E_i^w and e_i defined by (8) leads to

$$\begin{bmatrix} \underline{A}_i^T X \overline{I} + \overline{I}^T X \underline{A}_i + \overline{C}_i^T \overline{C}_i + \overline{E}_i^T W_i \overline{E}_i & \overline{I}^T X B_i + \overline{C}_i^T D_i + \overline{E}_i^T W_i E_i^w \\ B_i^T X \overline{I} + D_i^T \overline{C}_i + E_i^{wT} W_i \overline{E}_i & D_i^T D_i - \gamma^2 I + E_i^{wT} W_i E_i^w \end{bmatrix} \leq 0,$$

which no longer assures that the inequality of (9) always holds for $[x(t)^T\ w(t)^T]^T \notin \Omega_i$. If $||w(t)|| \leq \xi$ holds, (7) is obtained by additional utilization of the S-procedure with $\alpha_i(\xi^2 - ||w(t)||^2) \geq 0,\ \alpha_i \geq 0$ in the same way, and the proof is completed. ∎

B. Analysis via Piecewise Quadratic Storage Functions

Next, we consider analysis via piecewise quadratic storage functions.

1) Preliminaries: Each region Ω_i belongs to one of the following sets:

$$Cls_A \equiv \left\{\Omega_i \,\middle|\, N_\varepsilon([\hat{x}^T\ 0^T]^T) \subset \Omega_i,\ \exists \hat{x} \in \boldsymbol{R}^n,\ \exists \varepsilon > 0\right\}$$
$$Cls_B \equiv \{\Omega_i |\ \Omega_i \notin Cls_A\},$$

where $N_\varepsilon(v)$ stands for the ε-neighborhood of v defined as $\{\tilde{v}|\ ||\tilde{v} - v|| < \varepsilon\}$. Cls_A is the set of all the regions Ω_i which strictly intersect the x-hyperplane, and Cls_B is the set of all the other regions. Numerical computation enables to determine which of the two sets each region Ω_i belongs to. We describe the number of elements of Cls_A as $N(\leq N_O)$, and reassign the subscripts $\{1, \ldots, N\}$ and the remaining subscripts $\{N+1, \ldots, N_O\}$ to the elements of Cls_A and of Cls_B, respectively.

Regions classified into Cls_A make a partition on the x-hyperplane, and we define each region made by the partition as X_j (see Fig. 1), which satisfies

$$\operatorname{int} X_i \cap \operatorname{int} X_j = \phi,\ i \neq j,\ \forall i,\ j$$
$$\operatorname{int} X_j \neq \phi,\ \forall j, \quad \cup_{j=1}^N X_j = \boldsymbol{R}^n.$$

From (3), X_j is described as

$$X_j = \{x|\ [x^T\ 0^T]^T \in \Omega_j,\ \Omega_j \in Cls_A\}$$
$$= \{x|\ G_j^x x + g_j \geq 0\}.$$

For each subscript $j \in \{1, \ldots, N\}$, we define a set $\Gamma(j)$ as

$$\Gamma(j) \equiv \{\tilde{i}|\ [x^T\ \hat{w}^T]^T \in \operatorname{int} \Omega_{\tilde{i}},\ \exists \hat{w},\ \exists x \in X_j\}.$$

This is a set of the subscript of each region Ω_i whose projection onto the x-hyperplane strictly intersects X_j. For example, in Fig. 1, the subscripts 3, 4, 5, 7, 8 belong to $\Gamma(4)$ and 1, 2, 6 do not.

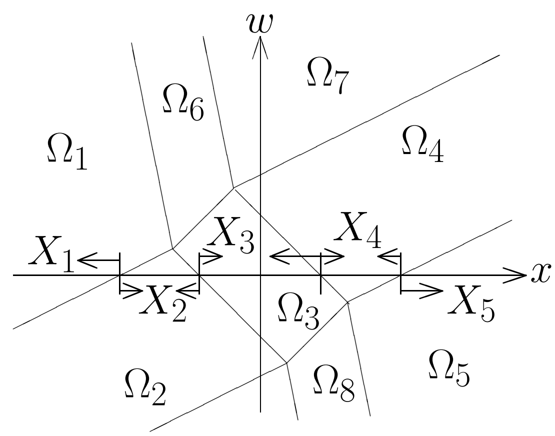

Fig. 1. Relation between X_i and Ω_i.

2) L_2-Gain Analysis Condition: In order to construct piecewise quadratic functions, we define matrices F_i and f_i as those which satisfy

$$F_i x + f_i = F_j x + f_j,\ \forall x \in X_i \cap X_j, \quad f_{i_0} = 0,\ \forall i_0 \in I_0 \qquad (10)$$

and define $\overline{F}_i$ as $[F_i\ \ f_i]$[4]. With these matrices and a symmetric matrix T, we define a function V_p as follows:

$$V_p(x(t)) \equiv \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T \overline{F}_j^T A \overline{F}_j \begin{bmatrix} x(t) \\ 1 \end{bmatrix},\ j \in \{\tilde{j}|\ x(t) \in X_{\tilde{j}}\}.$$

Then, $V_p(x)$ is a continuous and piecewise quadratic function. Furthermore, $V_p(x(t))$ is an absolutely continuous function about t[9], and is therefore differentiable almost everywhere in the arbitrary finite interval $\hat{I}$ defined as $[0, \hat{T}]$ for any fixed $\hat{T} \geq 0$. We describe the set of all the points at which $V_p(x(t))$ is differentiable as I_d, and assume V given by $V = V_p$ satisfies the following inequality:

$$\frac{dV}{dt}(t) - \gamma^2 w(t)^T w(t) + z(t)^T z(t) \leq 0, \quad \forall t \in I_d. \qquad (11)$$

This is a necessary and sufficient condition of (4) as well as (6), and a sufficient condition of $||\Upsilon||_{L_2} \leq \gamma$ if $V(x(t)) \geq V(x(0))$ holds for any $t \geq 0$. This argument leads to the following theorem.

Theorem 2: If, for all $j \in \{1, \ldots, N\}$ and $i \in \Gamma(j)$, there exist a symmetric matrix T, nonnegative numbers α_{ij}

and square matrices U_j, W_{ij}, Y_{ij} with nonnegative entries which satisfy the following inequalities, then $||\Upsilon||_{L_2,\xi} \le \gamma$ holds. Furthermore, if $\alpha_{ij} = 0,\ \forall i,\ j$ is satisfied as well, then $||\Upsilon||_{L_2} \le \gamma$ holds.

$$\begin{bmatrix} \overline{A}_i^T P_j + P_j\overline{A}_i + \overline{C}_i^T\overline{C}_i + \overline{E}_i^T W_{ij}\overline{E}_i + \overline{E}_j^T Y_{ij}\overline{E}_j + \alpha_{ij}\Xi^T\Xi & P_j\overline{B}_i + \overline{C}_i^T D_i + \overline{E}_i^T W_{ij} E_i^w \\ \overline{B}_i^T P_j + D_i^T\overline{C}_i + E_i^{wT} W_{ij}\overline{E}_i & D_i^T D_i - (\gamma^2 + \alpha_{ij})I + E_i^{wT} W_{ij} E_i^w \end{bmatrix} \le 0 \quad (12)$$

$$P_j - \overline{E}_j^T U_j \overline{E}_j \ge 0 \quad (13)$$

$$P_j \equiv \overline{F}_j^T T \overline{F}_j$$

$\overline{A}_i$ and $\overline{B}_i$ are defined as follows:

$$\overline{A}_i \equiv \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix},\ \overline{B}_i \equiv \begin{bmatrix} B_i \\ 0 \end{bmatrix}.$$

Proof: $V(x(t))$ satisfies

$$\frac{dV}{dt}(t) = 2\begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T P_j \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix},\ j \in \{\tilde{j}|\ x(t) \in X_{\tilde{j}}\}$$

on I_d[10]. With (1) and (2), the above equality gives

$$\begin{bmatrix} x(t) \\ 1 \\ w(t) \end{bmatrix}^T \hat{\Theta}(i,j) \begin{bmatrix} x(t) \\ 1 \\ w(t) \end{bmatrix} \le 0,$$

$$x(t) \in X_j,\ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \Omega_i,\ t \in I_d \quad (14)$$

as a necessary and sufficient condition of (11), where $\hat{\Theta}(i,j)$ is defined as

$$\hat{\Theta}(i,j) \equiv \begin{bmatrix} \overline{A}_i^T P_j + P_j\overline{A}_i + \overline{C}_i^T\overline{C}_i & P_j\overline{B}_i + \overline{C}_i^T D_i \\ \overline{B}_i^T P_j + D_i^T\overline{C}_i & D_i^T D_i - \gamma^2 I \end{bmatrix}.$$

Although $\hat{\Theta}(i,j) \le 0$, which is a sufficient condition of (14), can be solved by efficient computations, it would be quite conservative for the same reason described in the proof of Theorem 1. Hence, we use the S-procedure as well, and obtain

$$\begin{bmatrix} \overline{A}_i^T P_j + P_j\overline{A}_i + \overline{C}_i^T\overline{C}_i + \overline{E}_i^T W_{ij}\overline{E}_i + \overline{E}_j^T Y_{ij}\overline{E}_j & P_j\overline{B}_i + \overline{C}_i^T D_i + \overline{E}_i^T W_{ij} E_i^w \\ \overline{B}_i^T P_j + D_i^T\overline{C}_i + E_i^{wT} W_{ij}\overline{E}_i & D_i^T D_i - \gamma^2 I + E_i^{wT} W_{ij} E_i^w \end{bmatrix} \le 0$$

as a less conservative sufficient condition of (14). If $||w(t)|| \le \xi,\ \forall t$ holds, additional use of the S-procedure for $\alpha_{ij}(\xi^2 - ||w(t)||^2) \ge 0$ derives (12) itself as a sufficient condition of (14). Additionally, utilization of the S-procedure for $P_j \ge 0$, which is a sufficient condition of $V(x(t)) \ge V(x(0))$, leads to (13) by the same token. ■

Remark 2: At least one choice of the matrices F_i and f_i which satisfy (10) always exists[10]. Additionally, a systematic method for obtaining a choice of those matrices is described in [4].

Remark 3: Since the trajectory x may move on the common boundary of plural regions $\{X_j\}$, that is, sliding modes may exist, V_p is not always a piecewise C^1 function about t, though the piecewise quadratic function considered in [4] were referred to as a piecewise C^1 function.

Remark 4: In [9], we derived a similar L_2-gain analysis condition to Theorem 2. The difference between the condition and Theorem 2 is that the condition has the form of (12) with $W_{ij} = 0,\ \forall i,\ j$. This difference causes nonexistence of solutions of the condition, because the (1,1)-block of (12) is never negative semidefinite when $j \in I_0$ and $i \in I_1$ with $W_{ij} = 0,\ \forall i,\ j$[9]. In Theorem 2, however, (12) has possibility of existence of solutions, because the terms of W_{ij} exist.

3) Enhancement of Advantage of Theorem 2: The advantage of Theorem 2 over Theorem 1 is given by the following two points:

1) The number of combinations of j and i in (12) is reduced to that of j and elements of $\Gamma(j)$.
2) The S-procedure in (13) allows P_j not to be always positive semidefinite.

When Theorem 2 is used for a control system with input constraints which can be described as a PWA system, however, Theorem 2 may not have any advantage over Theorem 1 because of the following reasons:

- The dimension of constrained signals m, which satisfies $N_O \le 3^m$[9], [10], is smaller than $n + l$, and therefore, the number of elements of $\Gamma(j)$ is usually equal to N_O.
- Because I_0 includes just one element, that is, the origin of $\boldsymbol{R}^{n+l}$ and that of $\boldsymbol{R}^n$ is an interior point of just one region Ω_{i_0} and X_{i_0}, respectively, (13) does not hold if $U_{i_0} \ne 0$, that is, P_{i_0} must be always positive semidefinite.

In order to resolve this problem, we add hyperplanes which include the origin to the space $\boldsymbol{R}^{n+l}$ partitioned into N_O regions (see Fig. 2). This operation partitions the region

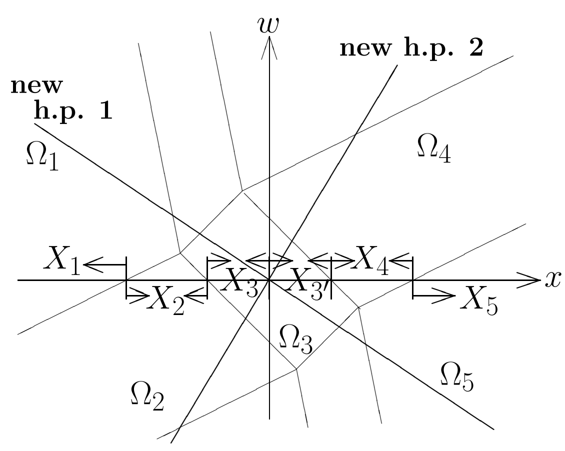

Fig. 2. Addition of hyperplanes.

Ω_{i_0}, which includes the origin as an interior point, into plural regions $\{\Omega_{i_0^k}\}$ which contain the origin as a boundary point. This may enable (13) to be satisfied with $U_{i_0^k} \neq 0$, in other words, $P_{i_0^k}$ which is not positive semidefinite may be acceptable. Furthermore, the number of elements of $\Gamma(j)$ may become smaller than N_O for each j in (12). This may achieve less conservatism in Theorem 2.

From these effects, addition of hyperplanes which include the origin may enhance advantage of utilization of piecewise quadratic storage functions, and may improve results by Theorem 2.

Remark 5: This operation has already suggested for the analysis conditions via piecewise quadratic functions in [4] to achieve less conservatism than those with the initial partition. In [4], the operation was suggested to improve results by the conditions via piecewise quadratic functions. In other words, the operation is not always necessary to achieve less conservatism than the condition via globally quadratic functions. On the other hand, in Theorem 2, the operation is sometimes necessary to obtain better results than Theorem 1 as explained above, though it is useful to improve the results as well.

IV. Numerical Example

In this section, we consider a system shown in Fig. 3 for numerical evaluation. P is a linear system and σ_0 is a standard saturation factor, which are defined as follows:

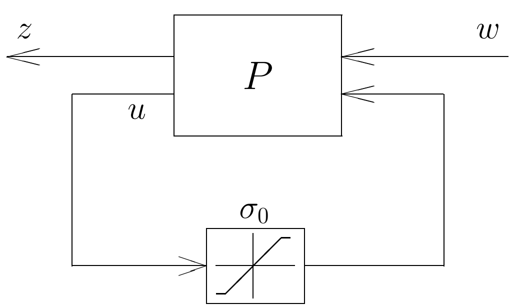

Fig. 3. A system with saturation nonlinearity.

$$P = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

$$\tilde{A} = \begin{bmatrix} -0.5 & 0 \\ -0.5 & -5.5 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 \\ -0.25 & 0.75 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} 1 & 0 \\ -0.25 & 0.75 \end{bmatrix}$$

$$(\sigma_0(u))_k = \begin{cases} 1, & u_k \geq 1 \\ u_k, & |u_k| < 1 \\ -1, & u_k \leq -1 \end{cases}.$$

The dynamic of this system can be equivalently transformed into the equations (1) and (2) with the following matrices:

$$A_1 = A_3 = \begin{bmatrix} -0.5 & 0 \\ -0.5 & -5.5 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 1.5 \\ -2 & -1 \end{bmatrix}$$
$$B_1 = B_3 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.5 \\ -2 \end{bmatrix}$$
$$a_1 = -a_3 = \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix},\ a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$C_1 = C_2 = C_3 = [1 \ \ 0]$$
$$D_1 = D_2 = D_3 = 1,\ c_1 = c_2 = c_3 = 0 \ .$$

H^∞ norm from w to z in the linear region in which any element of u is not saturated is around 1.253. Tables I–III show upper bounds of $||\Upsilon||_{L_2}$ and $||\Upsilon||_{L_2,\xi}$ analyzed by Theorems 1 and 2, with no, two and three additional hyperplanes, respectively.

TABLE I

Upper bounds of $||\Upsilon||_{L_2}$ and $||\Upsilon||_{L_2,\xi}$ analyzed without additional hyperplanes.

ξ	**Theorem 1**	**Theorem 2**
∞	1.614	1.614
5	1.614	1.614
0.5	1.614	1.614
0.3	1.564	1.564
0.1	1.330	1.330
0.05	1.299	1.299
0.005	1.278	1.278

TABLE II

Upper bounds of $||\Upsilon||_{L_2}$ and $||\Upsilon||_{L_2,\xi}$ analyzed with two additional hyperplanes.

ξ	**Theorem 1**	**Theorem 2**
∞	1.614	1.613
5	1.613	1.612
0.5	1.532	1.506
0.3	1.402	1.394
0.1	1.291	1.288
0.05	1.281	1.279
0.005	1.278	1.276

In Table I, results by Theorem 2 do not differ from those by Theorem 1. This is because this system involves the problems pointed out in section III-B.3. On the other hand, the results about both global and local L_2-gain in Tables II and III show that Theorem 2 achieves less conservatism than Theorem 1. This means that piecewise quadratic storage functions are indeed effective to reduce conservatism in performance analysis for PWA systems whose switching laws of dynamics depend on both their states and inputs, though addition of hyperplanes might be necessary for achieving the reduction.

TABLE III

UPPER BOUNDS OF $||\Upsilon||_{L_2}$ AND $||\Upsilon||_{L_2,\xi}$ ANALYZED WITH THREE ADDITIONAL HYPERPLANES.

ξ	**Theorem 1**	**Theorem 2**
∞	1.614	1.491
5	1.613	1.489
0.5	1.532	1.423
0.3	1.402	1.323
0.1	1.291	1.272
0.05	1.281	1.268
0.005	1.278	1.267

Additionally, in all of the tables, the analyzed upper bounds fall off as ξ, which is an upper bound of amplitude of input signals, decreases. This means that the local analysis of PWA systems with bounded inputs are significant as well as the global analysis.

Table IV shows the results by Theorems 1 and 2 with other three additional hyperplanes. The results by Theorem 2 of this table differ from those of Table III, and this shows that choice of additional hyperplanes plays an important role in analysis as well as the number of additional hyperplanes.

TABLE IV

UPPER BOUNDS OF $||\Upsilon||_{L_2}$ AND $||\Upsilon||_{L_2,\xi}$ ANALYZED WITH OTHER THREE ADDITIONAL HYPERPLANES.

ξ	**Theorem 1**	**Theorem 2**
∞	1.614	1.507
5	1.613	1.505
0.5	1.532	1.430
0.3	1.402	1.336
0.1	1.291	1.277
0.05	1.281	1.271
0.005	1.278	1.270

V. CONCLUSION

In this paper, we have discussed L_2-gain analysis of PWA systems whose switching laws of dynamics depend on both their states and inputs. Two analysis conditions have been derived: One is based on globally quadratic storage functions, and the other is based on piecewise quadratic storage functions. The latter condition achieves less conservatism than the former because of the use of piecewise quadratic storage functions. Although the advantage of utilizing piecewise quadratic functions has already shown in existing analysis methods, the derived condition is practically more significant, since the class of PWA systems considered in this paper includes much more control systems with input constraints, which is a crucial problem in control of real-world systems, than the class of PWA systems to which the existing methods can be applied.

The derived conditions, based on both global and piecewise quadratic storage functions, enable not only global but also local analysis in which amplitude of inputs is bounded. Since, in control of real-world systems, there are many cases where amplitude of inputs need not be large and its upper bound is known, the local analysis is practically significant as well. Effectiveness of the conditions both in the global and in the local analysis has been proved by a numerical example.

The conditions derived in this paper involve simultaneous LMIs, which can be efficiently solved by numerical computations. They, however, tend to involve intractable computations, as the size of states or inputs increase. The number of additional hyperplanes also increases the intractability. Therefore, this computational problem should be considered in future work. Additionally, the way to make a partition in the state space described in section III-B.1 may not be the best way, and the best way should be pursued. Furthermore, the best way to choose additional hyperplanes should be discussed, since the choice changes results by the presented analysis conditions.

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