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Control of Piece-wise Linear Systems with Piece-wise Linear Controls*

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Abstract - A pole-placement approach is described for piece-wise linear systems. It is shown that if the resulting piecewise linear control achieves local stability, it also achieves global stability. This property holds for piece-wise linear controls whether based on state feedback or on dynamic output feedback. Observer based designs and low-order controllers based on projective controls are discussed

I. INTRODUCTION

Systems exhibiting switching characteristics in the plant, or containing switching characteristics in the controller are of considerable interest. We consider a class of linear systems with switching characteristics and show that local stability properties induce global stability properties. As a consequence, it suffices to concentrate on local stability in designing globally asymptotically stable (GAS) controllers.

The pole-placement approach [1],[2] applied to piece-wise linear systems results in piece-wise linear state feedback control. Similarly, if the pole-placement approach is used to design the observer gain for a piece-wise linear system, the resulting observer will have piece-wise linear characteristics. And finally, the pole-placement design produces, as a by-product, the point-wise (state dependent) eigenvectors, and so allows the application of projective controls approach [12]-[14] to design low order dynamic output feedback controllers for nonlinear systems [3]. By construction such controllers will be piece-wise linear when designed for piece-wise linear systems, and will in all cases produce a piece-wise linear closed-loop system. Guaranteeing local stability then suffices to guarantee global asymptotic stability.

The class of piecewise linear systems considered here is defined in Section 2, and the main result, the global extension of local stability results, is presented in Section 3. The pole-placement design, and its repercussions when applied to piece-wise linear systems is discussed in Section 4, and examples of state feedback design are presented in Section 5. Observer based design is discussed in Section 6. Projective control design of low order controllers [3], is reviewed in Section 7 together with the repercussions of the approach when applied to piece-wise linear systems. An example of the designs of state feedback and output feedback controls is presented in Section 8.

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2. PROBLEM FORMULATION

Piece-wise linear systems may occur because of switching characteristics in the system, or because linear switching controls are used to control linear systems. Some examples of controlled systems that fall within the category considered here are provided below.

Example 1: Piece-wise linear system, such as

$$\begin{aligned}\dot{x}_1 &= x_1 + |2x_1 - 3x_2| + 2x_3\\ \dot{x}_2 &= -x_3 + |x_2 + 3x_3|\\ \dot{x}_3 &= u\end{aligned} \tag{1}$$

Example 2: A system with saturation characteristics, e.g.

$$\begin{aligned}\dot{x}_1 &= x_2 + x_4\\ \dot{x}_2 &= -x_1\varphi(z) + x_3\\ \dot{x}_3 &= x_1 + x_4\\ \dot{x}_4 &= u\\ \varphi(z) &= \varphi(x_1 - x_3)\end{aligned} \tag{2}$$

in which the nonlinear characteristic is a saturation type nonlinearity, such as defined by a linear characteristic with a limit, $\varphi(z) = k\,\mathrm{sat}_\beta(\alpha z)$, or defined by a nonlinear function, such as $\varphi(z) = k\tanh(\alpha z)$.

Example 3: Linear jump systems, e.g.

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12}s_1 & a_{13}\\ a_{21} & a_{220} + a_{221}s_2 & a_{23}\\ 0 & 0 & 0\end{bmatrix}x + \begin{bmatrix}0\\0\\1\end{bmatrix}u \tag{3}$$

$$s_1 = \mathrm{sign}(g_1^T x - \gamma_1),\ \ s_2 = \mathrm{sign}(g_2^T x - \gamma_2)$$

Example 4: Quantized systems of the form

$$\dot{x} = A_\sigma x + B_\sigma u \tag{4}$$

where $\{A_\sigma, B_\sigma\} = \{A_i, B_i\}$ if $a_{i-1} < h \le a_i$, $h = g^T x + \omega$, $i = 1,..,K$.

These are specific examples of systems of the following general form

$$\begin{aligned}&\dot{x} = A(\eta)x + B(\eta)u,\ \ x \in R^n,\ \eta \in R^q,\\ &\eta = [\eta_1\ \ \eta_2\ \dots\ \eta_q],\ \ \ \eta_i = \mathrm{sign}(h_i),\\ &h_i = g_i^T x - \bar{\gamma}_i,\ \ h = (G^T x - \bar{\gamma})\end{aligned} \tag{5}$$

with Example 2 well approximated by (5) during design.

The primary issue is to determine state feedback and output feedback controls to make the system GAS. There are currently no available methodologies to approach this class of control problems, although special cases have been considered, and procedures to treat subclasses of switching, or hybrid systems possessing particular properties have

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been treated (see fore example [4]-[11]). The goal here is to combine global properties of piece-wise linear systems with the properties of the pole-placement approach to produce piece-wise linear controls, which will result in piece-wise linear closed-loop system. Then, the homogeneity property of piece-wise linear systems implies that local stability results in the global stability of the close-loop system. State feedback controllers, full order observer based controllers and low-order controllers based on the Projective controls approach[3],[12]-[14] are discussed.

3. GLOBAL EXTENSIONS OF LOCAL PROPERTIES

It is well known that linear system are characterized by the superposition principle. Consider the linear time-invariant system $\dot{x} = Ax + Bu$,with u = 0. Let $x(t, x_1^0)$ be the response of a linear system with initial condition (IC) x_0^1, and let $x(t, x_2^0)$ be the response with the IC x_0^2. Then

$$x(t, \alpha x_0^1 + \beta x_0^2) = \alpha x(t, x_1^0) + \beta x(t, x_2^0)$$

Piece-wise linear system do not satisfy the superposition principle. However, piece-wise linear systems satisfy the homogeneity part of the superposition principle:

Property. Let the piece-wise linear system (5) with $\bar{\gamma} = 0$ have the response $x(t, x_0)$ for the IC x_0. Then,

$$x(t, \alpha x_0) = \alpha x(t, x_1^0).$$

This property is instrumental in establishing an important property of piece-wise linear systems. Let u = - K(η)x be a well defined piece-wise linear control over R^n, where $\eta \in R^q$ and $\eta_i = \text{sign}(h_i)$, $h_i = g_i^T x$, i =1,..,q, as in (5), with $\bar{\gamma} = 0$. Let the resulting close-loop system be

$$\dot{x} = [A(\eta) - B(\eta)K(\eta)]x = F(\eta)x \qquad (6)$$

Theorem 1. Suppose the control u = - K(η)x , results in a LAS closed-loop system. Then, the system is GAS.

Proof. Let x_0 be in the vicinity of the origin so that the trajectory from this initial condition converges to the origin, i.e. $x(t,x_0) \to 0$. Since the system is LAS the implication is that x_0 can be an arbitrary point on a closed surface S surrounding the origin. Consider now the trajectory from an initial point $\hat{x}_0 = Mx_0, M > 0$. Define the state $z(t) = (1/M)x(t)$. This leads to

$$\frac{1}{M}\dot{x} = [A(\eta) - B(\eta)K(\eta)]\frac{1}{M}x, \quad \frac{1}{M}\hat{x}(0) = \frac{1}{M}Mx_0,$$

and

$$\eta = \text{sign}(M^q \frac{1}{M^q} h) = \text{sign}(M^q \prod_{i=1}^{q} g_i^T \frac{1}{M} x)$$

or

$$\dot{z} = [A(\eta) - B(\eta)K(\eta)]z, \quad \hat{z}_0 = x_0, \quad \eta = \text{sign}(\prod_{i=1}^{q} g_i^T z)$$

Thus, $z(t, \hat{x}_0) = x(t,x_0)$, and so $\hat{x}(t) = x(t, \hat{x}_0) = Mz(t) = Mx(t, x_0)$. Hence, the trajectory converges. Since this is true for all $x_0 \in S$, and all M >1, the system is GAS.

The result can be generalized to $\bar{\gamma} \neq 0$ as in (5). To distinguish the two cases define the switching functions to be $\mu = \text{sign}(\tilde{h})$ with

$$\tilde{h}_i = \prod_{j=1}^{q_i} (g_j^T x - \bar{\gamma}_j), \quad q_i = 1,.., q,$$

Applying u = - K(μ)x results in the closed-loop system $\dot{x} = F(\mu)x$. Let it have a region of attraction *A*, and suppose $x_0 \in A$. There is then a closed surface $S \subseteq A$ in R^n encompassing the origin so that for all $x_0 \in S$ trajectories converge to the origin.

Assumption 1. Let the control $u^\gamma = - K^\gamma(\mu)x$ insure that S is in the region of attraction of the origin for all $0 \le |\gamma_i| \le |\bar{\gamma}_i|$, i = 1,…,q.

Theorem 2. Suppose Assumption 1 holds, Then the closed-loop system is GAS.

Proof: The proof follows that of Theorem 1 and uses the fact that $\frac{1}{M}\bar{\gamma}_j < \bar{\gamma}_j$ for M > 1. Specifically,

$$\tilde{h}(\tilde{\mu}) = \prod_{j=1}^{q_i} (Mg_j^T \frac{1}{M}\tilde{x} - \bar{\gamma}_j) = M^{q_i} \prod_{j=1}^{q_i} (g_j^T z - \frac{1}{M}\bar{\gamma}_j) \quad (7)$$

Since M > 1, then by Assumption 1, this system converges to the origin. Therefore, repeating the steps in the proof of Theorem 1 it again follows that if the system is LAS, it is GAS.

Consider now output feedback control for piece-wise linear systems. Given the system (5) with $\bar{\gamma} = 0$, with an output y = Cx, construct a piece-wise linear observer

$$\begin{aligned} \dot{z} &= A(\hat{\eta})z + B(\hat{\eta})u + H(y - Cz) \\ u &= -K(\hat{\eta})z, \quad \eta = \text{sat}(\hat{h}), \quad \hat{h} = G^T z \end{aligned} \qquad (8)$$

Define the observer error, $e = z - x$. Then

$$\dot{e} = (A(\hat{\eta}) - HC)e + [A(\hat{\eta}) - A(\eta)]x + [B(\hat{\eta}) - B(\eta)]u$$

and when B is constant

$$\dot{e} = (A(\hat{\eta}) - HC)e + [A(\hat{\eta}) - A(\eta)]x$$

If all state components appearing in the definition of η are measured, then $\hat{\eta} = \eta$, and the error dynamics reduces to

$$\dot{e} = (A(\eta) - HC)e \quad (9)$$

In this case choosing $H(\eta)$ to make (9) GAS will achieve convergence of the observer state if the system is point-wise observable (dual to the definition of point-wise controllability [2]). If the observer gain is designed using the pole-placement approach then $H(\eta)$ is piece-wise linear. Combining this with a state feedback control that stabilizes (5) will result in a piece-wise linear controller and a piece-wise linear closed-loop system. And so, with the state error converging, and local stability implying global stability of the closed loop system, the result is a GAS closed-loop system.

Conditions under which an observer can be constructed when not all components of the state that define η are not measured are not known. The following results, however, holds.

Theorem 3. Suppose the closed-loop system

$$\begin{aligned}
&\dot{x} = A(\eta)x + B(\eta)u, \quad x \in R^n, \ \eta \in R^q, \ 1 \le q \le n \\
&\eta = [\eta_1 \ \ \eta_2 \ \dots \ \eta_q], \quad \eta_i = \text{sign}(h_i), \ h_i = g_i^T x, \ \ h = G^T x \\
&\dot{z} = A(\hat{\eta})z + B(\hat{\eta})u + H(y - Cz) \\
&u = -K(\hat{\eta})z, \quad \hat{\eta} = \text{sat}(\hat{h}), \quad \hat{h} = G^T z
\end{aligned} \quad (10)$$

is LAS. Then the system is also GAS.

Proof: This follows directly from the fact that the closed-loop system has the form $\dot{\tilde{x}} = \tilde{F}(\tilde{\eta})\tilde{x}, \ \ \tilde{x}^T = [x^T \ z^T]$, with $\tilde{\eta} = [\eta^T \ \hat{\eta}^T]$, and is piece-wise linear. Theorem 1 now applies to prove the result.

Theorem 4. Suppose the closed-loop system

$$\begin{aligned}
&\dot{x} = A(\eta)x + B(\eta)u, \quad x \in R^n, \ \eta \in R^q, \ 1 \le q \le n \\
&\eta = [\eta_1 \ \eta_2 \dots \eta_q], \eta_i = \text{sign}(\tilde{h}_i), \\
&\tilde{h}_i = g_i^T x - \bar{\gamma}_i, \ h = G^T x - \bar{\gamma} \\
&\dot{z} = A(\hat{\eta})z + B(\hat{\eta})u + H(y - Cz) \\
&u = -K(\hat{\eta})z, \quad \eta = \text{sat}(\hat{h}), \quad \hat{h} = G^T z - \bar{\gamma}
\end{aligned}$$

is LAS for all $0 \le |\gamma_i| \le |\bar{\gamma}_i|$. Then it is GAS.

Proof: Follows in complete analogy from Assumption 1 and Theorem 2.

These results carry over to low order controllers, but are stated separately because a different procedure is used to develop the low order dynamic controller with required properties. Again, assume first that $\bar{\gamma} = 0$.

Assumption 2. The signals $h = G^T x$ are in the set of measured outputs of the system.

Theorem 3'. Suppose there exists a low order controller of the form

$$\begin{aligned}
&\dot{z} = H(\eta)z + D(\eta)y, y \in R^r \\
&u = -N_d(\eta)z - K_d(\eta)y, \ \eta = \text{sat}(h), \ \ h = G^T x
\end{aligned} \quad (11)$$

with $z \in R^p$, $p < n-r$, which makes the closed-loop system (1) and (11) LAS. Then the system is also GAS.

Assumption 1'. Let the control (11) with $h = G^T x - \gamma$ insure there is a closed region S containing the origin is in the region for all $0 \le |\gamma_i| \le \bar{\gamma}_i$, $i = 1,\dots,q$.

Theorem 4'. Suppose Assumptions 1' and 2 hold, and there exists a controller

$$\begin{aligned}
&\dot{z} = H(\eta)z + D(\eta)y, \ \ y \in R^r, \ \ z \in R^p, \ \ p < n - r \\
&u = -N_d(\eta)z - K_d(\eta)y, \eta = \text{sat}(\tilde{h}), \ \tilde{h} = G^T z - \bar{\gamma}
\end{aligned} \quad (12)$$

which makes the closed-loop system (6) with controller (12) LAS for all $0 \le |\gamma_i| \le |\bar{\gamma}_i|$. Then the system is GAS.

4. POLE-PLACEMENT DESIGN OF LOCALLY STABILIZING CONTROLS

Properties of the resulting closed-loop system will hold if one can design candidate controls that will maintain the piece-wise linear property of the closed-loop system. For state feedback controls the following approach supplies a constructive procedure.

Theorem 7. Suppose $u = -K(x)x$ is a pole-placement control with arbitrary desired eigenvalues that are piece-wise constant (e.g. function of only η) in each region of the partition induced by the switching manifolds. Then the resulting system is GAS if it is LAS.

Proof: Let $\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\}$ be the piece-wise constant desired eigenvalues in region π^k of the partition induced by the switching manifolds. Let a^k be the vector of coefficients of the corresponding characteristic polynomial in region k, let $\alpha(\eta) = [\alpha_1(\eta) \ \dots . \alpha_{n-1}(\eta)]$ be the coefficients of the open-loop characteristic polynomial of $A(\eta)$, and let

$$M(\eta) = \left[B(\eta) \ \ A(\eta)B(\eta) \ \dots . \ A(\eta)^{n-1}B(\eta)\right],$$

$$S(\eta) = \begin{bmatrix} \alpha_2(\eta) & \alpha_2(\eta) & . & \alpha_{n-1}(\eta) & 1 \\ \alpha_2(\eta) & . & . & 1 & 0 \\ . & \alpha_{n-1}(\eta) & . & . & . \\ \alpha_{n-1}(\eta) & 1 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \end{bmatrix} \quad (13)$$

The pole-placement control is then defined by

$$u = -K(x)x = -[a(\eta) - \alpha(\eta)]S(\eta)^{-1} M(\eta)^{-1} x \, . \quad (14)$$

The result is now a direct consequence of Theorem 1 and the fact that by construction the control is piece-wise linear and results in a piece-wise linear closed-loop system which has the properties required in Theorem 1.

It is observed that the partition introduced for the piecewise constant desired eigenvalues need not coincide with the partition induced by the piecewise linear system.

No assumption is made on the nature of the control for a given initial condition. The stabilizing control itself may remain a switching control, or there may be periods of time when it enters a sliding regime. On the other hand if a given control does not stabilize the system divergence could occur with a switching regime accompanied by an infinite, but denumerable, number of switches, or in the sliding regime, when trajectories can diverge along the sliding manifold. The payoff is that if a modified control is sought to insure stability it is sufficient to show local stability to deduce global stability of the system. In practice, it is quite simple to detect local stability, or instability via simulations. We have found it easy to construct state feedback controls that locally stabilize piece-wise linear systems in the many examples that have been considered when the piece-wise linear system was controllable [2]. However, the question of the necessary and/or sufficient conditions under which this can be achieved when point-wise stabilizability [2] is lost along the sliding manifolds is open, and needs further study.

5. EXAMPLES OF STATE FEEDBACK DESIGN

Implications of these results will be illustrated on the 3rd order system of the type in Example 3, Section 2, with parameters $a_{11}=-1, a_{12}=1, a_{13}=-2, a_{21}=1, a_{220}=-1,$ $a_{221}=1,\ \ a_{23}=1\ \ (a_{22}(s_2)=a_{220}+a_{221}s_2)$ and with switching functions s_1=sign(h_1), s2 = sign(h_2) where

$$h_1=x_2+x_3,\ \ h_2=x_1+x_3 \qquad (15)$$

The SR of this system is characterized by the open-loop characteristic polynomial

$$\phi(s)=s^3-(a_{11}+a_{22}(s_2))s^2+[a_{11}a_{22}(s_2)-a_{21}a_{12}s_1]s\,, \quad (16)$$

leading to $\alpha_0=0,\ \alpha_1=a_{11}[a_{220}+a_{221}s_2]-a_{21}a_{12}s_1$, and $\alpha_2=-a_{11}-a_{220}-a_{221}s_2]$. Thus,

$$S=\begin{bmatrix}\alpha_1 & \alpha_2 & 1\\ \alpha_2 & 1 & 0\\ 1 & 0 & 0\end{bmatrix},\ S^{-1}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -\alpha_2\\ 1 & -\alpha_2 & \alpha_2^2-\alpha_1\end{bmatrix}$$

$$M=\begin{bmatrix}0 & a_{13} & a_{11}a_{13}+a_{12}a_{23}s_1\\ 0 & a_{23} & a_{21}a_{13}+a_{22}(s_2)a_{23}\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & m_{12} & m_{13}\\ 0 & m_{22} & m_{23}\\ 1 & 0 & 0\end{bmatrix},$$

$$\det M=a_{21}a_{13}^2+a_{22}(s_2)a_{23}a_{13}-a_{11}a_{13}a_{23}-a_{12}a_{23}^2s_1$$

Then, from (4), $K(\eta)=[k_1\ \ k_2\ \ k_3]$ where $k_3=a_2-\alpha_2$ and

$$k_1=\frac{1}{\det M}[-(a_0-\alpha_0)m_{22}+(a_1-\alpha_1)(m_{23}+\alpha_2 m_{22})$$
$$-(a_2-\alpha_2)(\alpha_2 m_{23}+(\alpha_2^2-\alpha_1)m_{22})])]$$

$$k_2=\frac{1}{\det M}[(a_0-\alpha_0)m_{12}-(a_1-\alpha_1)(m_{13}+\alpha_2 m_{12})$$
$$+(a_2-\alpha_2)(\alpha_2 m_{13}+(\alpha_2^2-\alpha_1)m_{12})]$$

Choosing, for example, the constant eigenvalues {-3,-4,-5}results in LAS, and thus GAS, behavior. A typical response is shown in Figure 1, with part of the response, after the large transient, blown-up in Figure 2. From the time histories of s_1, s_2, shown in Figure 2, high frequency switching occurs around $h_2=0$ for $t\in[1.75,2.15]$ sec, and around $h_1=0$ for $t\in[3.4,5]$ sec. Note that $\det M=4-2s_2-s_1$, implying $\det M\neq 0$, and so the system is controllable [2].

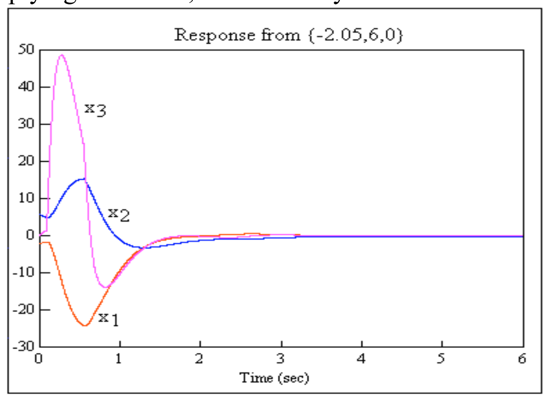

Fig. 1. Response with state feedback

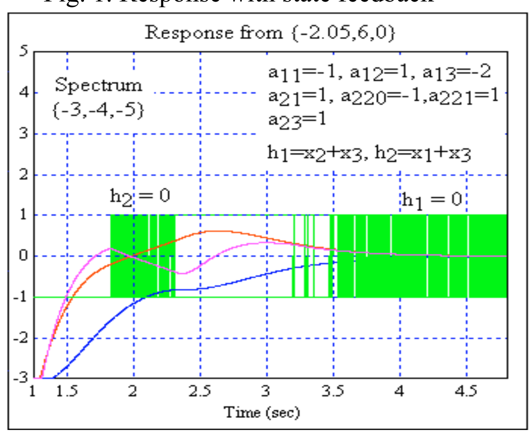

Fig.2. Response contains both switching and sliding modes

It is noted that a modified system characterized by the data $a_{11}=1, a_{12}=1, a_{13}=-2, a_{21}=1,\ \ a_{220}=1, a_{221}=1,\ \ a_{220}=1,$ $a_{221}=1,\ \ a_{23}=1,$ with $h_1=x_2-x_3, h_2=x_2+x_3$, is not open-loop stable for any of the possible instance of $A(s_1,s_2)$. However, $\det M=4-2s_2-s_1$ still holds, and the system is controllable. The sufficient condition for controllability for the general case of Example 3 is

$$|a_{21}a_{13}^2+(a_{220}-a_{11})a_{13}a_{23}|>|a_{221}a_{23}a_{13}|+|a_{12}a_{23}^2|,$$

and it was simple to find stabilizing control laws whenever this condition was satisfied.

6. OBSERVER BASED OUTPUT FEEDBACK DESIGN

The system described by

$$a_{11}=1,\ a_{12}=1, a_{13}=-2, a_{21}=1,\ a_{220}=1,\ a_{221}=1,\ a_{23}=1 \quad (17)$$

with

$$h_1 = x_2 - x_3,\ \ h_2 = x_2 + x_3 \quad (18)$$

will be used here to illustrate the design of observer based controls for piece-wise linear systems, assuming the second state is measured, $y = x_2$, (thus, not all the states appearing in the definition of the switching characteristics are measured). This implies that $\hat{\eta} \neq \eta$ because the estimate of the switching nonlinearities $\{\hat{s}_1, \hat{s}_2\}$ are different from the actual switching nonlinearities $\{s_1, s_2\}$. The observer based closed-loop system is, thus, described by (8) , and was implemented by performing a pole-placement design on the pair $\{A^T, C^T\}$ while choosing an observer spectrum {-3,-5,-8}. Combining the observer design with the state feedback pole-placement design, with desired spectrum {1, -2,-3}, resulted in a GAS piece-wise linear system.

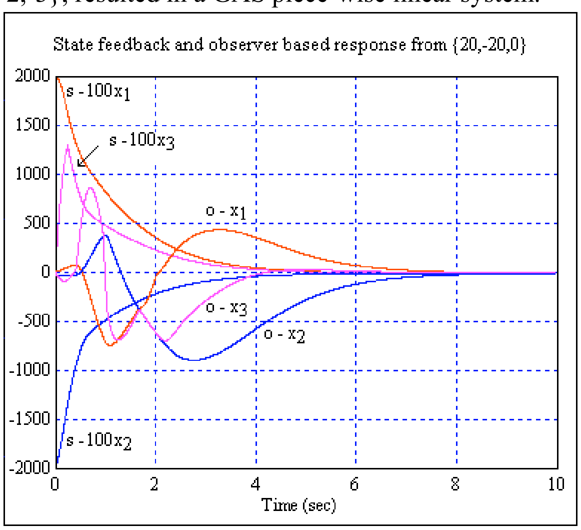

Figure 3. Observer based output feedback Response

The response from the initial condition x_0 = {20,-20,0} is shown in Figure 3. The significantly larger transient than with state feedback control for this initial condition is a combined effect of the observer initial conditions being set to zero and the effect of unsynchronized switching in the system and the observer.

7. LOW ORDER CONTROLLER DESIGN BY PROJECTIVE CONTROLS

The Projective control approach can be used to design an output feedback controller of low order. Consider (5), with $\overline{\gamma} = 0$, and suppose there is a controller of the form

$$\begin{aligned} &\dot{z} = H(\hat{\eta})z + D(\hat{\eta})y \\ &u = -N_d(\hat{\eta})z - K_d(\hat{\eta}),\ \ \hat{\eta} = \mathrm{sat}(\hat{h}),\ \ \hat{h} = G^T z, \end{aligned} \quad (19)$$

with $\hat{h} \in R^q$, which makes the closed-loop system LAS. Then, it is clear from above constructions that the system will also be GAS. The question is how to identify controllers which will make the system LAS. In the case of observer based output feedback control one can assume z to be an estimate of the state, while in the case of low order controller there is no such natural relation with the state. When Assumption 2 is made, z = x, and the projective control idea [12]-[14] can be used to develop candidate controllers.

Possibility to extend the Projective controls approach to nonlinear systems based on the pole-placement approach is described in [3]. To develop a controller using the Projective controls approach, assume a state feedback control given by (14) has been determined. The projective controller can then be constructed as follows. Let $y \in R^r$, $r \geq q$, and let y contain all components of $h = G^T x - \gamma$ (and so $\hat{\eta} = \eta$). Let the resulting closed-loop system given by (6) represent the reference dynamics, let $X_r(\eta)$ be r real eigenvectors of the reference dynamics associated with r eigenvalues collected into Λ_r, and let $X_p(\eta)$ be additional p eigenvectors of the reference dynamics, associated with Λ_p, where p is the dimension of the low order controller in (19), and $r+p \leq n$. Define

$$\begin{aligned} &N_0(\eta) = X_r(\eta)(CX_r(\eta))^{-1} \\ &B_0(\eta) = (I_n - N_0(\eta)C)X_p(\eta) \\ &V(\eta) = Y^{\#} - Y^{\#}N_0(\eta)C,\ \ Y^{\#} = (Y^TY)^{-1}Y^T \end{aligned} \quad (20)$$

The set of p-dimensional controllers which retain $\{\Lambda_r, X_r(\eta)\}$ and $\{\Lambda_p, X_p(\eta)\}$ in the point-wise spectrum of the resulting closed-loop system is then given by (19) with the matrices $\{H(\eta), D(\eta), N_d(\eta), K_d(\eta)\}$ parameterized by the free parameter matrix $P \in R^{pxr}$ as

$$\begin{aligned} &H(\eta) = \Lambda_p(\eta) + PCF(\eta)B_0(\eta) \\ &D(\eta) = PCF(\eta)(N_0(\eta) - B_0(\eta)P) - \Lambda_p P \\ &N_d(\eta) = K(\eta)B_0(\eta) \\ &K_d(\eta) = K(\eta)(N_0(\eta) - B_0(\eta)P) \end{aligned} \quad (21)$$

The residual dynamics characterized by point-wise eigenvalues of the matrix

$$A_{re} = VAY + Y^T B_0 PCAY = A_r + \widetilde{B}_r P \widetilde{C}_r \quad (22)$$

implying that the residual eigenvalues are functions of η. Here $Y \in {}^{R(n-r)xn}$ satisfies CY = 0. The residual dynamics supplies the remaining point-wise eigenvalues of the closed-loop system, and so the freedom in P is initially used to achieve a point-wise closed-loop spectrum with all eigenvalues having negative real parts for all {x,z}.

For the considered third ordered system, if p = 1 the residual dynamics is of first order , $\widetilde{B}_r$ is a scalar variable,

$P = [p_1\ p_2]$, and $\widetilde{C}_r = CAY = [a_{21}\quad 0]^T$, implying a single residual eigenvalue, $\lambda_3 = \lambda_3(\eta)$. The approach was applied to the system defined by (3), (17), (18) and a reference dynamics associated with the desired spectrum {-1,-2,-3}. In accordance with Assumption 2 it is assumed that the states x_2 and x_3 are measured, and so $y_1 = x_2$, $y_2 = x_3$. For this system a first order dynamic controller was used (static controllers can not stabilize the system). The details of the design (the analytical expressions for the controller matrices, and for the residual dynamics A_{re}) can be found in [15]. Choosing

$$\Lambda_r = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \Lambda_p = -3, \quad P = \left[-50\operatorname{sign}(\widetilde{B}_r) \quad 1\right]$$

a locally stable closed-loop system was obtained, which is then by Theorem 5 also GAS. The response from x_0 = {20, -20, 0} is shown in Figure 4. Also shown are the responses when the state feedback control (s) is used, and when the observer based control (o) is used. (Zero initial condition were used in the observer and projective controller.)

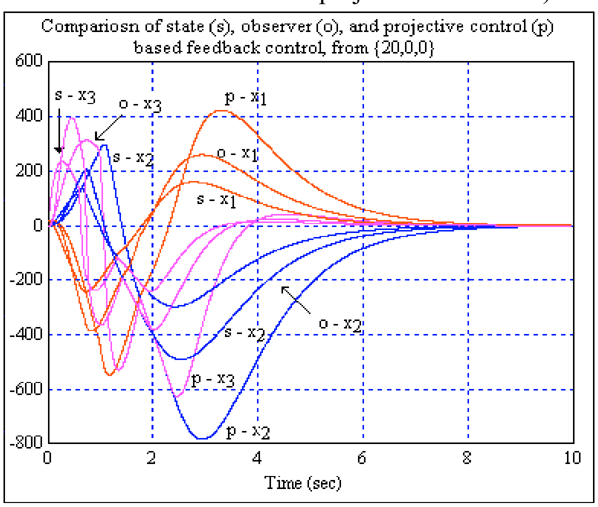

Figure 4. Response with the projective control, State, and observer based feedback

8. CONCLUSION

While the described approach per se does not guarantee that a design characterized by some desired closed-loop spectrum, and observer spectrum, or with a particular choice of the free parameters in the Projective controller will guarantee GAS behavior, if one finds a controller that guarantees the system is LAS it will then also be GAS. The pole-placement based approach, thus, provides a promising avenue for the design of controls for piece-wise linear systems using state feedback or output based feedback. Experience has shown that the approach is also applicable to systems in which there exist saturation characteristics, which are approximated by switching characteristics. This, and other extensions , are being pursued.

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