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Robust multiple-fault detection and isolation: A gradient flow approach

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Abstract— **In this paper a novel solution to the fault detection and isolation observer design problem for LTI systems is presented. A gradient flow approach is proposed for synthesizing an optimal observer under eigenstructure assignment which is robust against unstructured uncertainties. This is achieved by minimizing the spectral condition number of the observer eigenvector matrix. Convergence properties of the gradient flow solution are proved and its efficiency demonstrated via a numerical example.**

I. Introduction

Fault detection and isolation (FDI) techniques via analytical redundancy involve the generation and the evaluation of fault signals on the basis of available measurements and of a mathematical model of the plant. A popular approach to analytical redundancy is the detection filter which was first introduced by Beard [1] and refined by Jones [2]. In the last two decades many other contributions involving similar ideas were proposed in the literature [3]-[8]. Massoumnia [3] presented a geometric approach for fault detection filters by assuming that only one fault at a time may occur, while in [4], limitedly to disturbance-free systems, a rigorous framework for FDI problems was developed by imposing that the signature images of the faults to be isolated lie in the unobservable subspace of the residuals and by checking if the faults to be detected are input-observable. In deterministic robust diagnostic observer design problems, eigenstructure assignment is used to decouple the unknown inputs from the residuals (see [5],[6]) whereas the unknown input observer approach decouples the unknown inputs from the observer states used for the residual generation [7]. Recently, in [8] a robust multiple-fault detection and identification algorithm was formulated as an optimization problem. The approach is based on the projection theory [9]. Specifically, the output error is projected onto several subspaces each one corresponding to one specific fault. In each subspace the corresponding output error is maximized whereas the error component due to all other faults, noises and initial conditional error, is minimized.

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This paper concerns with the design of robust FDI filters which achieve multiple-fault isolation under eigenstructure assignment in the presence of unstructured uncertainties. The solution is based on the theory of gradient flow models that have been proposed recently for the resolution of a wide variety of real-time optimization problems (see [10] and [11] and references therein). In control theory, gradient flow models have been proposed in [10]-[13] for robust pole assignment in state and output control feedback design problems. Specifically, the feedback matrix gain that solves the given pole assignment problem is obtained by minimizing the spectral condition number of the closed-loop eigenvector matrix. The relevance of the spectral condition number of a matrix M, denoted as $\kappa_2(M)$, in robust state-feedback synthesis problems arises from its connection to the sensitivity of the eigenvalues of M to perturbations (see [14]). Let Δ be a matrix and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of M. Then the perturbed closed-loop matrix $M + \Delta$ remains stable for all matrices Δ such that $\|\Delta\|_2 \leq \min_j Re(-\lambda_j)/\kappa_2(M)$ (continuous-time case) or $\|\Delta\|_2 \leq \min_j (1 - |\lambda_j|)/\kappa_2(M)$ (discrete-time case).

Then, robustness is achieved here by minimizing the spectral condition number of the observer eigenvector matrix and and by imposing to the filter the corresponding optimal eigenstructure. Moreover, disturbance decoupling and isolation conditions, expressed in terms of eigenvector conditions (disturbance decoupling) and Markov parameters equalities (isolation process), have also been considered as additional constraints. The main feature of the approach relies on its capability to converge to a constant observer gain which achieves, if possible, robust multiple-fault detection and isolation against unstructured uncertainty.

The paper is organized as follows: in Section II the classical FDI problem is outlined. Then, a formulation in terms of a constrained optimization problem under eigenstructure assignment is proposed, where the objective function is the spectral condition number. Section III describes the approach and proves its convergence properties. In Section IV a numerical example is used to show the effectiveness of the proposed method. Finally, some conclusive remarks end the paper.

II. Problem Formulation

Consider the following state space description

$$\left\{\begin{array}{lcl}\dot{x}(t) & = & Ax(t)+Bu(t)+Ed(t)+Rf(t)\\ y(t) & = & Cx(t)\end{array}\right. \quad (1)$$

where $x(t)\in\mathcal{R}^n$ is the state vector, $u(t)\in\mathcal{R}^m$ the input vector, $y(t)\in\mathcal{R}^p$ the measurement output vector, $d(t)\in\mathcal{R}^d$ the unknown disturbance vector, $f(t)\in\mathcal{R}^q$ the fault to be detected and isolated. A, B, C, E, R are known real matrices of appropriate dimensions. In particular, R denotes the fault distribution matrix. Without loss of generality, we shall assume that C is full row rank and the couple (A,C) is observable. Notice that system formulation (1) is not restrictive: sensor faults can also be represented by actuator faults as indicated in [15].

We consider a full-order state-observer defined as follows

$$\left\{\begin{array}{lcl}\dot{\hat{x}}(t) & = & (A+KC)\hat{x}(t)+Bu(t)+Ky(t)\\ \hat{y}(t) & = & C\hat{x}(t)\\ r(t) & = & Q(y(t)-\hat{y}(t))\end{array}\right. \quad (2)$$

where $r(t)$ is the residual vector, $\hat{x}(t)$ and $\hat{y}(t)$ the state and, respectively, the output estimations. The matrix $Q\in\mathcal{R}^{q\times p}$ is the residual weighting factor. When the residual generator represented by (2) is applied to the system described by (1), the state estimation error $e(t):=x(t)-\hat{x}(t)$ and the residual $r(t)$ are governed by the following equations:

$$\left\{\begin{array}{lcl}\dot{e}(t) & = & (A+KC)e(t)+Ed(t)+Rf(t)\\ r(t) & = & He(t)\end{array}\right. \quad (3)$$

where $H=QC$. Consider the following problem:

P1 - Exact Fault Detection- *Determine, if there exist, matrices K and Q such that (2) is asymptotically stable and the residual is insensitive to disturbances.* □

Essentially, we look for matrices K and Q such that:

- $$A+KC \text{ is asympotically stable}$$
- $$G_{rd}(s)=H\left(sI-A-KC\right)^{-1}E=0 \quad (4)$$

Condition (4) ensures that the diagnostic observer is unaffected by unknown inputs. Necessary and sufficient conditions for disturbance decoupling can be found in [16]. Here, for design purposes, we shall use the following sufficient conditions [17].

Lemma 1 *A sufficient condition for satisfying the disturbance decoupling requirement (4) is that both*

(1) $QCE=0$;
(2) each column of the matrix E is a right eigenvector of $A+KC$, corresponding to a specific eigenvalue;

are satisfied. □

Problem **P1** consists of an eigenstructure assignment problem subject to algebraic constraints (4). A numerical procedure for its solution can be derived by considering

P1′- Exact Fault Detection- *Let $\Lambda=\{\lambda_1,\ldots,\lambda_n\}$ be a symmetric set of n complex numbers with $Re(\lambda_i)<0, i=1,\ldots,n$. Find, if there exist, matrices K and Q such that:*

$$AZ-Z\Lambda+KCZ=0 \quad (5)$$

$$QCE=0 \quad (6)$$

$$E_i\in\{Z_1,Z_2,\ldots,Z_n\} \quad (7)$$

where $Z=[Z_1,\,Z_2,\,\ldots,Z_n]$ is the right eigenvector matrix of $(A+KC)$ and E_i is the i-th column of the matrix E. □

When the more difficult problem of exact fault detection and isolation is of interest, **P1** has to be modified in:

P2 - Exact Fault Detection and Isolation- *Determine, if there exist, matrices K and Q such that (2) is asymptotically stable, the residual is insensitive to disturbances and the transfer matrix from faults to residuals is diagonal.* □

Essentially, we look for matrices K and Q such that:

- $$A+KC \text{ is asymptotically stable}$$
- $$\begin{array}{rcl}G_{rd}(s) & = & H\left(sI-A-KC\right)^{-1}E=0 \quad (8)\\ G_{rf}(s) & = & H\left(sI-A-KC\right)^{-1}R\\ & = & diag(g_{r_1,f_1},\ldots,g_{r_q,f_q}) \quad (9)\end{array}$$

This problem has been well studied in the past and the main results related to its solvability are described in [16], [18]. It is straightforward to observe that a solution can be obtained only if the number of measurements is larger than or equal to the number of faults plus the number of disturbances

$$p\geq q+d \quad (10)$$

A further obvious necessary condition for $G_{rf}(s)$ being made diagonal, along with $G_{rd}(s)=0$, is that

$$[E\ R] \text{ is full column rank} \quad (11)$$

Finally, by taking into account Lemma 1, the following sufficient conditions for **P2** arise.

Lemma 2 *A sufficient condition for solving the exact fault detection and isolation problem* **P2** *is that all*

(1) $QCE=0$;
(2) each column of the matrix E is a right eigenvector of $A+KC$, corresponding to a specific eigenvalue;
(3) the first n Markov parameters $(HZYR,\,HZ\Lambda YR,\ldots,HZ\Lambda^{n-1}YR)$ of the transfer matrix $G_{rf}(s)$ are diagonal matrices, viz.

$$\begin{array}{l}e_i^T HZYRe_j=0,\; e_i^T HZ\Lambda YRe_j=0,\;\ldots,\\ \ldots,e_i^T HZ\Lambda^{n-1}YRe_j=0,\; i,j=1,\ldots,q,\;\; i\neq j\end{array} \quad (12)$$

where e_i, $i=1,\ldots,q$ represent the column vectors of the canonical basis of $\mathcal{R}^q$.

are satisfied.

Proof. Points (1)-(2) are sufficient conditions for disturbance decoupling (see Lemma 1). Point (3) ensures diagonalization of $G_{rf}(s)$ by standard Markov parameters results [19]. □

Then based on Lemma 2, a solution of **P2** can be obtained by considering

P2$'$- Exact Detection and Fault Isolation- *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *be a symmetric set of* n *complex numbers with* $Re(\lambda_i) < 0, i = 1, \ldots, n$*, find, if there exist, matrices* K *and* Q *such that:*

$$AZ - Z\Lambda + KCZ = 0 \tag{13}$$

$$QCE = 0 \tag{14}$$

$$E_i \in \{Z_1, Z_2, \ldots, Z_n\} \tag{15}$$

$$e_i^T QCZ\Lambda^k YRe_j = 0 \tag{16}$$

$$k = 0, \ldots, n-1;\ i, j = 1, \ldots, q,\ \ i \neq j$$

□

Problems **P1$'$** and **P2$'$** are formulated as eigenstructure assignment problems with equality constraints. Finding a numerical solution of such problems is more difficult w.r.t. the basic assignment problem due to the fact that the free parameter space (redundancy) is largely reduced. As a consequence, the algorithms based on the eigenstructure assignment involve huge and laborious algebraic computation (see for instance [6], [20]).

A. Robustness against unstructured uncertainties

An important issue concerning a reliable FDI scheme consists of improving the diagnostic robustness against both structured and unstructured uncertainties. The assignment problem (13) has a nonunique solution for a multivariable system and the minimization of an objective function can improve the performance of the observer. In this paper, the model uncertainty is expressed via the maximum unstructured time-invariant uncertainty Δ affecting A such that the stability of the matrix $A + \Delta$ is retained. In the case of state feedback, Kausky et al. [14] showed that if the matrix A is subject to a perturbation Δ, the closed-loop matrix $A + \Delta + BK$ remains stable if

$$\|\Delta\|_2 < \min_j Re(-\lambda_j)/\kappa_2(Z) \tag{17}$$

where $\kappa_2(Z) = \|Z\|_2 \|Z^{-1}\|_2$ is the spectral condition number of the assigned eigensystem. As a consequence, the minimization of the spectral condition number can be considered a key figure for robust design. Moreover, as in [23] and [12], the Frobenius condition number $\kappa_F(Z) = \|Z\|_F \|Z^{-1}\|_F$, $\kappa_2(Z) \leq \kappa_F(Z)$, is preferred to $\kappa_2(Z)$ because $\kappa_2(Z)$ is not differentiable with respect to Z and it has been proved in [12] that any Z minimizing $\|Z\|_F^2 + \|Z^{-1}\|_F^2$ also minimizes $\kappa_F(Z)$.

Hereafter for brevity, we consider Problem **P2$'$** only. For the latter, the design of a robust residual generator can be rewritten as into the following optimization problem:

P3- Robust Fault Detection and Isolation-

$$\min_{Z,Y,K,Q} \ \|Z\|_F^2 + \|Y\|_F^2 \tag{18}$$

subject to

$$AZ - Z\Lambda + KCZ = 0 \tag{19}$$

$$ZY = I \tag{20}$$

$$Ze_i - E_i = 0,\ i = 1, \ldots, d \tag{21}$$

$$QCE = 0 \tag{22}$$

$$e_i^T QCZ\Lambda^k YRe_j = 0 \tag{23}$$

$$k = 0, \ldots, n-1;\ i, j = 1, \ldots, q,\ \ i \neq j$$

where the new variable Y has been introduced for imposing the nonsingularity of the eigenvector matrix Z via the condition (20).

The solution of **P3** allows one to reduce as much as possible the conditioning of the observer eigenvector matrix. Therefore, a well-conditioned eigenstructure assignment is obtained and the resulting diagnostic observer is intrisically robust against all unstructured uncertainties satisfying (17).

III. FDI DESIGN VIA GRADIENT FLOW

The application of the gradient flow approach of [11] to problem **P3** leads to consider the following energy function $(V[t] \triangleq V[Z(t), Y(t), K(t), Q(t)])$:

$$\begin{aligned} V[t] \triangleq \ & T(t)\left(\|Z(t)\|_F^2 + \|Y(t)\|_F^2\right) \\ & + \|AZ(t) - Z(t)\Lambda + K(t)CZ(t)\|_F^2 \\ & + \|Z(t)Y(t) - I\|_F^2 \\ & + \sum_{i=1}^{d} \|Z(t)e_i - E_i\|_F^2 + \|Q(t)CE\|_F^2 \\ & + \sum_{i \neq j} \sum_{k=0}^{n-1} \|e_i^T Q(t)CZ(t)\Lambda^k Y(t)Re_j\|_F^2 \end{aligned} \tag{24}$$

where $Z(t) \in \mathcal{R}^{n \times n}, Y(t) \in \mathcal{R}^{n \times n}, K(t) \in \mathcal{R}^{n \times p}$ and $Q(t) \in \mathcal{R}^{q \times p}$ are activation matrices corresponding to the unknowns Z, Y, K and Q respectively. $T(t)$ in (24) is a scalar temperature parameter which satisfies the following conditions: $T(t) \geq 0,\ \forall t$ and $\lim_{t \to \infty} T(t) = 0$ if and only if (Z, Y, K, Q) represents a feasible solution to the optimization problem (18)-(23). By considering T as a parameter and letting the time derivative of the state variables be directly proportional to the gradient of the energy function with respect to the variables Z, Y, K and Q, one gets the following dynamic equations:

$$\begin{aligned} \frac{dZ}{dt} \ & = -\mu\big(TZ + A^T(AZ - Z\Lambda + KCZ) \\ & \quad -(AZ - Z\Lambda + KCZ)\Lambda^T \\ & \quad +C^T K^T(AZ - Z\Lambda + KCZ) + (ZY - I)Y^T \\ & \quad + \sum_{i=1}^{d}(Ze_i - E_i)e_i^T \\ & \quad + \sum_{i \neq j} \sum_{k=0}^{n-1} (e_i^T QC)^T (e_i^T QCZ\Lambda^k YRe_j)(\Lambda^k YRe_j)^T\Big) \\ & \triangleq -\mu(TZ + H_1(Z, Y, K, Q),\ \ Z(0) = Z_0, \end{aligned} \tag{25}$$

$$\begin{aligned}\frac{dY}{dt} &= -\mu\Big(TY + Z^T(ZY-I) \\ &+ \sum_{i\neq j}\sum_{k=0}^{n-1}(e_i^T QCZ\Lambda^k)^T(e_i^T QCZ\Lambda^k YRe_j)(Re_j)^T\Big) \\ &\triangleq -\mu(TY + H_2(Z,Y,Q)),\;\; Y(0)=Y_0,\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{dK}{dt} &= -\mu\,(AZ - Z\Lambda + KCZ)\,Z^T C^T \\ &\triangleq -\mu H_3(Z,K),\;\; K(0)=K_0,\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{dQ}{dt} &= -\mu\Big(QCE(CE)^T + \\ &\sum_{i\neq j}\sum_{k=0}^{n-1} e_i(e_i^T QCZ\Lambda^k YRe_j)(CZ\Lambda^k Re_j)^T\Big) \\ &\triangleq -\mu H_4(Z,Y,Q),\;\; Q(0)=Q_0,\end{aligned} \tag{28}$$

where μ is a positive scaling parameter. Finally, the "temperature" parameter $T(t)$ is defined as

$$\begin{aligned}T(t) &= \eta(\|AZ(t)-Z(t)\Lambda+K(t)CZ(t)\|_F^2 \\ &+\|Z(t)Y(t)-I\|_F^2 + \sum_{i=1}^{d}\|Z(t)e_i - E_i\|_F^2 + \\ &\|Q(t)CE\|_F^2 + \sum_{i\neq j}\sum_{k=0}^{n-1}\|e_i^T QCZ\Lambda^k YRe_j\|_F^2) \\ &\triangleq \eta\hat{T}(t),\;\; T(0)=T_0\end{aligned} \tag{29}$$

where $\eta > 0$ is a parameter design, which affects the rate of convergence of T.

Remark 1- The main features of the proposed gradient flow approach can be summarized as follows:

1) detection and isolation of multiple actuator and sensor faults can be achieved
2) the robustness of the filter against unstructured perturbations is taken into account by the minimization of the spectral condition number. □

Convergence properties of the proposed gradient flow for the robust residual generator design are reported in the next Theorem 1:

Theorem 1 *Given the initial state $(Z_0, Y_0, K_0, Q_0) \notin \Omega$, where Ω is the set of equilibrium points of the gradient flow, viz. each state (Z, Y, K, Q) satisfying the conditions (19)-(23), it is an equilibrium point of Eqs. (25)-(28). Then*

(a) *The state $(Z(t), Y(t), K(t), Q(t))$ of the gradient flow (25)-(28) and (29) converges to an equilibrium point $(\bar{Z}, \bar{Y}, \bar{K}, \bar{Q}) \in \Omega$.*
(b) *If $(\bar{Z}, \bar{Y}, \bar{K}, \bar{Q}) \in \Omega^*$, where $\Omega^* := \{(Z,Y,K,Q)\,|\,(Z,Y,K,Q)\,\text{satisfying}\,(19)-(23)\} \subseteq \Omega$, then the convergence is exponential.*

Proof. See [21].

A. A "singular" case

When the number of independent disturbances and fault signals exceeds the number of measurements, the necessary condition (10) is not satisfied and the fault detection and isolation problem cannot be jointly solved, because the transfer matrix $G_{rf}(s)$ cannot be made diagonal. In such a case, it is possible to relax the isolation requirements by satisfying them only approximately. Here, the gradient flow developed in the previous section is straightforward generalized in order to include such a case. The idea can be briefly described as follows: let $G_{rf}(s)$ be a strictly proper transfer matrix, expressed in terms of its Markov parameters:

$$G_{rf}(s) = G_1 s^{-1} + \ldots + G_2 s^{-k} + \ldots = \sum_{k=1}^{\infty} G_k s^{-k} \tag{30}$$

Via the Caley-Hamilton theorem, it is possible to show that only the first n Markov parameters are necessary to completely define the transfer matrix $G_{rf}(s)$. Then, a pre-specified structure can be assigned for $G_{rf}(s)$ by imposing that

$$QCZ\Lambda^k YR = G^*_{k+1},\;\; k=0,\ldots,n-1. \tag{31}$$

instead of (12) in **P3**, with $G_i^* \in \mathcal{R}^{q\times q}$, $i=0,\ldots,n-1$ selected by the designer.

As a consequence, the gradient flow of Section 3.1 is modified as follows: the energy function becomes

$$V[t] \triangleq T(t)\left(\|Z\|_F^2 + |Y\|_F^2\right) \tag{32}$$
$$+\,\|AZ(t) - Z(t)\Lambda + K(t)CZ(t)\|_F^2 \tag{33}$$
$$+\,\|Z(t)Y(t) - I\|_F^2 \tag{34}$$
$$+\sum_{i=1}^{d}\|Z(t)e_i - E_i\|_F^2 + \|Q(t)CE\|_F^2 \tag{35}$$
$$+\sum_{k=0}^{n-1}\left\|Q(t)CZ(t)\Lambda^k Y(t)R - G_{k+1}\right\|_F^2 \tag{36}$$

the dynamic equations (25-28):

$$\begin{aligned}\frac{dZ}{dt} &= -\mu\Big(TZ + A^T(AZ - Z\Lambda + KCZ) \\ &-(AZ - Z\Lambda + KCZ)\Lambda^T \\ &+ C^T K^T(AZ - Z\Lambda + KCZ) + (ZY - I)Y^T \\ &+\sum_{i=1}^{d}(Ze_i - E_i)e_i^T \\ &+\sum_{k=0}^{n-1}(QC)^T\left(QCZ\Lambda^k YR - G_k\right)\left(\Lambda^k YR\right)^T\Big) \\ &\triangleq -\mu(TZ + \tilde{H}_1(Z,Y,K,Q),\;\; Z(0)=Z_0,\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{dY}{dt} &= -\mu\Big(TY + Z^T(ZY - I) \\ &+\sum_{k=0}^{n-1}\left(QCZ\Lambda^k\right)^T\left(QCZ\Lambda^k YR - G_{k+1}\right)(R)^T\Big) \\ &\triangleq -\mu(TY + \tilde{H}_2(Z,Y,Q),\;\; Y(0)=Y_0,\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{dK}{dt} &= -\mu\,(AZ - Z\Lambda + KCZ)\,Z^T C^T \\ &\triangleq -\mu\tilde{H}_3(Z,K),\;\; K(0)=K_0,\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{dQ}{dt} &= -\mu\Big(QCE(CE)^T + \sum_{k=0}^{n-1}\left(QCZ\Lambda^k YR - G_{k+1}\right) \\ &\left(CZ\Lambda^k R\right)^T\Big) \\ &\triangleq -\mu(\tilde{H}_4(Z,Y,Q),\;\; Q(0)=Q_0,\end{aligned} \tag{40}$$

$$
\begin{aligned}
T(t) \triangleq\ & \eta \Big(\|AZ(t) - Z(t)\Lambda + K(t)CZ(t)\|_F^2 \\
& + \|Z(t)Y(t) - I\|_F^2 + \textstyle\sum_{i=1}^{d} \|Z(t)e_i - E_i\|_F^2 \\
& + \|Q(t)CE\|_F^2 + \\
& \sum_{k=0}^{n-1} \left\|Q(t)CZ(t)\Lambda^k Y(t)R - G_{k+1}\right\|_F^2 \Big) \\
\triangleq\ & \eta \hat{T}(t), \ \ T(0) = T_0.
\end{aligned}
\tag{41}
$$

Theorem 2 *Given the initial state $(Z_0, Y_0, K_0, Q_0) \notin \overline{\Omega}$, where $\overline{\Omega}$ is the set of equilibrium points of the gradient flow, viz. each state (Z, Y, K, Q) satisfying the conditions (19)-(22) and (31), it is an equilibrium point of Eqs. (37)-(40). Then*

(a) *The state $(Z(t), Y(t), K(t), Q(t))$ of the gradient flow (37)-(40) and (41) converges to an equilibrium point $(\bar{Z}, \bar{Y}, \bar{K}, \bar{Q}) \in \overline{\Omega}$.*

(b) *If $(\bar{Z}, \bar{Y}, \bar{K}, \bar{Q}) \in \overline{\Omega}^*$, where $\overline{\Omega}^* := \{(Z, Y, K, Q) \,|\, (Z, Y, K, Q)\, satisfying\, (19) - (22)\, and\, (31)\} \subseteq \overline{\Omega}$, then the convergence is exponential.*

Proof See [21].

It is worth pointing out that the design parameters μ and η strongly affect the stationary solution. Specifically, η influences the decay rate of the temperature parameter T, while μ affects that of the flow. Numerical experiences (see Section IV) seem to reveal that a sort of trade-off arises between μ and η, in that the higher the rate of convergence of the flow, the lower that of T. Observe also that the property of convergence of the flow to a feasible solution may depend on the rate of T, the higher the worse.

IV. NUMERICAL EXAMPLE

This example, considered in [22], deals with the robust FDI procedure developed in Section 3.2. The system is a double-effect pilot evaporator plant described by the following matrices:

$$
A = \begin{bmatrix} 0 & 0 & -0.0034 & 0 & 0 \\ 0 & -0.0410 & 0.0013 & 0 & 0 \\ 0 & 0 & -1.1471 & 0 & 0 \\ 0 & 0 & -0.0360 & 0 & 0 \\ 0 & 0.0940 & 0.0057 & 0 & -0.0510 \end{bmatrix},
$$

$$
B = \begin{bmatrix} -1.0000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.9480 \\ 0.9160 & -1.0000 & 0 \\ -0.5980 & 0 & 0 \end{bmatrix}
$$

$$
E = \begin{bmatrix} 0 & 1.0000 \\ 0.0620 & -0.1320 \\ 0 & -7.1890 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

Let us consider the faults of the first and third inputs, so that the fault entry matrix is chosen in this case as $R = [col_1(B)\ col_3(B)]$. Due to the fact that there are three measurements and four independent disturbances and faults, the necessary condition (10) does not hold true. Therefore joint disturbance decoupling and fault isolation cannot be achieved. As a consequence, the method in [6] is not applicable, while other algorithms (e.g. [20]) allow one to derive robust fault detection observers only. Notice that the system is detectable and the dimension of the unobservable subspace is $dim(\mathcal{X}_{NO}) = 2$. Therefore, an admissible observer spectrum is $\Lambda = \Lambda_N \cup \Lambda_V$ where $\Lambda_N = \{-2, -3\}$ (the subset of the new observer eigenvalues) and $\Lambda_V = \{-0.0510, -0.0410, -1.1471\}$ (a subset of the open-loop eigenvalues). Finally, the Markov parameters of (30) were imposed according to the following structure of the fault-residual transfer matrix:

$$
G^*_{rf}(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{0.001}{s+2} \\ \frac{0.001}{s+3} & \frac{1}{s+3} \end{bmatrix}.
$$

We choose again as initial values of the computation

$$
Z(0) = Y(0) = I_5,\ K(0) = 0_{5\times 4}, Q(0) = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}
$$

Applying the gradient flow (32-40) and $T(t)$ in (41) with $\mu = 10^{11}$ and $\eta = 10^{-7}$, we obtained

$$
\bar{Z} = \begin{bmatrix} 0.00013446091 & 0.0517298606 \\ 0.0619965095 & 0.1021744161 \\ 0.0000180818 & 0.0109866527 \\ -0.0001687437 & -0.0487201859 \\ 0.0000000119 & 0.9880044007 \end{bmatrix}
$$

$$
\begin{matrix} 0.088268324 & 0.999992964 & -0.0000123462 \\ 0.04704487274553 & -0.1320000624 & 0.0758774744 \\ 0.02219193779655 & -7.18900097756 & -0.000000691 \\ -0.00587760277504 & -0.00001137315 & 0.0000154567 \\ 0.63084402828304 & 0.00000000665 & 0.766273607 \end{matrix} \Bigg]
$$

$$
\bar{K} = \begin{bmatrix} -3.01015918837239 & -0.25575050251047 \\ -1.34975175539159 & -0.20936471215718 \\ -0.47084588449908 & -0.06549434430870 \\ 0.06920862552386 & 0.04562817771611 \\ -19.20729734829293 & -2.67919847626693 \end{bmatrix}
$$

$$
\begin{matrix} -1.13101671089989 \\ 2.62829908201686 \\ -0.32237039884921 \\ -1.92432103302015 \\ 18.72450504950933 \end{matrix} \Bigg]
$$

$$
\bar{Q} = \begin{bmatrix} 0.07553994520407 & 0.01050817932394 \\ 7.58287560565125 & 1.05478877431106 \end{bmatrix}
$$

$$
\begin{matrix} 1.17415748997201 \\ 8.28916472026641 \end{matrix} \Bigg]
$$

and $\kappa_2(\bar{Z}) = 230.6786$. The resulting computed disturbance-residual and fault-residual transfer matrices are

$$
G_{rd}(s) = \begin{bmatrix} 0 & \frac{-3.356e-006\,(s+5.532)}{(s+2)(s+1.147)} \\ 0 & \frac{-8.9287e-007(s+311.5)}{(s+3)\,(s+1.147)} \end{bmatrix},
$$

$$
G_{rf}(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{0.0099618}{s+2} \\ \frac{0.0099993\,(s+2.007)(s+1.148)}{(s+3)(s+2)(s+1.147)} & \frac{1}{s+3} \end{bmatrix}.
$$

In next Fig. 3, for the sake of completeness, the step responses of residual-fault transfer matrices, computed via the algorithm of Section III.A and, respectively, the SCB-based design algorithm of [20], are shown. As expected, our flow gradient solution has determined a reasonable "detection and isolation degree" in those cases when the exact multiple-fault isolation problem cannot be solved. As a matter of fact the approach of [20], which imposes the fault detection conditions, is not capable to achieve any degree of fault isolation (see Fig. 3 right).

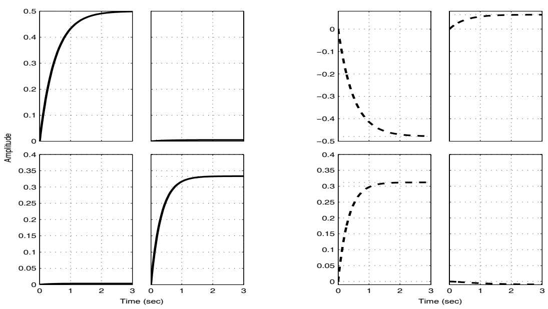

Fig. 1. Step responses of $G_{rf}(s)$ computed via gradient flow algorithm of Section 3.1 (continuous line) and via SCB-based design algorithm of [20](dashed line)

Remark 2- As far as the computational burdens of the proposed algorithms are concerned, it is worth pointing out that they involve a number of ordinary differential equations which is equal to $2 * n^2 + n * p + q * p$. An interesting mathematical and computational analysis for a suite of programs for solving ordinary differential equations in MATLAB can be found in [24] where, amongst many, a *stiff* problem is considered: the classical *Brusselator* system modelling diffusion of a chemical reactor of $2\,N$ equations. It has been showed that such a problem can be solved by the *ode15s MATLAB function with the* **SparseJ** *option* up to a 1000 sized system. As a consequence, taking also into consideration that our problem formulation leads to *initial value problems* of simpler structures w.r.t. a *stiff* ones, the proposed algorithm can hopefully be applied to reasonable large sized FDI problems.

V. Conclusions

In this paper a novel solution to robust multiple-fault detection and isolation has been presented. The problem is formulated as an optimization one where the objective function is the spectral condition number of the observer eigenvector matrix and the fault isolability conditions, expressed in terms of Markov parameters, are imposed as constraints. A gradient flow model has been derived and its convergence properties proved. The main feature of such an approach relies on its capability to both efficiently isolate failures and be robust against unstructured model uncertainties. A second flow gradient algorithm, derived simply by modifying the energy function of the first flow model, is presented in order to consider also situations where the exact solvability conditions are not met. The numerical experiments demonstrate the effectiveness of the approach.

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