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A Higher Order Stokes-Dirac Structure for Distributed-Parameter Port-Hamiltonian Systems

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***Abstract*— This paper studies an extension of a Stokes-Dirac structure which is treated in a port-Hamiltonian formulation of distributed-parameter systems for a higher order. The extended structure does not only use exterior derivative operators but Hodge star operators and their composite operators to relate flows with efforts. The structure represents a differential relation between energy variables and it shows clearly some geometric properties.**

I. INTRODUCTION

A generalized Hamiltonian formulation provides a useful framework to describe a physical network. A port-Hamiltonian system consists of an interconnection of some subsystems, which are represented as Hamiltonian systems with boundary ports. The input-output relation is given by a power-conserving interconnection called a Dirac structure.

Recently, in the framework a treatment of distributed-parameter systems has been proposed [1]. The Dirac structure is defined by differential forms on a spatial domain of the system and its boundary. The definition based on Stokes' theorem is called a Stokes-Dirac Structure. The structure means the power-conserving property, namely the change of the interior energy is equal to the power supplied to the system through its boundary.

In this paper, an extension of the Stokes-Dirac structure is presented. The standard Stokes-Dirac structure is defined by using an exterior derivative operator to describe the relation between flows and efforts. The proposed extension allows the structure to use not only exterior derivative operators, but Hodge star operators and their composite operators also. The structure means that there are higher order differential relations between energy variables, and it makes some geometric properties clear.

Finally, two examples are presented. One is about Euler-Bernoulli equation. This is the case that a system representation is both an asymmetry and an infinite, then the proof of the Stokes-Dirac structure can not be performed. We show that the Stokes-Dirac structure with the higher order gives a symmetric form. And it expresses same geometric relations. Another example is a treatment of Electromagnetic wave equations.

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II. PRELIMINARIES

In the modeling of a power-conserving interconnection, the relation between elements is described such as the total incoming power into the interconnection is always zero through the boundary. The power-conserving interconnection is formalized by a Dirac structure as follows [1].

Definition 2.1: Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces with a bilinear operation $\langle\ |\ \rangle : \mathcal{F} \times \mathcal{E} \to L$ called a pairing such as

$$\langle\ e|f\ \rangle \in L\,,\ \ f \in \mathcal{F}\,,\ \ e \in \mathcal{E}\,. \tag{1}$$

By symmetrizing the pairing we obtain a symmetric bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathcal{F} \times \mathcal{E}$ defined as

$$\langle\!\langle\ (f^1, e^1), (f^2, e^2)\ \rangle\!\rangle := \langle\ e^1|f^2\ \rangle + \langle\ e^2|f^1\ \rangle\,. \tag{2}$$

Definition 2.2: Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces with the pairing $\langle\ |\ \rangle$. We denote an orthogonal complement with respect to the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ by $\perp$. A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} = \mathcal{D}^{\perp}$.

Definition 2.3: Let Z be an n-dimensional smooth manifold with a smooth $(n-1)$-dimensional boundary ∂Z, representing the space of energy variables. Let $\Omega^k(Z)$ be differential k-forms on Z. A pairing between $\alpha \in \Omega^k(Z)$ and $\beta \in \Omega^{n-k}(Z)$ is given by

$$\langle\ \beta|\alpha\ \rangle\ := \int_Z \beta \wedge \alpha\,. \tag{3}$$

Similarly, there is a pairing between $\alpha \in \Omega^k(\partial Z)$ and $\beta \in \Omega^{n-k-1}(\partial Z)$ is given by

$$\langle\ \beta|\alpha\ \rangle\ := \int_{\partial Z} \beta \wedge \alpha\,. \tag{4}$$

Definition 2.4: Let $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ be linear spaces satisfying $p + q = n + 1$ given by

$$\begin{aligned}\mathcal{F}_{p,q} &:= \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)\,,\\ \mathcal{E}_{p,q} &:= \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z)\,.\end{aligned} \tag{5}$$

Let $(f_p, f_q, f_b) \in \mathcal{F}_{p,q}$ and $(e_p, e_q, e_b) \in \mathcal{E}_{p,q}$. By (3) and (4), (2) yields the bilinear form

$$\begin{aligned}&\langle\!\langle\ (f_p^1, f_q^1, e_p^1, e_q^1, f_b^1, e_b^1), (f_p^2, f_q^2, e_p^2, e_q^2, f_b^2, e_b^2)\ \rangle\!\rangle\\ &:= \int_Z \left(e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\right)\\ &\quad + \int_{\partial Z} \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1\right)\,.\end{aligned} \tag{6}$$

III. Incidental problems

A general modeling method with a Stokes-Dirac structure has not been introduced yet. The difficulty may be close to an existence of a Hamiltonian of systems in terms of analytical mechanics. Then we intend to expand and to formalize it first. In this section we introduce a more systematic treatment of multi variables systems and a definition procedure of the higher order differential energy variable. And we discuss about these properties.

A. Multi variables systems

Generally a practical system consists of some power ports. Then let us consider a formulation of multi variables systems with the Stokes-Dirac structure.

Lemma 3.1: Let $\mathcal{F}'$ and $\mathcal{E}'$ be linear spaces satisfying $p+q=n+1$ given by

$$\begin{aligned}\mathcal{F}' &:= \Big(\Omega^p(Z)\times\Omega^q(Z)\Big)^m \times \Big(\Omega^{n-p}(\partial Z)\Big)^l , \\ \mathcal{E}' &:= \Big(\Omega^{n-p}(Z)\times\Omega^{n-q}(Z)\Big)^m \times \Big(\Omega^{n-q}(\partial Z)\Big)^l .\end{aligned} \tag{7}$$

Consider m pairs of energy variables and its dual on Z $(f_i, e_i) \in \mathcal{F}'\times\mathcal{E}'$ $(\, i = 1,\cdots,2m\,)$. Consider l pairs of energy variables and its dual on ∂Z $(f_{bi}, e_{bi}) \in \mathcal{F}'\times\mathcal{E}'$ $(\, i = 1,\cdots,l\,)$. These variables are related by

$$F = A\cdot E\,,\quad \begin{bmatrix}F_b\\E_b\end{bmatrix} = B\cdot E\,, \tag{8}$$

where

$$\begin{aligned}F &= [f_1,\cdots,f_{2m}]^\top,\ E = [e_1,\cdots,e_{2m}]^\top ; \\ F_b &= [f_{b1},\cdots,f_{bl}]^\top,\ E_b = [e_{b1},\cdots,e_{bl}]^\top .\end{aligned} \tag{9}$$

A is a $2m\times 2m$ matrix containing exterior derivative operators, Hodge star operators and their composite operators as elements. B is a $2l\times 2m$ matrix containing exterior derivative operators, Hodge star operators, their composite operators and real numbers as elements. Diagonal elements of A are equal to zero.

Let a_{ij} be a (i,j) element of the matrix A. we consider the small matrix $\widetilde{A}_{ij} = \left[\begin{smallmatrix}0 & a_{ij}\\ a_{ji} & 0\end{smallmatrix}\right]$. We have $\widetilde{A}_{ij}$ is satisfied a Dirac structure on two pairs (f_i, e_i) and (f_j, e_j) with appropriate boundary variables for all i and j, if and only if, the linear subspace which is defined by (8) is satisfied a Dirac structure.

Proof: Let $\boldsymbol{f}$ be a set of a flow and $\boldsymbol{e}$ be a set of an effort such as

$$\begin{aligned}\boldsymbol{f}^\# &= \{f_1^\#,\cdots,f_{2m}^\#\}\ ,\ \boldsymbol{e}^\# = \{e_1^\#,\cdots,e_{2m}^\#\} ; \\ \boldsymbol{f}_b^\# &= \{f_{b1}^\#,\cdots,f_{bl}^\#\}\ ,\ \boldsymbol{e}_b^\# = \{e_{b1}^\#,\cdots,e_{bl}^\#\} .\end{aligned} \tag{10}$$

By (1) and (2), we have

$$\begin{aligned}&\langle\!\langle\ (\boldsymbol{f}^1,\boldsymbol{e}^1,\boldsymbol{f}_b^1,\boldsymbol{e}_b^1),(\boldsymbol{f}^2,\boldsymbol{e}^2,\boldsymbol{f}_b^2,\boldsymbol{e}_b^2)\ \rangle\!\rangle \\ &= \int_Z \big(e_1^1\wedge f_1^2+\cdots+e_{2m}^1\wedge f_{2m}^2+e_1^2\wedge f_1^1+\cdots+e_{2m}^2\wedge f_{2m}^1\big) \\ &\qquad + \int_{\partial Z}\big(e_{b1}^1\wedge f_{b1}^2+\cdots+e_{bl}^1\wedge f_{bl}^2 \\ &\qquad\qquad + e_{b1}^2\wedge f_{b1}^1+\cdots+e_{bl}^2\wedge f_{bl}^1\big) .\end{aligned} \tag{11}$$

By the first equation of (8), each term of (11) is written as

$$\begin{aligned}e_i\wedge f_i &= e_i\wedge\big(a_{i1}e_1+\cdots+a_{i\,2m}\,e_{2m}\big) \\ &= e_i\wedge a_{i1}e_1+\cdots+e_i\wedge a_{i\,2m}\,e_{2m} .\end{aligned} \tag{12}$$

We can immediately know that the diagonal element of the matrix A is zero all. By substitution (12) into (11) we have

$$\begin{aligned}\sum_{i=1}^{2m}\sum_{j=i+1}^{2m}\Bigg[&\int_Z\big(e_i^1\wedge a_{ij}e_j^2+e_j^1\wedge a_{ji}e_i^2+e_i^2\wedge a_{ij}e_j^1 \\ &+e_j^2\wedge a_{ji}e_i^1\big)+\int_{\partial Z}\big(e_{bk}^1\wedge f_{bk}^2+e_{bk}^2\wedge f_{bk}^1\big)\Bigg] ,\end{aligned} \tag{13}$$

where $k = 2m(i-1)+j+i(i+1)/2$. Each non-zero term which is determined by (i,j) in (13) is corresponding to one Stokes-Dirac structure.

Indeed, if all terms of (13) satisfy the Dirac structure then the sufficient condition is satisfied.

Assume that (8) holds Dirac structure and that at least one term of (13) is not satisfied. Then (11) must be nonzero. But this is contradiction for the assumption. Thus this shows the "only if" part. ■

B. Definition of energy variables

We now derive a procedure of a higher order differentiation of energy variables. We assume that the exterior derivative operator relates to spatial variables only.

Energy variables are described by differential forms. Then, simple multiple operations of the exterior derivative 'd' do not generate these higher order differentiations. Because $d\circ d = 0$.

If ω is an exact p-form which is an element of the image $d(\Omega^{p-1}(M))\subset\Omega^p$, then $d\omega = 0$. Even though ω is exact, $*\omega$ is not exact i.e. $d*\omega\neq 0$ necessarily, where $*$ is a Hodge star operator. Hence we define the higher order differentiation with the alternate action of d and $*$.

We assume that energy variables are defined as the flow $f_1, \bar{f}_1 \in \Omega^k(M)$ first. There are two sequences according to the first operation as the following diagram:

$$\begin{tikzcd}[column sep=small]
 & e_2 \arrow[d, phantom, "\pitchfork"] & f_1\ ,\ \bar{f}_1 \arrow[d, phantom, "\pitchfork"] & \bar{e}_2 \arrow[d, phantom, "\pitchfork"] & \\
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, shift right, "*"'] & \Omega^{k+1}(M) \arrow[r] \arrow[d, "*"'] & \cdots \\
\cdots & \Omega^{n-k+1}(M) \arrow[l] \arrow[u, "*"] & \Omega^{n-k}(M) \arrow[l, "d"'] \arrow[u, shift right, "*"'] & \Omega^{n-k-1}(M) \arrow[l, "d"'] & \cdots \arrow[l] \\
 & f_2 \arrow[u, phantom, "\cup"] & e_1\ ,\ \bar{e}_3 \arrow[u, phantom, "\cup"] & \bar{f}_2 \arrow[u, phantom, "\cup"] &
\end{tikzcd} \tag{14}$$

Namely, one of sequences takes $*$ first.

$$f_1 \xrightarrow{*} e_1 \xrightarrow{d} f_2 \xrightarrow{*} e_2 \xrightarrow{d} f_3 \rightarrow \cdots \,. \tag{15}$$

The other takes d first.

$$\bar{f}_1 \xrightarrow{d} \bar{e}_2 \xrightarrow{*} \bar{f}_2 \xrightarrow{d} \bar{e}_3 \xrightarrow{*} \bar{f}_3 \rightarrow \cdots \,. \tag{16}$$

Remark 3.1: If energy variables are one-parameter functions, or 1-forms and 0-forms on a smooth 1-dimensional manifold, these are unique forms on the manifold respectively. In the case of the others the flow and the effort do not take same forms until $*d*d$ or $d*d*$ is operated.

C. Arbitrariness of energy variables

We consider three types of an equivalent class which provides freedom of representations about energy variables.

One of freedoms exists on the definition of higher order energy variables. If $\omega \in \Omega^k(M)$ is an effort, then the flow is defined by $d\omega \in \Omega^{k+1}(M)$ in (15). On the other hand we consider that $\omega + d\eta \in \Omega^k(M)$ is a effort, then the flow is equal to the previous result $d\omega \in \Omega^{k+1}(M)$.

A set of such closed k-forms $\omega \in \Omega^k(M)$ is identified as de Rham cohomology class denoted by

$$[\omega] = \omega + d\eta \in \Omega^k(M)\,,\ \eta \in \Omega^{k-1}(M)\,. \tag{17}$$

Remark 3.2: We can see that this result corresponds to a gauge transformation[4]. It is known as a degree of freedom such that the vector potential is not defined uniquely in Maxwell's equation for example. This transformation gives it equivalent physical meanings as long as the relation holds after applying a exterior derivative.

Another freedom appears in the case that a result of an interior product between a flow and an effort is zero equivalently.

Example 3.1: One of the example is treated in [1] as the special case of the standard Stokes-Dirac structure for an ideal isentropic fluid. This is called a modified Stokes-Dirac structure.

$$\begin{aligned} \begin{bmatrix} f_\rho \\ f_v \end{bmatrix} &= \begin{bmatrix} de_v \\ de_\rho + \frac{1}{*\rho} * \big((*dv)\wedge(*e_v)\big) \end{bmatrix} \\ \rightarrow e_v \wedge f_v &= e_v \wedge de_\rho + e_v \wedge \left(\tfrac{1}{*\rho} * \big((*dv)\wedge(*e_v)\big)\right) \\ &= e_v \wedge de_\rho\,. \end{aligned} \tag{18}$$

The last freedom exists on a modeling of systems with higher order energy variables. That is, there are some combinations to represent higher order terms (e.g. $w_{xxx}dx = dw_{xx} = d*dw_x = d*w_{xx}dx =$...). We will show it as a example in section V.

IV. Main Results

From *Lemma 3.1* we only have to verify a Stokes-Dirac structure for decomposed structures independently instead of a whole system. Then we consider the Stokes-Dirac structure for such a minimum structure. The extension allows the structure to use not only exterior derivative operators, but Hodge star operators and their composite operators also.

A. Definitions

Lemma 4.1: Let ω be a differential k-form on a smooth n-dimensional manifold M. Then $(*d)^m\omega$ is a σ-form, where

$$\sigma := \begin{cases} k\,, & \text{when } m \text{ is an even number;} \\ n-k-1\,, & \text{when } m \text{ is an odd number.} \end{cases} \tag{19}$$

Proof: The following diagram summarizes the relationships between spaces of differential forms.

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \longrightarrow \cdots \\ & *\big\uparrow & & *\big\downarrow & \\ \cdots \longleftarrow & \Omega^{n-k}(M) & \xleftarrow{d} & \Omega^{n-k-1}(M) & \longleftarrow \cdots \end{array} \tag{20}$$

■

Definition 4.1: Let $\bar{\sigma}$ be an integer number defined by the converse case of σ such as

$$\bar{\sigma} := \begin{cases} n-k-1\,, & \text{when } m \text{ is an even number;} \\ k\,, & \text{when } m \text{ is an odd number.} \end{cases} \tag{21}$$

Definition 4.2: Let σ and $\bar{\sigma}$ be integer numbers defined by (19) and (21). Let ϵ_m and $\bar{\epsilon}_m$ be binary numbers which are equal to 1 or -1 defined by

$$\epsilon_m := \begin{cases} -(-1)^{\sigma\bar{\sigma}}\,, & \text{when } m \text{ is an even number;} \\ 1\,, & \text{when } m \text{ is an odd number,} \end{cases} \tag{22}$$

$$\bar{\epsilon}_m := \begin{cases} -1\,, & \text{when } m \text{ is an even number;} \\ (-1)^{\sigma\bar{\sigma}}\,, & \text{when } m \text{ is an odd number.} \end{cases} \tag{23}$$

Lemma 4.2: If we consider equations

$$\begin{aligned} d(*d)^m\alpha\wedge\beta = &\sum_{i=0}^{m}\epsilon_{m-i}\, d\big((*d)^{m-i}\alpha\wedge(*d)^i\beta\big) \\ &+ \zeta_0\alpha\wedge d(*d)^m\beta\,, \end{aligned} \tag{24}$$

$$\begin{aligned} d(*d)^m\beta\wedge\alpha = &\sum_{i=0}^{m}\epsilon_{m-i}\, d\big((*d)^{m-i}\beta\wedge(*d)^i\alpha\big) \\ &+ \bar{\zeta}_0\beta\wedge d(*d)^m\alpha\,, \end{aligned} \tag{25}$$

then we have following relations.

(i) the case that n is an even number

Let α be a k-form and β be a $(n-k-1)$-form. If $(*d)^m\alpha$ is a σ-form and $(*d)^m\beta$ is a $\bar{\sigma}$-form, then we have

$$\zeta_0 = -(-1)^{\sigma}\,,\ \bar{\zeta}_0 = -(-1)^{\bar{\sigma}}\,. \tag{26}$$

(ii) the case that n is an odd number

Let α and β be k-forms. If $(*d)^m\alpha$ and $(*d)^m\beta$ are $\bar{\sigma}$-forms, then we have

$$\zeta_0 = \bar{\zeta}_0 = (-1)^{\sigma(\bar{\sigma}+1)}\,. \tag{27}$$

Proof: The summary proof is showed only. Let n be an even number. Then $(*d)^m\alpha$ is a σ-form and $(*d)^m\beta$ is a

$\bar{\sigma}$-form. It is easy to show above statements using following properties:

$$\begin{aligned} d(*d)^m\alpha \wedge (*d)^{n-m}\beta \\ &= d\big((*d)^m\alpha \wedge (*d)^{n-m}\beta\big) \\ &\quad - (-1)^{\sigma}(*d)^m\alpha \wedge d(*d)^{n-m}\beta\,, \end{aligned} \tag{28}$$

$$\begin{aligned} (*d)^m\alpha \wedge d(*d)^{n-m}\beta \\ &= *d(*d)^{m-1}\alpha \wedge d(*d)^{n-m}\beta \\ &= (*d)^{n-m+1}\beta \wedge d(*d)^{m-1}\alpha \\ &= (-1)^{\sigma(\bar{\sigma}+1)}d(*d)^{m-1}\alpha \wedge (*d)^{n-m+1}\beta\,. \end{aligned} \tag{29}$$

■

B. Extensions of the Stokes-Dirac structure

We show three types of extensions for the Stokes-Dirac structure as follows.

Theorem 4.3 ($d(*d)^m$-type): Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces satisfying $p+q=n+1$ given by

$$\begin{aligned} \mathcal{F} &:= \Omega^p(Z)\times\Omega^q(Z)\times\Big(\Omega^{n-p}(\partial Z)\Big)^m\,, \\ \mathcal{E} &:= \Omega^{n-p}(Z)\times\Omega^{n-q}(Z)\times\Big(\Omega^{n-q}(\partial Z)\Big)^m\,. \end{aligned} \tag{30}$$

Let $\mathcal{D}$ be a linear subspace of $\mathcal{F}\times\mathcal{E}$ such as

$$\begin{aligned} \mathcal{D} = \Big\{&(f_p, f_q, e_p, e_q, f_{b1}, \cdots, f_{bm}, e_{b1}, \cdots, e_{bm}) \in \mathcal{F}\times\mathcal{E} \;\Big| \\ &\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & \gamma_q\, d(*d)^m \\ \gamma_p\, d(*d)^m & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ &\begin{bmatrix} f_{b1} \\ f_{b2} \\ \vdots \\ f_{bm} \end{bmatrix} = \begin{bmatrix} \bar{\epsilon}_0 e_p|_{\partial Z} \\ \bar{\epsilon}_1 (*d) e_p|_{\partial Z} \\ \vdots \\ \bar{\epsilon}_m (*d)^m e_p|_{\partial Z} \end{bmatrix}, \begin{bmatrix} e_{b1} \\ e_{b2} \\ \vdots \\ e_{bm} \end{bmatrix} = \begin{bmatrix} (*d)^m e_q|_{\partial Z} \\ (*d)^{m-1} e_q|_{\partial Z} \\ \vdots \\ e_q|_{\partial Z} \end{bmatrix} \Big\}\,, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \gamma_p &:= \begin{cases} -(-1)^{\bar{\sigma}(\sigma+1)}\,, & \text{when } n \text{ is an even number;} \\ -(-1)^{\sigma(\bar{\sigma}+1)}\,, & \text{when } n \text{ is an odd number,} \end{cases} \\ \gamma_q &:= \zeta_0\,. \end{aligned} \tag{32}$$

Then $\mathcal{D}$ is a Dirac structure.

Theorem 4.4 ($*(d*)^m$-type): Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces satisfying $2k=n+1$ given by

$$\begin{aligned} \mathcal{F} &:= \Omega^k(Z)\times\Omega^k(Z)\times\Big(\Omega^{n-k}(\partial Z)\Big)^m\,, \\ \mathcal{E} &:= \Omega^{n-k}(Z)\times\Omega^{n-k}(Z)\times\Big(\Omega^{n-k}(\partial Z)\Big)^m\,. \end{aligned} \tag{33}$$

Let $\mathcal{D}$ be a linear subspace of $\mathcal{F}\times\mathcal{E}$ such as

$$\begin{aligned} \mathcal{D} = \Big\{&(f_p, f_q, e_p, e_q, f_{b1}, \cdots, f_{bm}, e_{b1}, \cdots, e_{bm}) \in \mathcal{F}\times\mathcal{E} \;\Big| \\ &\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & \hat{\gamma}_q\, *(d*)^{m+1} \\ \hat{\gamma}_p\, *(d*)^{m+1} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ &\begin{bmatrix} f_{b1} \\ f_{b2} \\ \vdots \\ f_{bm} \end{bmatrix} = \begin{bmatrix} \bar{\epsilon}_0 e_p|_{\partial Z} \\ \bar{\epsilon}_1 (*d) e_p|_{\partial Z} \\ \vdots \\ \bar{\epsilon}_m (*d)^m e_p|_{\partial Z} \end{bmatrix}, \begin{bmatrix} e_{b1} \\ e_{b2} \\ \vdots \\ e_{bm} \end{bmatrix} = \begin{bmatrix} (*d)^m e_q|_{\partial Z} \\ (*d)^{m-1} e_q|_{\partial Z} \\ \vdots \\ e_q|_{\partial Z} \end{bmatrix} \Big\}\,, \end{aligned} \tag{34}$$

where $\hat{\sigma} = \sigma|_{k=n-k}$, $\hat{\bar{\sigma}} = \bar{\sigma}|_{k=n-k}$, $\hat{\gamma}_p = (-1)^{(n-k)k}\gamma_p|_{\hat{\sigma},\hat{\bar{\sigma}}}$, $\hat{\gamma}_q = (-1)^{(n-k)k}\gamma_q|_{\hat{\sigma},\hat{\bar{\sigma}}}$. Then $\mathcal{D}$ is a Dirac structure.

Proof: We consider that $\hat{p}=n-k$ and $\hat{q}=n-k$ in (30). If we replace $*(d*)^{m+1}$ in (31) with $d(*d)^m$, that is,

$$*(d*)^{m+1}\alpha\wedge\beta = (-1)^{(n-k)k}d(*d)^m *\alpha \wedge *\beta\,, \tag{35}$$

then we only have to consider the same relation. ■

Theorem 4.5 ($*$-type): Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces satisfying $2k=n+1$ given by

$$\begin{aligned} \mathcal{F} &:= \Omega^k(Z)\times\Omega^k(Z)\,, \\ \mathcal{E} &:= \Omega^{n-k}(Z)\times\Omega^{n-k}(Z)\,. \end{aligned} \tag{36}$$

If a linear subspace of $\mathcal{F}\times\mathcal{E}$ is

$$\mathcal{D} = \Big\{(f_p, f_q, e_p, e_q)\in\mathcal{F}\times\mathcal{E} \;\Big|\; \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -* \\ * & 0 \end{bmatrix}\begin{bmatrix} e_p \\ e_q \end{bmatrix}\Big\}, \tag{37}$$

then $\mathcal{D}$ is a Dirac structure.

Proof: The proof is the same way as previous subsection. ■

This structure has been induced in [2] and [3] as a part of Timoshenko beam models. The structure has no boundary variables.

V. Examples

In this section, two concrete examples are presented to clarify the usage of the extended Stokes-Dirac structure.

A. Euler-Bernoulli beam equation

Consider Euler-Bernoulli beam equation

$$\rho w_{tt} = -EI w_{xxxx}\,, \tag{38}$$

where w is the displacement of the beam at the spatial variable x at time t, ρ is the mass density and EI is the elasticity modulus. The total stored energy is written as

$$\mathcal{H}(t) = \frac{1}{2}\int_0^L \rho\big(w_t\big)^2 + EI\big(w_{xx}\big)^2 dx\,. \tag{39}$$

Let $\alpha_p = w_{xx}dx$ ($f_{p1} = -\partial_t\alpha_p$) be a potential elastic energy of the bending and let $\sigma_p = *\alpha_p$ ($= e_{p1}$) be an associated co-energy variable, that is, the stress. Let $\alpha_q = \kappa\, w_t dx$ ($f_{q1} = -\partial_t\alpha_q$) be a kinetic energy, that is, the translational momentum and let $v_q = \kappa^{-1} *\alpha_q$ ($= e_{q1}$) be a co-energy velocity, where $\kappa = \frac{\rho}{EI}$.

Now we consider the representation of (38) with the standard Stokes-Dirac structure. But it is impossible to construct the right-hand side of (38) with only these energy variables corresponding to the left-hand side $f_{q1} = -\kappa\, w_{tt}dx$. If a new energy variable $e_{p2} = w_{xxx}$ is defined, then (38) is written as $-\kappa\, w_{tt}dx = d\cdot w_{xxx}$.

Next the dual $f_{p2} = -w_{txxx}dx$ of the energy variable e_{p2} must be defined as well. This yields naturally the effort e_{q2} which is related by f_{p2}. Furthermore the sequence

$$e_{p2} \xrightarrow{*} f_{p2} \xrightarrow{d} e_{q2} \xrightarrow{*} f_{q2} \xrightarrow{d} e_{p3} \rightarrow \cdots \tag{40}$$

is derived by recursive operations. The system representation given by this procedure is written as

$$\left[\begin{array}{c} \kappa\, w_{tt}dx \\ \kappa\, w_{txx}dx \\ \kappa\, w_{ttxx}dx \\ \vdots \\ \hline w_{txx}dx \\ w_{txxx}dx \\ w_{txxxx}dx \\ \vdots \end{array}\right] = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} w_t \\ w_{tx} \\ w_{txx} \\ \vdots \\ \hline w_{xx} \\ w_{xxx} \\ w_{xxxx} \\ \vdots \end{array}\right]. \tag{41}$$

This is not satisfied with the standard Stokes-Dirac structure. Now, by the proposed extension, (41) is written as

$$\begin{bmatrix} \kappa\, w_{tt}dx \\ w_{txx}dx \end{bmatrix} = \begin{bmatrix} 0 & -d{*}d \\ d{*}d & 0 \end{bmatrix} \begin{bmatrix} w_t \\ w_{xx} \end{bmatrix} = \begin{bmatrix} 0 & -{*}\Delta \\ {*}\Delta & 0 \end{bmatrix} \begin{bmatrix} w_t \\ w_{xx} \end{bmatrix}, \tag{42}$$

where $\Delta = d\delta + \delta d \ : \Omega^p(M) \to \Omega^p(M)$ is Laplace-Beltrami operator and $\delta = (-1)^{np+n+1} * d * \ : \Omega^p(M) \to \Omega^{p-1}(M)$ is an adjoint operator of d, then $-{*}\Delta \equiv * * d * d = (-1)^{k(n-k)} d * d$. For example we can apply a control method to (42) using Casimir functional [3].

Remark 5.1: In the case above Δ is considered for a 0-form. Then $\Delta = \delta d$, namely we used only $d(*d)^m$-type. Generally, we have to consider both $d(*d)^m$-type and $*(d*)^m$-type for a calculation of Δ.

A differential form ω which is satisfying $\Delta\omega = 0$ is called a harmonic. This means that a value of a point is equal to a mean value around the neighborhood. In (42) there are relations [flow] $= *\Delta\omega$. These can be modified to $\Delta\omega =$ [effort]. It is considered that ω is a harmonic form if all efforts are equal to zero.

B. Electromagnetic wave equations

Maxwell's equations contains an important physical result that an electromagnetic field travels into a space as a wave. The formulation in terms of distributed-parameter port-Hamiltonian systems of Maxwell's equations has been given already [1].

Let Z be a 3-dimensional manifold with a boundary ∂Z. Now we identify contravariant k-tensor fields and k-forms. The energy variables are both the electric field induction $D \in \Omega^2(Z)$ and the magnetic field induction $B \in \Omega^2(Z)$. The co-energy variables are both the electric field intensity $E \in \Omega^1(Z)$ and the magnetic field intensity $H \in \Omega^1(Z)$. And $J \in \Omega^2(Z)$ is the current density and $\hat{\rho} \in \Omega^3(Z)$ is the charge density.

Maxwell's equations are

$$\hat{\varepsilon}_0 \delta E = -\hat{\rho}\,, \tag{43}$$

$$dE = -\frac{\partial B}{\partial t}\,, \tag{44}$$

$$dB = 0\,, \tag{45}$$

$$c^2 \hat{\varepsilon}_0 \delta B = J + \hat{\varepsilon}_0 \frac{\partial E}{\partial t}\,, \tag{46}$$

where $\hat{\varepsilon}_0$ is the electric permittivity ($\|\hat{\varepsilon}_0\| = \varepsilon_0, *\hat{\varepsilon}_0 u = \varepsilon_0 u, u \in \Omega^1(Z)$) and c is the speed of light.

Now we introduce the constitutive relations of the medium: $H = c^2\hat{\varepsilon}_0 B, D = \hat{\varepsilon}_0 E$. Then (43) can be written as

$$dD = \hat{\rho}\,. \tag{47}$$

Applying δ to (47) we have $\delta dD = *d\rho$. And we have

$$d\delta D = -\frac{1}{c^2}\frac{\partial J}{\partial t} - \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2}\,. \tag{48}$$

By the calculation of $\Delta = d\delta + \delta d$ we have

$$\frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} + \Delta D = *d\rho - \frac{1}{c^2}\frac{\partial J}{\partial t}\,. \tag{49}$$

Applying $d*$ to (46), (44) yields

$$c^2 \varepsilon_0 d\delta B = d{*}J - \varepsilon_0 \frac{\partial^2 B}{\partial t^2}\,. \tag{50}$$

Then we have

$$\frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} + \Delta B = \frac{1}{c^2\varepsilon_0} d{*}J\,. \tag{51}$$

In free space, (49) and (51) can be written as

$$\begin{bmatrix} \partial_t(dH) \\ \partial_t(dE) \end{bmatrix} = c^2 \begin{bmatrix} 0 & d{*}d \\ -d{*}d & 0 \end{bmatrix} \begin{bmatrix} *B \\ -*D \end{bmatrix}, \tag{52}$$

$$\begin{bmatrix} f_{b1} \\ f_{b2} \end{bmatrix} = \begin{bmatrix} -c^2 * B \\ -c^2(*d) * B \end{bmatrix}, \ \begin{bmatrix} e_{b1} \\ e_{b2} \end{bmatrix} = \begin{bmatrix} -(*d)*D \\ -*D \end{bmatrix}. \tag{53}$$

The energy balance is

$$\frac{d\mathcal{E}}{dt} = \int_{\partial Z} c^2 \varepsilon_0 B \wedge \frac{\partial B}{\partial t} + \varepsilon_0 E \wedge \frac{\partial E}{\partial t}\,. \tag{54}$$

It is considered that $\mathcal{E}$ of (54) is a energy density of Poynting's theorem.

VI. Conclusions

The extended Stokes-Dirac structure can represent higher order differential relations between energy variables and make some geometric properties clear.

The other advantage is the property which decomposes the whole structure symmetrically into independent ones for the Stokes-Dirac structure.

Appendix

Proof of Theorem 4.3: The statement follows from the proof of the standard Stokes-Dirac structure [1]. We will denote the linear subspace by

$$\begin{aligned} \Psi_1 &:= (f_p^1, f_q^1, e_p^1, e_q^1, f_{b1}^1, \cdots, f_{bm}^1, e_{b1}^1, \cdots, e_{bm}^1)\,, \\ \Psi_2 &:= (f_p^2, f_q^2, e_p^2, e_q^2, f_{b1}^2, \cdots, f_{bm}^2, e_{b1}^2, \cdots, e_{bm}^2)\,. \end{aligned} \tag{55}$$

First, $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is calculated for $\Psi_1, \Psi_2 \in \mathcal{D}$. If this result is zero, then $\Psi_1 \in \mathcal{D}^\perp$ showing $\mathcal{D} \subset \mathcal{D}^\perp$. Secondly, we consider a condition of $\Psi_1 \in \mathcal{D}^\perp$ such that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is zero for all $\Psi_2 \in \mathcal{D}$. Namely, $\mathcal{D}^\perp \subset \mathcal{D}$, that is, $\mathcal{D} = \mathcal{D}^\perp$.

(i) $\mathcal{D} \subset \mathcal{D}^\perp$

Let $\Psi_1 \in \mathcal{D}$, and consider any $\Psi_2 \in \mathcal{D}$. By substitution of (31) into (6) the right-hand side of (6), transpositions of both the second and the fourth term in the first integral yield

$$\int_Z \Big[\gamma_q e_p^1 \wedge d(*d)^m e_q^2 - d(*d)^m e_p^2 \wedge e_q^1 + \gamma_q e_p^2 \wedge d(*d)^m e_q^1 - d(*d)^m e_p^1 \wedge e_q^2\Big] + \int_{\partial Z} \sum_{i=0}^{m} \bar{\epsilon}_{m-i} \Big[(*d)^{m-i} e_q^1 \wedge (*d)^i e_p^2 + (*d)^{m-i} e_q^2 \wedge (*d)^i e_p^1\Big]. \quad (56)$$

By *Lemma 4.2*, we have

$$d(*d)^m e_p^2 \wedge e_q^1 = \sum_{i=0}^{m} \epsilon_{m-i}\, d\big((*d)^{m-i} e_p^2 \wedge (*d)^i e_q^1\big) + \zeta_0 e_p^2 \wedge d(*d)^m e_q^1 \,, \quad (57)$$

$$d(*d)^m e_p^1 \wedge e_q^2 = \sum_{i=0}^{m} \epsilon_{m-i}\, d\big((*d)^{m-i} e_p^1 \wedge (*d)^i e_q^2\big) + \zeta_0 e_p^1 \wedge d(*d)^m e_q^2 \,. \quad (58)$$

Substitution of (57) and (58) in the first term in (56) yields

$$-\int_Z \Big[\sum_{i=0}^{m} \epsilon_{m-i}\, d\big((*d)^{m-i} e_p^2 \wedge (*d)^i e_q^1\big) + \sum_{i=0}^{m} \epsilon_{m-i}\, d\big((*d)^{m-i} e_p^1 \wedge (*d)^i e_q^2\big)\Big] + \int_{\partial Z} \sum_{i=0}^{m} \epsilon_{m-i} \Big[(*d)^{m-i} e_p^1 \wedge (*d)^i e_q^2 + (*d)^{m-i} e_p^2 \wedge (*d)^i e_q^1\Big]. \quad (59)$$

By Stokes' theorem we have that (59) is equal to zero.

(ii) $\mathcal{D}^\perp \subset \mathcal{D}$
We consider a condition of $\Psi_1 \in \mathcal{D}^\perp$ such that the right-hand side of (6) is zero for all elements $\Psi_2 \in \mathcal{D}$. Hence by substitution of (31) we have

$$\int_Z \Big[\gamma_q e_p^1 \wedge d(*d)^m e_q^2 + \gamma_p e_q^1 \wedge d(*d)^m e_p^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\Big] + \int_{\partial Z} \sum_{i=0}^{m} \Big[\bar{\epsilon}_{m-i} e_b^1 \wedge (*d)^i e_p^2 + (*d)^{m-i} e_q^2 \wedge f_b^1\Big] = 0 \,, \quad (60)$$

for all e_p^2, e_q^2. Now we consider a condition such that e_p^2, e_q^2 of the first integral is zero on the boundary ∂Z, that is,

$$\int_Z \Big[d(*d)^m e_q^2 \wedge e_p^1 - d(*d)^m e_p^2 \wedge e_q^1 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\Big] = 0 \,. \quad (61)$$

By *Lemma 4.2* we have

$$d(*d)^m e_q^2 \wedge e_p^1 = \sum_{i=0}^{m} \epsilon_{m-i}\, d\big((*d)^{m-i} e_q^2 \wedge (*d)^i e_p^1\big) + \bar{\zeta}_0 e_q^2 \wedge d(*d)^m e_p^1 \,. \quad (62)$$

Since $e_p^2|_{\partial Z} = e_q^2|_{\partial Z} = 0$, if we substitute both (62) and (57) into (61), then by Stokes' theorem we have

$$\int_Z \Big[\gamma_p e_q^2 \wedge d(*d)^m e_p^1 + \gamma_q e_p^2 \wedge d(*d)^m e_q^1 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\Big] = 0 \,, \quad (63)$$

for all e_p^2, e_q^2 with $e_p^2|_{\partial Z} = e_q^2|_{\partial Z} = 0$. Clearly, this implies

$$f_p^1 = \gamma_q d(*d)^m e_q^1 \ , \ f_q^1 = \gamma_p d(*d)^m e_p^1 \,. \quad (64)$$

Finally, by the substitution (64) into (60) we have

$$\int_Z \Big[\gamma_q e_p^1 \wedge d(*d)^m e_q^2 + \gamma_p e_q^1 \wedge d(*d)^m e_p^2 + \gamma_q e_p^2 \wedge d(*d)^m e_q^1 + \gamma_p e_q^2 \wedge d(*d)^m e_p^1\Big] + \int_{\partial Z} \sum_{i=0}^{m} \Big[\bar{\epsilon}_{m-i} e_b^1 \wedge (*d)^i e_p^2 + (*d)^{m-i} e_q^2 \wedge f_b^1\Big] = 0 \,, \quad (65)$$

for all e_p^2, e_q^2. The substitution of both (62) and (57) into (65) yields

$$-\int_Z \bar{\epsilon}_{m-i} \Big[\sum_{i=0}^{m} d\big((*d)^{m-i} e_q^2 \wedge (*d)^i e_p^1\big) + \sum_{i=0}^{m} d\big((*d)^{m-i} e_q^1 \wedge (*d)^i e_p^2\big)\Big] + \int_{\partial Z} \sum_{i=0}^{m} \Big[\bar{\epsilon}_{m-i} e_b^1 \wedge (*d)^i e_p^2 + (*d)^{m-i} e_q^2 \wedge f_b^1\Big] = 0 \,. \quad (66)$$

Hence by Stokes' theorem we have

$$\int_{\partial Z} \sum_{i=0}^{m} \Big[\bar{\epsilon}_{m-i} (*d)^{m-i} e_q^2 \wedge (*d)^i e_p^1 + \bar{\epsilon}_{m-i} (*d)^{m-i} e_q^1 \wedge (*d)^i e_p^2 - \big(\bar{\epsilon}_{m-i} e_b^1 \wedge (*d)^i e_p^2 + (*d)^{m-i} e_q^2 \wedge f_b^1\big)\Big] = 0 \,, \quad (67)$$

for all e_p^2, e_q^2, that is,

$$f_b^1 = \bar{\epsilon}_{m-i} (*d)^i e_p^1|_{\partial Z} \ , \ e_b^1 = (*d)^{m-i} e_q^1|_{\partial Z} \,. \quad (68)$$

It shows indeed $\Psi_1 \in \mathcal{D}$. ∎

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