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Stability Analysis and H_∞ Synthesis for Linear Systems With Time-Varying Delays

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***Abstract*— This paper is devoted to stability analysis and synthesis of the linear systems with time-varying delays. Some new stability conditions are developed for the systems based on Lyapunov-Razumikhin theorem. Then a design method for the general state feedback controllers is proposed for the systems with multiple delays by the LMI optimization based approach. Numerical examples are used to demonstrate the effectiveness of the proposed design technique.**

I. INTRODUCTION

An ever-growing number of internet-connected devices is now accessible to a multitude of users. Being a ubiquitous communication means, the internet could allow any user to reach and command any device connected to the network. The internet has become the preferred form of interactive communication, with new applications in multiplayer games, teleconferencing, and telerobotics being developed and tested every day [11]. Internet based control has become an important means in the control systems design. A very important problem in this kind of systems is the delays due to the signal communication. Actually, nonlinear systems with time-delay constitute basic mathematical models of real phenomena, for instance in circuits theory, economics and mechanics. Not only dynamical systems with time-delay are common in chemical processes and long transmission lines in pneumatic, hydraulic, or rolling mill systems, but computer controlled systems requiring numerical computation have time-delays in control loops. The presence of time-delays in control loops usually degrades system performance and complicates the analysis and design of feedback controllers. Stability analysis and synthesis of retarded systems is an important issue addressed by many authors and for which surveys can be found in several monographs (see *e.g.* [7], [6]).

Most of these previous work on time delays has focused on unknown but constant time delays. In practical systems, the time delays are variable due to the motion of the slave systems, for example, space-based or underwater telerobotic applications involve moving vehicles and thus experience changing transmission times to and from the stationary operator. Moreover, some systems possess rapidly and possibly randomly varying transmission delays. A obvious example is the satellite-based transmission through varying relay sites. In the internet, which has frequently been used as a means for creating teleoperation systems between a variety of remote sites, information is transmitted in small packets and is routed in real-time through a possibly large number of intermediate stops. Although average latencies may be low, the instantaneous delays may increase suddenly due to rerouting or other network traffic. In the extreme, the connection may be temporarily blocked. It could be expected that the above mentioned methods are applicable even in the case of random time-varying delay, by designing the control for the maximum value of the delay (worst case controller). However, it is shown in [8] that a control algorithm designed for a fixed, maximum delay T may not stabilize the system when the delay varies between 0 and T. Niemeyer and Slotine [10] also showed some stability problems due to internet transmission.

Stability criteria for linear system with time delays can be classified into two categories considering their dependence from time delays. Delay-independent stability conditions are independent of the size of the delays (*i.e.*, the time delays are allowed to be arbitrarily large) and thus, in general, are conservative, especially in situations where delays are small. It can be used to study the systems without any information on the time delays. Delay-dependent results [2], [1], [3] are usually used to determine a maximum value for the time delays which guarantees stability. They are expected to be less conservative. To facilitate the computation process, the linear matrix inequality (LMI) approach is employed in the development. Some work has been devoted to extend these results to the systems with time-varying delays. However, the time-varying information of the delay is required and the stability conditions always require the upper bound of the time derivative of delays less than 1 [2], [3], [4].

It is well known that the choice of an appropriate Lyapunov functional is the key-point for deriving of stability criteria. In this paper, we will present a new method for the stability analysis of the systems with time-varying delays by the Lyapunov-Razumikhin function approach. We will propose some delay-independent conditions and delay-dependent conditions which will be used to test the stability for the various control systems without time-varying information of delays. The requirement that the upper bound of the time derivative of delays is less than 1 in the above mentioned papers will be removed in these conditions.

The paper is organized as follows. Section 2 gives the problem description. Delay-independent and delay-

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dependnet stability analysis and design will be addressed in Sections 3 and 4 respectively. H_∞ control will be studied in Section 5. We will give some numerical examples to show the feasibility of the result in Section 6. Finally, Section 7 will conclude the paper.

II. Problem Statement

Consider linear systems with time-varying delays

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{r} A_i x(t-d_i(t)) + B_0 u(t), \quad (1)\\ x(t) &= \psi(t), \quad t \in [-\tau, 0], \quad (2)\end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, $0 < d_i(t) \leq \tau_i \leq \tau$ the time-varying delays and A, A_i and B_0 are appropriately dimensioned real-valued matrices. Assume that the initial condition ψ is a continuous vector-valued function, *i.e.*, $\psi \in \mathcal{C}_{n,\tau}$. We use $x_t \in \mathcal{C}_{n,\tau}$ to denote the restriction of $x(t)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0]$.

In this paper, we will be interested in the stability analysis and design for the system (1)-(2). We will also consider the control of the system (1) by the instantaneous state feedback

$$u(t) = F_0 x(t), \quad (3)$$

and the delayed state feedback

$$u(t) = \sum_{i=1}^{r} F_i x(t-d_i(t)), \quad (4)$$

where $F_0, F_i \in \mathbb{R}^{m\times n}$. But in what follows, we will combine the instantaneous state feedback (3) and the delayed state feedback (4) as

$$u(t) = \sum_{i=0}^{r} F_i x(t-d_i(t)), \quad (5)$$

with $d_0(t) = 0$. Then the closed-loop system under the above state feedback (5) can be written as

$$\dot{x}(t) = \sum_{i=0}^{r} (A_i + B_0 F_i) x(t-d_i(t)). \quad (6)$$

III. Delay-Independent Stability Analysis

In this section, we will first give methods on delay-independent stability for the system (1)-(2) with $u = 0$. We will then present a state feedback controller design such that the closed-loop system is delay-independent stable.

Define $y(t) = \dot{x}(t)$. We can then rewrite system (1) with the following form

$$A_0 x(t) - y(t) + \sum_{i=1}^{r} A_i x(t-d_i(t)) = 0. \quad (7)$$

For simplicity, in what follows, we will use the following notation:

$$\begin{aligned} x_d &= \begin{bmatrix} x^T(t-d_1(t)) & \ldots & x^T(t-d_r(t)) \end{bmatrix}^T,\\ \bar{x}(t) &= \begin{bmatrix} x^T(t) & y^T(t) & x_d^T \end{bmatrix}^T,\\ A_d &= \begin{bmatrix} A_1 & \ldots & A_r \end{bmatrix}.\end{aligned}$$

Then, (7) can be rewritten as

$$A_0 x(t) - y(t) + A_d x_d = 0.$$

Hence, we have

$$2(P_1 x + P_2 y + P_d x_d)^T (A_0 x - y + A_d x_d) = 0 \quad (8)$$

for any weighting matrices P_1, P_2 and P_d with compatible dimensions.

A. Delay-independent stability condition

Theorem 1: Consider system (1)-(2) with $u \equiv 0$, if there exist matrices $P_0 > 0, P_1, P_2, P_d$ and $Q > 0$ of compatible dimensions such that

$$\begin{bmatrix} P_1^T A_0 + A_0^T P_1 + rP_0 & * & * \\ P_2^T A_0 - P_1 + P_0 & -P_2^T - P_2 & * \\ P_d^T A_0 + A_d^T P_1 & A_d^T P_2 - P_d^T & \Lambda_{33} \end{bmatrix} < 0, \quad (9)$$

$$Q - \mathrm{diag}\{P_0, P_0, \ldots, P_0\} < 0, \quad (10)$$

where

$$\Lambda_{33} = P_d^T A_d + A_d^T P_d - Q,$$

then the solution $x(t) \equiv 0$ is delay-independently asymptotically stable for any time-varying time-delays $d_i(t) > 0$, for $i = 1, 2, \ldots, r$.

Proof: Given $P_0 > 0$, consider a quadratic Lyapunov function candidate as

$$V(x(t)) = x^T(t) P_0 x(t).$$

First, we have

$$\varepsilon_1 ||x||^2 \leq V(x) \leq \varepsilon_2 ||x||^2,$$

where $\varepsilon_1 = \lambda_{\min}(P_0),\ \varepsilon_2 = \lambda_{\max}(P_0)$. The derivative of $V(x(t))$ along the solutions of (1)-(2) is

$$\dot{V}(x(t)) = 2x^T(t) P_0 y(t).$$

By (8), we have,

$$\dot{V}(x(t)) = \bar{x}^T(t) \Omega \bar{x}(t),$$

where $\Omega =$

$$\begin{bmatrix} P_1^T A_0 + A_0^T P_1 & * & * \\ P_2^T A_0 - P_1 + P_0 & -P_2^T - P_2 & * \\ P_d^T A_0 + A_d^T P_1 & A_d^T P_2 - P_d^T & P_d^T A_d + A_d^T P_d \end{bmatrix}.$$

Define

$$\xi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \ \Gamma = \begin{bmatrix} P_1^T A_d + A_0^T P_d \\ P_2^T A_d - P_d \end{bmatrix}$$

Note that

$$2x_d^T \Gamma^T \xi(t) \leq \xi^T(t) \Gamma \tilde{Q}^{-1} \Gamma^T \xi(t) + x_d^T \tilde{Q} x_d,$$

where $\tilde{Q} = Q - (P_d^T A_d + A_d^T P_d) > 0$. We then have

$$\begin{aligned} \dot{V}(x(t)) &\leq \xi^T(t) \left\{ \Lambda + \Gamma \tilde{Q}^{-1} \Gamma^T \right\} \xi(t) + x_d^T Q x_d \\ &\leq \xi^T(t) \left\{ \Lambda + \Gamma \tilde{Q}^{-1} \Gamma^T \right\} \xi(t) \\ &\quad + \sum_{i=1}^{r} x^T(t-d_i) P_0 x(t-d_i), \end{aligned}$$

if $Q < \text{diag}\{P_0, P_0, \ldots, P_0\}$, where

$$\Lambda = \begin{bmatrix} P_1^T A_0 + A_0^T P_1 & A_0^T P_2 - P_1^T + P_0 \\ P_2^T A_0 - P_1 + P_0 & -P_2^T - P_2 \end{bmatrix}.$$

By using Razumikhin theorem, we assume that there exists a real $v > 1$ such that

$$V(x(t-\theta)) < vV(x(t)), \quad \text{for} \quad \theta \in [0, \tau]$$

then

$$\begin{aligned} \dot{V}(x(t)) &\leq \xi^T(t)\left(\Lambda + \Gamma\tilde{Q}^{-1}\Gamma^T\right)\xi(t) + vrV(x(t)) \\ &= \xi^T(t)(\Omega + vr\bar{P}_0 + \Gamma\tilde{Q}^{-1}\Gamma^T)\xi(t) \end{aligned}$$

where

$$\bar{P}_0 = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously, if (9) holds, by continuity, we can always find a $v > 1$ such that

$$\begin{bmatrix} P_1^T A_0 + A_0^T P_1 + vrP_0 & * & * \\ P_2^T A_0 - P_1 + P_0 & -P_2^T - P_2 & * \\ P_d^T A_0 + A_d^T P_1 & A_d^T P_2 - P_d^T & \Lambda_{33} \end{bmatrix} < 0.$$

By Schur complements, we have $\dot{V}(x(t)) < 0$ for $\forall x \neq 0$. This completes the proof. ■

Note that the condition of Theorem 1 does not include any information of time-delay, *i.e.*, the theorem provides a delay-independent condition for stability of linear time-delay systems with time-varying delays in terms of the solvability of several linear matrix inequalities.

Remark 1: In some of the early references by the Lyapunov-Krasovskii functional approach, for example [2], [4], the upper bound of the derivative information of the time-varying delay is required to be known and then some LMI conditions involving this upper bound were derived. Based on the Lyapunov-Razumikhin theorem and the special equality (8), we obtained a new LMI condition on the stability of the systems with time-varying time-delays, which also works in the case that the time-delays are time-varying and the time-varying information is not available.

B. Controller design

With the state feedback (5), the closed-loop system can be described by (6). By Theorem 1, the closed-loop system is delay-independently stable for all time-delays if there exist matrices $P_0 > 0, P_1, P_2, P_d$ and $Q_i > 0$ of compatible dimensions satisfying (10) and

$$\begin{bmatrix} P_1^T \hat{A}_0 + \hat{A}_0^T P_1 + rP_0 & * & * \\ P_2^T \hat{A}_0 - P_1 + P_0 & -P_2^T - P_2 & * \\ P_d^T \hat{A}_0 + \hat{A}_d^T P_1 & \hat{A}_d^T P_2 - P_d^T & \bar{\Lambda}_{33} \end{bmatrix} < 0, \tag{11}$$

where $\bar{\Lambda}_{33} = P_d^T \hat{A}_d + \hat{A}_d^T P_d - Q$, *i.e.*

$$\begin{bmatrix} P^T \hat{A} + \hat{A}^T P + r\Gamma_2 P_0 \Gamma_2^T & P^T \Gamma_1 \hat{A}_d + \hat{A}^T \Gamma_1 P_d \\ \hat{A}_d^T \Gamma_1^T P + P_d^T \Gamma_1^T \hat{A} & P_d^T \hat{A}_d + \hat{A}_d^T P_d - Q \end{bmatrix} < 0.$$

where

$$\begin{aligned} \hat{A}_d &= \begin{bmatrix} \hat{A}_1 & \hat{A}_2 & \ldots & \hat{A}_r \end{bmatrix}, \\ P_d &= \begin{bmatrix} P_{d1} & P_{d2} & \ldots & P_{dr} \end{bmatrix}, \\ \hat{A} &= \begin{bmatrix} 0 & I \\ \hat{A}_0 & -I \end{bmatrix}, P = \begin{bmatrix} P_0 & 0 \\ P_1 & P_2 \end{bmatrix}, \\ \Gamma_1 &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \Gamma_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}. \end{aligned}$$

Let

$$\begin{aligned} X &= P^{-1} = \begin{bmatrix} X_0 & 0 \\ X_1 & X_2 \end{bmatrix}, X_{di} = P_{di}^{-1}, \\ X_d &= \text{diag}\{X_{d1}, X_{d2}, \ldots, X_{dr}\}, \\ \Gamma_3 &= P_d X_d = \begin{bmatrix} I & I & \ldots & I \end{bmatrix}, \\ \Theta &= \text{diag}\{X,\ X_d\}. \end{aligned}$$

Right- and left-Multiplying (11) by Θ and Θ^T respectively, we obtain

$$\begin{bmatrix} X_1 + X_1^T + rX_0 P_0 X_0 & * & * \\ \hat{A}_0 X_0 - X_1 + X_2^T & -X_2 - X_2^T & * \\ \Gamma_3^T(\hat{A}_0 X_0 - X_1) & X_d^T \hat{A}_d - \Gamma_3^T X_2 & \Xi_{33} \end{bmatrix} < 0,$$

where $\Xi_{33} = \Gamma_3^T \hat{A}_d X_d + X_d^T \hat{A}_d^T \Gamma_3 - X_d^T Q X_d$. By substituting $X_{di} = X_0,\ Y_0 = F_0 P_0^{-1}$ and $Y_i = F_i P_0^{-1}$, it is easy to obtain the following result.

Theorem 2: Consider system (1)-(2), if there exist matrices $X_0 > 0, X_1, X_2, Q > 0, Y_0$ and $Y_i,\ i = 1, 2, \ldots, r$, of compatible dimensions such that

$$\begin{bmatrix} X_1 + X_1^T + rX_0 & * & * \\ A_0 X_0 + BY_0 - X_1 + X_2^T & -X_2 - X_2^T & * \\ \Gamma_3^T(A_0 X_0 + BY_0 - X_1) & \Xi_{32} & \bar{\Xi}_{33} \end{bmatrix} < 0,$$

$$Q - \text{diag}\{X_0, X_0, \ldots, X_0\} < 0,$$

where

$$\begin{aligned} &\Xi_{32} = X_0^T A_d^T + Y_d^T B^T - \Gamma_3^T X_2, \\ &\bar{\Xi}_{33} = \Gamma_3^T(A_d X_0 + BY_d) + (X_0^T A_d^T + Y_d^T B^T)\Gamma_3 - Q \\ &Y_d = \begin{bmatrix} Y_1 & Y_2 & \ldots & Y_r \end{bmatrix}, \end{aligned}$$

then the delayed state feedback (5) with $F_i = Y_i X_0^{-1}, i = 0, 1, \ldots, r$, stabilizes the system (1)-(2) for any time-delays $d_i(t) > 0$.

IV. Delay-Dependent Stability Analysis

To reduce conservativeness in the analysis when the size on the delay is available, in this section, we will establish a new delay-dependent stability condition for the system (1)-(2). Recently, a new descriptor model transformation and a corresponding Lyapunov-Krasovskii functional have been introduced for stability of systems with constant delays [3]. The advantage of this transformation is to transform the original system to an equivalent descriptor form representation and will not introduce additional dynamics in the sense defined in [5]. In this section, based on the Lyapunov-Razumikhin theorem, we will present a new method for the stability analysis of the systems with time-varying delays, which also does not introduce any additional dynamics.

A. Delay-dependent stability condition

Rewrite (1) with the following equivalent formulation

$$y(t) \quad = \quad A_0 x(t) + \sum_{i=1}^{r} A_i x(t-d_i(t)). \qquad (12)$$

Since $x(t)$ is continuously differentiable for $t \geq 0$, using the Leibniz-Newton formula, one can write

$$x(t-d_i(t)) = x(t) - \int_{t-d_i(t)}^{t} y(s)ds$$

for $t \geq d_i(t)$. Then equation (12) can be represented in the following form with discrete and distributed delays in y

$$0 = -y(t) + Ax(t) - \sum_{i=1}^{r} A_i \int_{t-d_i(t)}^{t} y(s)ds, \qquad (13)$$

where $A = A_0 + \sum_{i=1}^{r} A_i$. In what follows, we will denote

$$\begin{aligned} M_i &= \begin{bmatrix} M_{11i} & M_{12i} & M_{13i} \\ M_{12i}^T & M_{22i} & M_{23i} \\ M_{13i}^T & M_{23i}^T & M_{33i} \end{bmatrix}, \\ \bar{N}_i &= \begin{bmatrix} N_{1i} & N_{2i} & N_{3i} \end{bmatrix}. \end{aligned}$$

Then (13) can be rewritten as

$$\begin{aligned} & Ax(t) - y(t) + A_d y_d = 0, \\ & y_d = \begin{bmatrix} \int_{t-d_1(t)}^{t} y^T(s)ds & \ldots & \int_{t-d_r(t)}^{t} y^T(s)ds \end{bmatrix}^T. \end{aligned}$$

Theorem 3: Consider system (1)-(2) with $u \equiv 0$, if there exist constants τ_i and matrices $P_0 > 0$, $\bar{P} = \begin{bmatrix} P_1 & P_2 & P_d \end{bmatrix}$, $M_i > 0$, $\bar{N}_i$, $Q > 0, Q_{2i} > 0, i = 1, 2, \ldots, r$, of compatible dimensions, satisfying

$$\begin{bmatrix} T_{11} + rP_0 & T_{12} & T_{13} \\ T_{12}^T & T_{22} + \sum_{i=1}^{r} \tau_i Q_{2i} & T_{23} \\ T_{13}^T & T_{23}^T & T_{33} - Q \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} M_i & \bar{N}_i^T \\ \bar{N}_i & Q_{2i} \end{bmatrix} > 0, \quad (15)$$

$$Q - \operatorname{diag}\{P_0, P_0, \ldots, P_0\} < 0. \quad (16)$$

where

$$\begin{aligned} T_{11} &= A_0^T P_1 + P_1^T A_0 + \sum_{i=1}^{r} (\tau_i M_{11i} + N_{1i}^T + N_{1i}), \\ T_{12} &= P_0 - P_1^T + A_0^T P_2 + \sum_{i=1}^{r} \left(\tau_i M_{12i} + N_{2i}\right), \\ T_{22} &= -P_2 - P_2^T + \sum_{i=1}^{r} \tau_i M_{22i}, \\ T_{13} &= P_1^T A_d - N_1^T + A_0^T P_d + \sum_{i=1}^{r} \left(\tau_i M_{13i} + N_{3i}\right) \\ T_{23} &= P_2^T A_d - N_2^T - P_d + \sum_{i=1}^{r} \tau_i M_{23i} \\ T_{33} &= A_d^T P_d + P_d^T A_d - N_3 - N_3^T + \sum_{i=1}^{r} \tau_i M_{33i}, \end{aligned}$$

$$\begin{aligned} N_1 &= \begin{bmatrix} N_{11}^T & N_{12}^T & \ldots & N_{1r}^T \end{bmatrix}^T, \\ N_2 &= \begin{bmatrix} N_{21}^T & N_{22}^T & \ldots & N_{2r}^T \end{bmatrix}^T, \\ N_3 &= \begin{bmatrix} N_{31}^T & N_{32}^T & \ldots & N_{3r}^T \end{bmatrix}^T. \end{aligned}$$

then the solution $x(t) \equiv 0$ is asymptotically stable for all time-varying delays $d_i(t) \leq \tau_i$.

Proof: We choose the Lyapunov candidate as

$$V(x(t), t) \quad = \quad x^T(t) P_0 x(t),$$

It is easy to see, $\dot{V}(x, t)$

$$\begin{aligned} &= 2x^T P_0 y + 2(P_1 x + P_2 y + P_d x_d)^T (Ax - y - A_d y_d) \\ &= x^T (A^T P_1 + P_1^T A)x + y^T(-P_2 - P_2^T)y \\ &\quad + 2x^T (P_0 - P_1^T + A^T P_2)y + 2x_d^T P_d^T Ax - 2x_d^T P_d^T y \\ &\quad - 2\bar{x}^T \bar{P}^T (\sum_{i=1}^{r} \int_{t-d_i(t)}^{t} A_i y(s)ds). \end{aligned}$$

Note that [9]

$$\begin{aligned} & -2\bar{x}^T \bar{P}^T \int_{t-d_i}^{t} A_i y(s)ds \\ \leq \; & \tau_i \bar{x}^T M_i \bar{x} + 2\bar{x}^T (\bar{N}_i^T - \bar{P}^T A_i) \int_{t-d_i}^{t} y(s)ds \\ & + \int_{t-d_i}^{t} y^T(s) Q_{2i} y(s)ds \\ \leq \; & \tau_i \bar{x}^T M_i \bar{x} + 2\bar{x}^T (\bar{N}_i^T - \bar{P}^T A_i)(x - x(t-d_i)) \\ & + \int_{t-\tau_i}^{t} y^T(s) Q_{2i} y(s)ds \end{aligned}$$

Then we have

$$\begin{aligned} \dot{V}(x, t) \quad \leq \quad & \bar{x}^T \left(\Theta + \sum_{i=1}^{r} \tau_i M_i \right) \bar{x} \\ & + \sum_{i=1}^{r} \int_{t-\tau_i}^{t} y^T(s) Q_{2i} y(s)ds, \end{aligned}$$

where $\Theta =$

$$\begin{bmatrix} P_1^T A_0 + A_0^T P_1 + \sum_{i=1}^{r} (N_{1i}^T + N_{1i}) & * & * \\ P_0 + P_2^T A_0 - P_1 + \sum_{i=1}^{r} N_{2i}^T & \Lambda_{22} & * \\ A_d^T P_1 - N_1 + P_d^T A_0 + \sum_{i=1}^{r} N_{3i}^T & \Lambda_{32} & \Lambda_{33} \end{bmatrix},$$

and $\qquad \Lambda_{22} = -P_2 - P_2^T$,

$$\begin{aligned} \Lambda_{32} &= A_d^T P_2 - N_2 - P_d^T, \\ \Lambda_{33} &= A_d^T P_d + P_d^T A_d - N_3 - N_3^T. \end{aligned}$$

To apply Razumikhin theorem, we assume that there exist a constant $\mu_1 > 1$ such that $V(x(t+\theta), t+\theta) \leq \mu_1 V(x(t), t)$, $\theta \in [-\tau, 0]$, i.e.

$$x^T(t+\theta) P_0 x(t+\theta) \leq \mu_1 x^T(t) P_0 x(t), \; \theta \in [-\tau, 0],$$

From (1), there exists a constant $\mu_2 > 1$ such that

$$y^T(t+\theta) Q_{2i} y(t+\theta) \leq \mu_2 y^T(t) Q_{2i} y(t), \; \theta \in [-\tau, 0].$$

Let $\nu = \max \mu_1, \mu_2$, we then have $\nu > 1$ such that

$$\begin{aligned} x^T(t+\theta)P_0x(t+\theta) &\leq \nu x^T(t)P_0x(t), \\ y^T(t+\theta)Q_{2i}y(t+\theta) &\leq \nu y^T(t)Q_{2i}y(t),\ \theta \in [-\tau, 0]. \end{aligned}$$

The remain proof is similar to Theorem 1. ■

B. Controller design

Theorem 4: Given system (1)-(2), if there exist constants τ_i and matrices $X_0 > 0, X_1, X_2, Q > 0, Q_{2i} > 0, M_i > 0, \bar{N}_i, Y_i, i = 0, 1, \dots, r$, satisfying

$$\begin{bmatrix} \hat{T}_{11} + rX_0 & \hat{T}_{12} & \hat{T}_{13} & X_1^T\hat{T}_{14} \\ \hat{T}_{12}^T & \hat{T}_{22} & \hat{T}_{23} & X_2^T\hat{T}_{14} \\ \hat{T}_{13}^T & \hat{T}_{23}^T & \hat{T}_{33} - Q & 0 \\ \hat{T}_{14}^TX_1 & \hat{T}_{14}^TX_2 & 0 & \hat{T}_{44} \end{bmatrix} < 0, \quad (17)$$

$$\begin{bmatrix} M_i & \bar{N}_i^T \\ \bar{N}_i & X_0^T + X_0 - Q_{2i} \end{bmatrix} > 0, \quad (18)$$

$$Q - \operatorname{diag}\{X_0, X_0, \dots, X_0\} < 0, \quad (19)$$

where

$$\begin{aligned}
\hat{T}_{11} &= X_1 + X_1^T + \sum_{i=1}^{r} \left(\tau_i M_{11i} + N_{1i} + N_{1i}^T\right), \\
\hat{T}_{12} &= (A_0X_0 + BY_0 - X_1)^T + X_2 + \sum_{i=1}^{r}(\tau_i M_{12i} + N_{1i}), \\
\hat{T}_{22} &= -X_2 - X_2^T + \sum_{i=1}^{r} \tau_i M_{22i}, \\
\hat{T}_{13} &= (A_0X_0 + BY_0 - X_1)^T\Gamma_3 - N_1^T + \sum_{i=1}^{r} \tau_i M_{13i}, \\
\hat{T}_{23} &= A_dX_0 + BY_d - X_2^T\Gamma_3 - N_2^T + \sum_{i=1}^{r} \tau_i M_{23i} \\
\hat{T}_{33} &= \Gamma_3^T(A_dX_1 + BY_d) + (X_1^TA_d^T + Y_d^TB^T)\Gamma_3 \\
&\quad -N_3 - N_3^T + \sum_{i=1}^{r} \tau_i M_{33i}, \\
\hat{T}_{14} &= \begin{bmatrix} \tau_1 I & \tau_2 I & \cdots & \tau_r I \end{bmatrix}, \\
\hat{T}_{44} &= -\operatorname{diag}\{\tau_1 Q_{21}, \tau_2 Q_{22}, \cdots, \tau_r Q_{2r}\},
\end{aligned}$$

then the state feedback (5) with $F_i = Y_iX_0^{-1}, i = 0, 1, \dots, r$, stabilizes the system (1)-(2) for all time-delays $0 \leq d_i(t) \leq \tau_i$.

Proof: By Theorem 3, the closed-loop system with state feedback (5) is stable for time-delays $d_i(t) \leq \tau_i$ if (15) and

$$\begin{aligned}
&\begin{bmatrix} \hat{A}_0^TP_1 + P_1^T\hat{A}_0 + rP_0 & * & * \\ P_2^T\hat{A}_0 - P_1 + P_0 & -P_2 - P_2^T & * \\ P_d^T\hat{A}_0 + \hat{A}_d^TP_1 & \hat{A}_d^TP_2 - P_d^T & \bar{\Lambda}_{33} \end{bmatrix} \\
&+ \sum_{i=1}^{r}(\tau_i M_i + \Pi_1\bar{N}_i + \bar{N}_i^T\Pi_1^T + \tau_i\Pi_2Q_{2i}\Pi_2^T) \\
&-\Pi_3N - N^T\Pi_3^T < 0, \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\Lambda}_{33} &= \hat{A}_d^TP_d + P_d^T\hat{A}_d - Q, \\
\Pi_1 &= \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T,\ \Pi_2 = \begin{bmatrix} 0 & I & 0 \end{bmatrix}^T, \\
\Pi_3 &= \begin{bmatrix} 0 & 0 & I \end{bmatrix}^T, \\
N &= \begin{bmatrix} \bar{N}_1^T & \bar{N}_2^T & \dots & \bar{N}_r^T \end{bmatrix}^T.
\end{aligned}$$

Right- and left-Multiplying the above inequality by Θ and Θ^T respectively, and substituting $M_i \Leftarrow \Theta^TM_i\Theta,\ \bar{N}_i \Leftarrow X_0\bar{N}_i\Theta, Q_{2i} \Leftarrow Q_{2i}^{-1}$, and $Q \Leftarrow X_dQX_d$, it is easy to find that (20) and (14) are equivalent to (17) and

$$\begin{bmatrix} M_i & \bar{N}_i^T \\ \bar{N}_i & X_0^TQ_{2i}^{-1}X_0 \end{bmatrix} > 0, \quad (21)$$

respectively. Note that for any matrix $X_i > 0$, we have

$$X_0^TQ_{2i}^{-1}X_0 \geq X_0^T + X_0 - Q_{2i},$$

Hence, (21) holds if (18) holds. ■

V. State Feedback H_∞ Controller Design

Now we consider the H_∞ controller design of the following system. Let us first consider linear systems with time-varying delays

$$\dot{x}(t) = \sum_{i=0}^{r} A_ix(t - d_i) + Bu(t) + B_1w(t), \quad (22)$$

$$z(t) = \sum_{i=0}^{r} C_ix(t - d_i) + Du(t) + D_1w(t), \quad (23)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, $w \in \mathcal{L}_2^q[0, \infty]$ the exogenous disturbance and $z(t) \in \mathbb{R}^p$ the output to be attenuated. For H_∞ control, we always consider the following performance index

$$J = \int_0^\infty (z^Tz - \gamma^2w^Tw)ds, \quad (24)$$

under zero initial condition, where $\gamma > 0$ is a prescribed constant. In what follows, we will consider the design of the state feedback controller (5) such that inequality $J < 0$ holds for all nonzero $w \in \mathcal{L}_2^q[0, \infty]$.

Theorem 5: Given autonomous system (22)-(23) with $u \equiv 0$, for a prescribed $\gamma > 0$, $J < 0$ holds for all nonzero $w \in \mathcal{L}_2^q[0, \infty]$ and all time-varying delays $d_i(t) \leq \tau_i$, if there exist matrices $P_0 > 0, \bar{P} = \begin{bmatrix} P_1 & P_2 & P_d \end{bmatrix},\ M_i > 0,\ \bar{N}_i,\ Q > 0,\ Q_{2i} > 0, i = 1, 2, \dots, r$, of compatible dimensions, satisfying LMI (14), (16) and

$$\begin{bmatrix} T_{11} + rP_0 & T_{12} & T_{13} & P_1^TB_1 & C_0^T \\ T_{12}^T & T_{22} + \bar{Q}_{2i} & T_{23} & P_2^TB_1 & 0 \\ T_{13}^T & T_{23}^T & T_{33} - Q & 0 & C_d^T \\ B_1^TP_1 & B_1^TP_2 & 0 & -\gamma^2I & D_1^T \\ C_0 & 0 & C_d & D_1 & -I \end{bmatrix} < 0 \quad (25)$$

where

$$\bar{Q}_{2i} = \sum_{i=1}^{r} \tau_iQ_{2i},\ C_d = \begin{bmatrix} C_1 & C_2 & \cdots & C_r \end{bmatrix}.$$

Theorem 6: Given system (22)-(23) and a prescribed $\gamma > 0$, $J < 0$ holds for all nonzero $w \in \mathcal{L}_2^q[0, \infty]$ and all time-varying delays $d_i(t) \le \tau_i$ if there exist matrices $X_0 > 0, X_1, X_2, Y_0, Y_i, M_i > 0,\ \bar{N}_i, Q > 0,\ Q_{2i} > 0, i = 1, 2, \ldots, r$, of compatible dimensions, satisfying (18), (19) and

$$\begin{bmatrix} \hat{T}_{11} + rX_0 & \hat{T}_{12} & \hat{T}_{13} & * & * & * \\ \hat{T}_{12}^T & \hat{T}_{22} & \hat{T}_{23} & * & * & * \\ \hat{T}_{13}^T & \hat{T}_{23}^T & \hat{T}_{33} - Q & 0 & * & * \\ \hat{T}_{14}^T X_1 & \hat{T}_{14}^T X_2 & 0 & \hat{T}_{44} & 0 & 0 \\ 0 & B_1^T & 0 & 0 & -\gamma^2 I & 0 \\ \hat{T}_{61}^T & 0 & \hat{T}_{63}^T & 0 & 0 & -I \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned} \hat{T}_{61}^T &= C_0 X_0 + D Y_0, \\ \hat{T}_{63}^T &= \begin{bmatrix} C_1 X_0 + D Y_1 & \cdots & C_r X_0 + D Y_r \end{bmatrix}, \end{aligned}$$

then the state feedback (5) with $F_i = Y_i X_0^{-1}, i = 0, 1, \ldots, r$, stabilizes the system (22)-(23) with $J < 0$.

VI. Numerical Examples

Example 1. Consider the example given in [4]. The system matrices are

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

This example is also discussed in many other references. The reference [4] gave a very detailed comparison with the known results. By Theorem 3, we find the upper bound of delay for the stability is $\tau \le \tau_0 = 4.472$, which is slightly better than the result of [4]. Moreover, our theory shows that the system is still stable even when the time-delay is time-varying for any $d(t) \le \tau_0 = 4.472$. However, the result of [4] can only guarantee the stability for the system with constant delay $\tau \le 4.47$. When time-delay is time-varying, the upper bound of the derivative of time-delay is involved in [4] and it has to be assumed to be less than 1.

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Example 2. Now we consider the following example

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C_0 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ D_0 &= \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

In [4], the stabilizability bounds was studied for the above system. The obtained maximum value in [4] is $\tau = 1.408$ for the system with constant delay. By Theorem 6, an upper bound $\tau = 1.684$ is obtained. This implies that the above system with any time-varying delay $d(t) \le 1.684$ can be stabilized by a memoryless state feedback. Actually, when $\tau = 1.684$, we obtain the state feedback gain

$$F = \begin{bmatrix} -0.2966 & -2.1653 \end{bmatrix}.$$

At this time, the minimum H_∞ performance bound is $\gamma = 5.6460$, which is obtained by Theorem 5.

VII. Conclusions

In this paper, both the delay-independent and delay-dependent stability are discussed by applying Lyapunov-Razumikhin theorem for the time-delay system with time-varying time delays. This paper also improved the stability conditions of the known references. We also discussed the stabilizability problem and the H_∞ control design for this class of systems with time-varying delays. Numerical examples are also proposed to show the effectiveness and the less conservativeness of the proposed method.

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